SOLUTION MANUAL

DYNAMICS OF STRUCTURES

THEORY AND APPLICATIONS TO EARTHQUAKE ENGINEERING

THIRD EDITION

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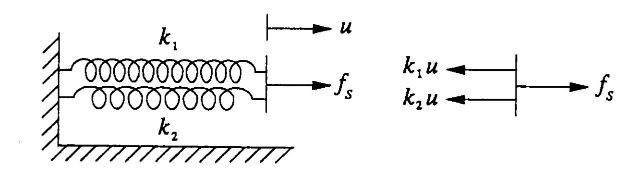
Problem 1.1

If k_e is the effective stiffness,

$$f_S = k_e u$$



Equilibrium of forces: $f_S = (k_1 + k_2)\,u$

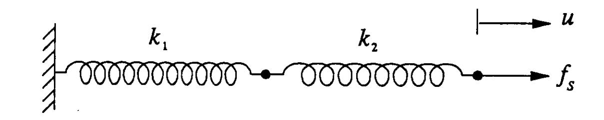
Effective stiffness: $k_e = f_S/u = k_1 + k_2$

Equation of motion: $m\ddot{u} + k_e u = p(t)$

Problem 1.2

If k_e is the effective stiffness,

$$f_S = k_e u \quad \text{(a)}$$



If the elongations of the two springs are u_1 and u_2,

$$u = u_1 + u_2 \quad \text{(b)}$$

Because the force in each spring is f_S,

$$f_S = k_1 u_1 \qquad f_S = k_2 u_2 \quad \text{(c)}$$

Solving for u_1 and u_2 and substituting in Eq. (b) gives

$$\frac{f_S}{k_e} = \frac{f_S}{k_1} + \frac{f_S}{k_2} \Rightarrow \frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \Rightarrow$$

$$k_e = \frac{k_1 k_2}{k_1 + k_2}$$

Equation of motion: $m\ddot{u} + k_e u = p(t)$

Problem 1.3

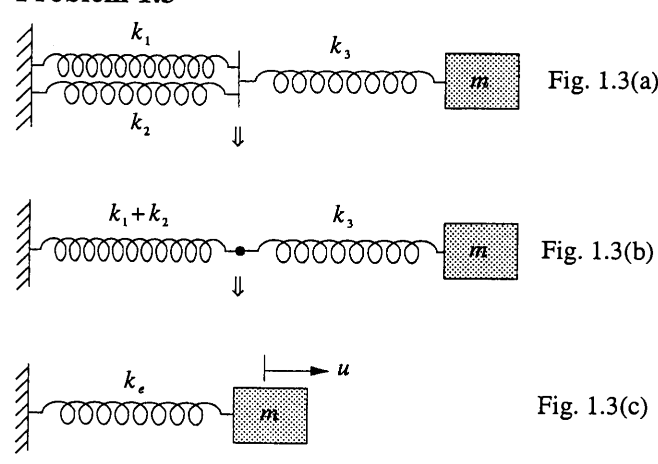


Fig. 1.3(a)

Fig. 1.3(b)

Fig. 1.3(c)

This problem can be solved either by starting from the definition of stiffness or by using the results of Problems P1.1 and P1.2. We adopt the latter approach to illustrate the procedure of reducing a system with several springs to a single equivalent spring.

First, using Problem 1.1, the parallel arrangement of k_1 and k_2 is replaced by a single spring, as shown in Fig. 1.3(b). Second, using the result of Problem 1.2, the series arrangement of springs in Fig. 1.3(b) is replaced by a single spring, as shown in Fig. 1.3(c):

$$\frac{1}{k_e} = \frac{1}{k_1 + k_2} + \frac{1}{k_3}$$

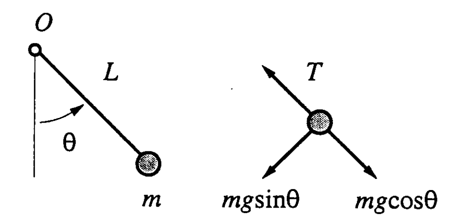
Therefore the effective stiffness is

$$k_e = \frac{(k_1 + k_2)\,k_3}{k_1 + k_2 + k_3}$$

The equation of motion is $m\ddot{u} + k_e u = p(t)$.

Problem 1.4

1. *Draw a free body diagram of the mass.*



2. *Write equation of motion in tangential direction.*

Method 1: By Newton's law.

$$-mg\sin\theta = ma$$

$$-mg\sin\theta = mL\ddot{\theta} \qquad \text{(a)}$$

$$mL\ddot{\theta} + mg\sin\theta = 0$$

This nonlinear differential equation governs the motion for any rotation θ.

Method 2: Equilibrium of moments about O yields

$$mL^2\ddot{\theta} = -mgL\sin\theta$$

or

$$mL\ddot{\theta} + mg\sin\theta = 0$$

3. *Linearize for small θ.*

For small θ, $\sin\theta \approx \theta$, and Eq. (a) becomes

$$mL\ddot{\theta} + mg\theta = 0$$

$$\ddot{\theta} + \left(\frac{g}{L}\right)\theta = 0 \qquad \text{(b)}$$

4. *Determine natural frequency.*

$$\omega_n = \sqrt{\frac{g}{L}}$$

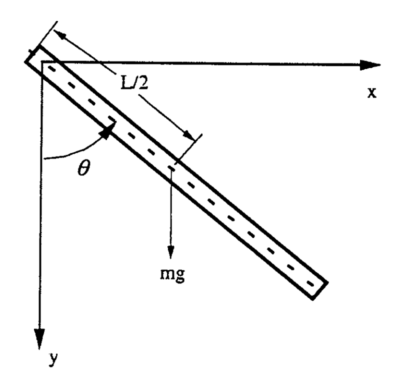
Problem 1.5

1. *Find the moment of inertia about O.*

From Appendix 8,

$$I_0 = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2$$

2. *Draw a free body diagram of the body in an arbitrary displaced position.*



3. *Write the equation of motion using Newton's second law of motion.*

$$\sum M_0 = I_0\ddot{\theta}$$

$$-mg\frac{L}{2}\sin\theta = \frac{1}{3}mL^2\ddot{\theta}$$

$$\frac{mL^2}{3}\ddot{\theta} + \frac{mgL}{2}\sin\theta = 0 \qquad \text{(a)}$$

4. *Specialize for small θ.*

For small θ, $sin\theta \cong \theta$ and Eq. (a) becomes

$$\frac{mL^2}{3}\ddot{\theta} + \frac{mgL}{2}\theta = 0$$

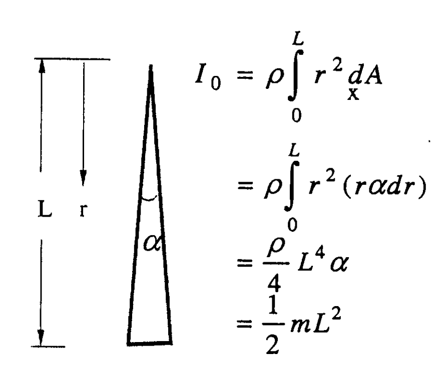
$$\ddot{\theta} + \frac{3}{2}\frac{g}{L} = 0 \qquad \text{(b)}$$

5. *Determine natural frequency.*

$$\omega_n = \sqrt{\frac{3}{2}\frac{g}{L}}$$

Problem 1.6

1. *Find the moment of inertia about about O.*



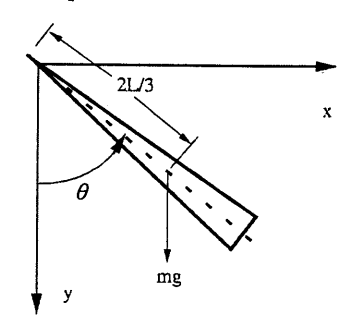
$$I_0 = \rho\int_0^L r^2 dA_x$$

$$= \rho\int_0^L r^2 (r\alpha dr)$$

$$= \frac{\rho}{4} L^4 \alpha$$

$$= \frac{1}{2} mL^2$$

2. *Draw a free body diagram of the body in an arbitrary displaced position.*



3. *Write the equation of motion using Newton's second law of motion.*

$$\sum M_0 = I_0\ddot{\theta}$$

$$-mg\frac{2L}{3}\sin\theta = \frac{1}{2}mL^2\ddot{\theta}$$

$$\frac{mL^2}{2}\ddot{\theta} + \frac{2mgL}{3}\sin\theta = 0 \qquad \text{(a)}$$

4. *Specialize for small θ.*

For small θ, $\sin\theta \cong \theta$, and Eq. (a) becomes

$$\frac{mL^2}{2}\ddot{\theta} + \frac{2mgL}{3}\theta = 0$$

or

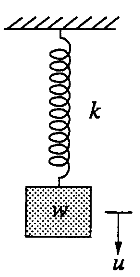
$$\ddot{\theta} + \frac{4}{3}\frac{g}{L} = 0 \qquad \text{(b)}$$

5. *Determine natural frequency.*

$$\omega_n = \sqrt{\frac{4}{3}\frac{g}{L}}$$

In each case the system is equivalent to the spring-mass system shown for which the equation of motion is

$$\left(\frac{w}{g}\right)\ddot{u} + ku = 0$$



The spring stiffness is determined from the deflection u under a vertical force f_S applied at the location of the lumped weight:

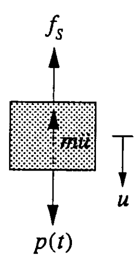
Simply-supported beam: $u = \dfrac{f_S L^3}{48EI} \Rightarrow k = \dfrac{48EI}{L^3}$

Cantilever beam: $u = \dfrac{f_S L^3}{3EI} \Rightarrow k = \dfrac{3EI}{L^3}$

Clamped beam: $u = \dfrac{f_S L^3}{192EI} \Rightarrow k = \dfrac{192EI}{L^3}$

Problem 1.7

Draw a free body diagram of the mass:



Write equation of dynamic equilibrium:

$$m\ddot{u} + f_S = p(t) \qquad \text{(a)}$$

Write the force-displacement relation:

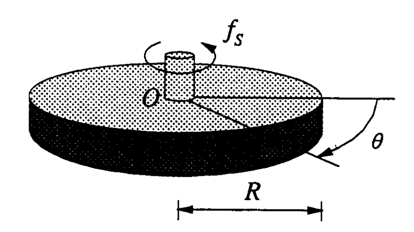
$$f_S = \left(\frac{AE}{L}\right)u \qquad \text{(b)}$$

Substitute Eq. (b) into Eq. (a) to obtain the equation of motion:

$$m\ddot{u} + \left(\frac{AE}{L}\right)u = p(t)$$

Problem 1.8

Show forces on the disk:



Write the equation of motion using Newton's second law of motion:

$$-f_S = I_O\ddot{\theta} \quad \text{where} \quad I_O = \frac{mR^2}{2} \qquad \text{(a)}$$

Write the torque-twist relation:

$$f_S = \left(\frac{GJ}{L}\right)\theta \quad \text{where} \quad J = \frac{\pi d^4}{32} \qquad \text{(b)}$$

Substitute Eq. (b) into Eq. (a):

$$I_O\ddot{\theta} + \left(\frac{GJ}{L}\right)\theta = 0$$

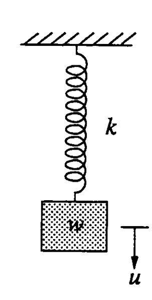
or,

$$\left(\frac{mR^2}{2}\right)\ddot{\theta} + \left(\frac{\pi d^4 G}{32L}\right)\theta = 0$$

Problems 1.9 through 1.11

In each case the system is equivalent to the spring-mass system shown for which the equation of motion is

$$\left(\frac{w}{g}\right)\ddot{u} + ku = 0$$



The spring stiffness is determined from the deflection u under a vertical force f_S applied at the location of the lumped weight:

Simply-supported beam: $u = \frac{f_S L^3}{48EI} \Rightarrow k = \frac{48EI}{L^3}$

Cantilever beam: $u = \frac{f_S L^3}{3EI} \Rightarrow k = \frac{3EI}{L^3}$

Clamped beam: $u = \frac{f_S L^3}{192EI} \Rightarrow k = \frac{192EI}{L^3}$

Problem 1.12

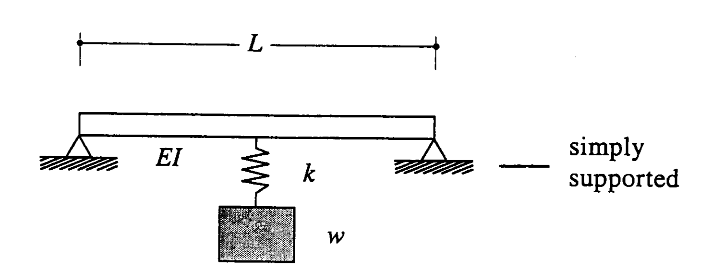


Fig. 1.12a

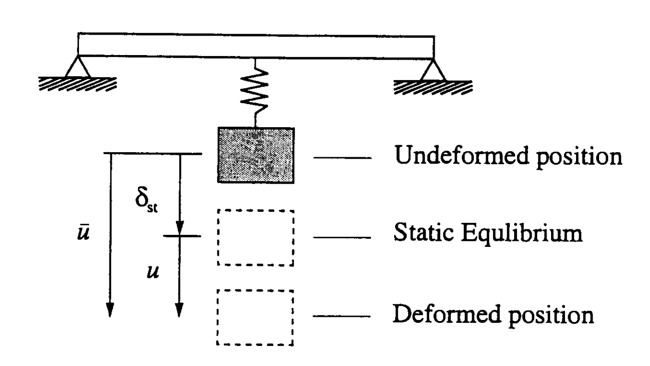


Fig. 1.12b

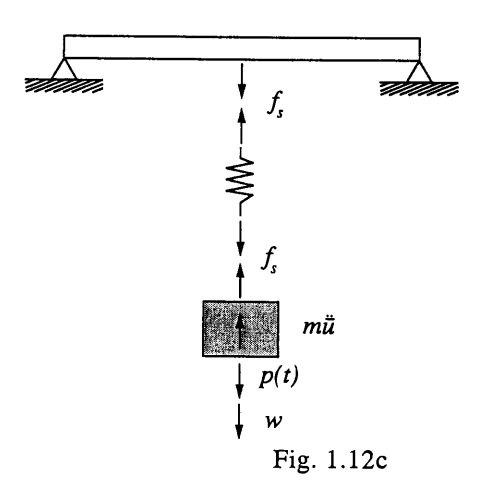


Fig. 1.12c

1. *Write the equation of motion.*

Equilibrium of forces in Fig. 1.12c gives

$$m\ddot{\bar{u}} + f_s = w + p(t) \quad \text{(a)}$$

where

$$f_s = k_e\bar{u} \quad \text{(b)}$$

The equation of motion is:

$$m\ddot{\bar{u}} + k_e\bar{u} = w + p(t) \quad \text{(c)}$$

2. *Determine the effective stiffness.*

$$f_S = k_e\bar{u} \quad \text{(d)}$$

where

$$\bar{u} = \delta_{spring} + \delta_{beam} \quad \text{(e)}$$

$$f_s = k\delta_{spring} = k_{beam}\,\delta_{beam} \quad \text{(f)}$$

Substitute for the δ's from Eq. (f) and for $\bar{u}$ from Eq. (d):

$$\frac{f_s}{k_e} = \frac{f_s}{k} + \frac{f_s}{k_{beam}}$$

$$k_e = \frac{kk_{beam}}{k + k_{beam}}$$

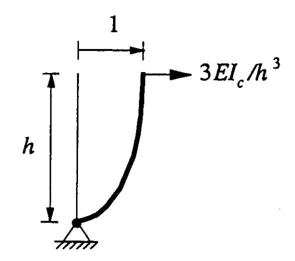
$$k_e = \frac{k\left(48EI/L^3\right)}{k + \dfrac{48EI}{L^3}}$$

3. *Determine the natural frequency.*

$$\omega_n = \sqrt{\frac{k_e}{m}}$$

Problem 1.13

Compute lateral stiffness:



$$k = 2 \times k_{column} = 2 \times \frac{3EI_c}{h^3} = \frac{6EI_c}{h^3}$$

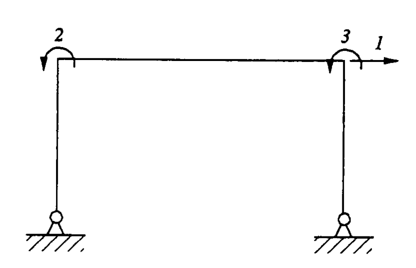
Equation of motion:

$$m\ddot{u} + ku = p(t)$$

Base fixity increases k by a factor of 4.

Problem 1.14

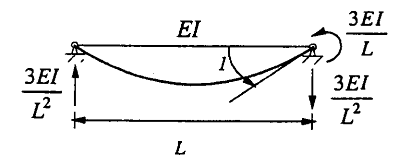
1. *Define degrees of freedom (DOF).*



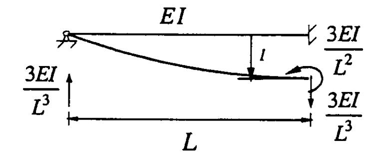
2. *Reduced stiffness coefficients.*

Since there are no external moments applied at the pinned supports, the following *reduced stiffness coefficients* are used for the columns.

Joint rotation:

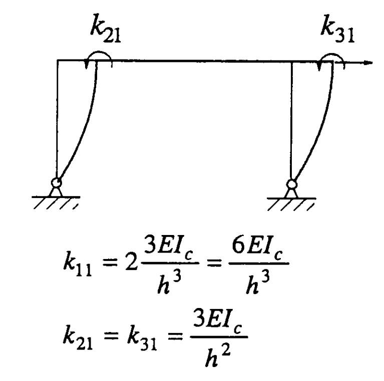


Joint translation:



3. *Form structural stiffness matrix.*

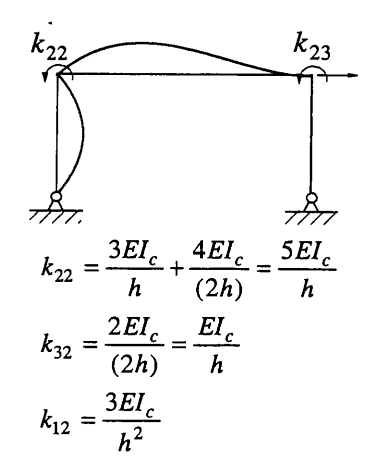
$u_1 = 1, \quad u_2 = u_3 = 0$



$$k_{11} = 2\frac{3EI_c}{h^3} = \frac{6EI_c}{h^3}$$

$$k_{21} = k_{31} = \frac{3EI_c}{h^2}$$

$u_2 = 1, \quad u_1 = u_3 = 0$

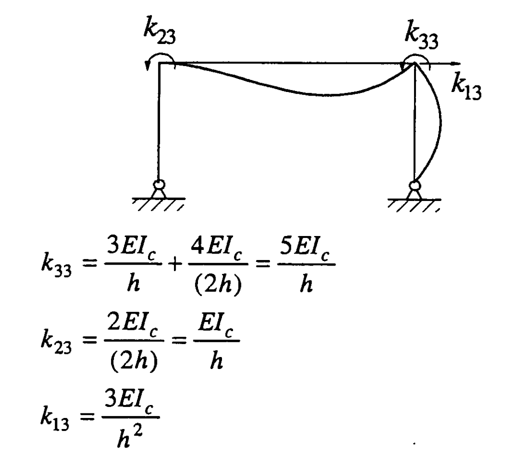


$$k_{22} = \frac{3EI_c}{h} + \frac{4EI_c}{(2h)} = \frac{5EI_c}{h}$$

$$k_{32} = \frac{2EI_c}{(2h)} = \frac{EI_c}{h}$$

$$k_{12} = \frac{3EI_c}{h^2}$$

$u_3 = 1, \quad u_1 = u_2 = 0$



$$k_{33} = \frac{3EI_c}{h} + \frac{4EI_c}{(2h)} = \frac{5EI_c}{h}$$

$$k_{23} = \frac{2EI_c}{(2h)} = \frac{EI_c}{h}$$

$$k_{13} = \frac{3EI_c}{h^2}$$

Hence

$$\mathbf{k} = \frac{EI_c}{h^3}\begin{bmatrix} 6 & 3h & 3h \\ 3h & 5h^2 & h^2 \\ 3h & h^2 & 5h^2 \end{bmatrix}$$

4. *Determine lateral stiffness.*

The lateral stiffness k of the frame can be obtained by static condensation since there is no force acting on DOF 2 and 3:

$$\frac{EI_c}{h^3}\begin{bmatrix} 6 & 3h & 3h \\ 3h & 5h^2 & h^2 \\ 3h & h^2 & 5h^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_S \\ 0 \\ 0 \end{Bmatrix}$$

First partition $\mathbf{k}$ as

$$\mathbf{k} = \frac{EI_c}{h^3}\left[\begin{array}{c|cc} 6 & 3h & 3h \\ \hline 3h & 5h^2 & h^2 \\ 3h & h^2 & 5h^2 \end{array}\right] = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{t0} & \mathbf{k}_{00} \end{bmatrix}$$

where

$$\mathbf{k}_{tt} = \frac{EI_c}{h^3}[6]$$

$$\mathbf{k}_{t0} = \frac{EI_c}{h^3}[3h \quad 3h]$$

$$\mathbf{k}_{00} = \frac{EI_c}{h^3}\begin{bmatrix} 5h^2 & h^2 \\ h^2 & 5h^2 \end{bmatrix}$$

Then compute the lateral stiffness k from

$$k = \mathbf{k}_{tt} - \mathbf{k}_{t0}\,\mathbf{k}_{00}^{-1}\,\mathbf{k}_{t0}^T$$

Since

$$\mathbf{k}_{00}^{-1} = \frac{h}{24EI_c}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

we get

$$k = \frac{6EI_c}{h^3} - \frac{EI_c}{h^3}[3h \quad 3h] \cdot \frac{h}{24EI_c}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \cdot \frac{EI_c}{h^3}\begin{bmatrix} 3h \\ 3h \end{bmatrix}$$

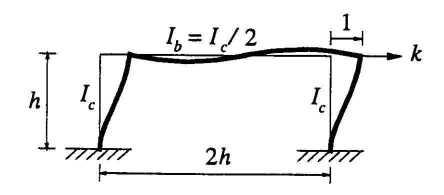
$$k = \frac{EI_c}{h^3}[6-3]$$

$$k = \frac{3EI_c}{h^3}$$

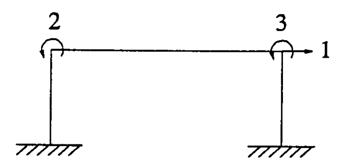
5. *Equation of motion.*

$$m\ddot{u} + \frac{3EI_c}{h^3}u = p(t)$$

Problem 1.15



Define degrees of freedom (DOF):



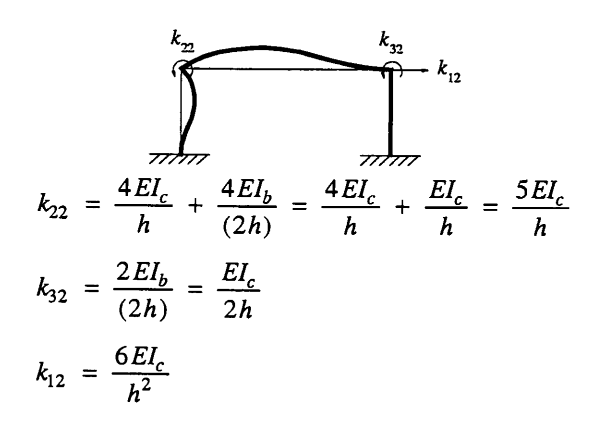
Form structural stiffness matrix:

$u_1 = 1, \quad u_2 = u_3 = 0$

$$k_{11} = 2\frac{12EI_c}{h^3} = \frac{24EI_c}{h^3}$$

$$k_{21} = k_{31} = \frac{6EI_c}{h^2}$$

$u_2 = 1, \quad u_1 = u_3 = 0$

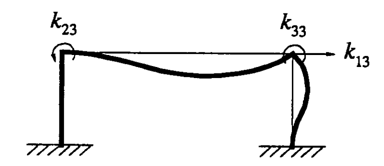


$$k_{22} = \frac{4EI_c}{h} + \frac{4EI_b}{(2h)} = \frac{4EI_c}{h} + \frac{EI_c}{h} = \frac{5EI_c}{h}$$

$$k_{32} = \frac{2EI_b}{(2h)} = \frac{EI_c}{2h}$$

$$k_{12} = \frac{6EI_c}{h^2}$$

$u_3 = 1, \quad u_1 = u_2 = 0$



$$k_{33} = \frac{4EI_c}{h} + \frac{4EI_b}{(2h)} = \frac{4EI_c}{h} + \frac{EI_c}{h} = \frac{5EI_c}{h}$$

$$k_{23} = \frac{2EI_b}{(2h)} = \frac{EI_c}{2h}$$

$$k_{13} = \frac{6EI_c}{h^2}$$

Hence

$$\mathbf{k} = \frac{EI_c}{h^3}\begin{bmatrix} 24 & 6h & 6h \\ 6h & 5h^2 & \frac{1}{2}h^2 \\ 6h & \frac{1}{2}h^2 & 5h^2 \end{bmatrix}$$

The lateral stiffness k of the frame can be obtained by static condensation since there is no force acting on DOF 2 and 3:

$$\frac{EI_c}{h^3}\begin{bmatrix} 24 & 6h & 6h \\ 6h & 5h^2 & \frac{1}{2}h^2 \\ 6h & \frac{1}{2}h^2 & 5h^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_S \\ 0 \\ 0 \end{Bmatrix}$$

First partition $\mathbf{k}$ as

$$\mathbf{k} = \frac{EI_c}{h^3}\left[\begin{array}{c|cc} 24 & 6h & 6h \\ \hline 6h & 5h^2 & \frac{1}{2}h^2 \\ 6h & \frac{1}{2}h^2 & 5h^2 \end{array}\right] = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{t0}^T & \mathbf{k}_{00} \end{bmatrix}$$

where

$$\mathbf{k}_{tt} = \frac{EI_c}{h^3}[24]$$

$$\mathbf{k}_{t0} = \frac{EI_c}{h^3}\begin{bmatrix} 6h & 6h \end{bmatrix}$$

$$\mathbf{k}_{00} = \frac{EI_c}{h^3}\begin{bmatrix} 5h^2 & \frac{1}{2}h^2 \\ \frac{1}{2}h^2 & 5h^2 \end{bmatrix}$$

Then compute the lateral stiffness k from

$$k = \mathbf{k}_{tt} - \mathbf{k}_{t0}\,\mathbf{k}_{00}^{-1}\,\mathbf{k}_{t0}^T$$

Since

$$\mathbf{k}_{00}^{-1} = \frac{4h}{99EI_c}\begin{bmatrix} 5 & -\frac{1}{2} \\ -\frac{1}{2} & 5 \end{bmatrix}$$

we get

$$k = \frac{24EI_c}{h^3} - \frac{EI_c}{h^3}\begin{bmatrix}6h & 6h\end{bmatrix} \cdot \frac{4h}{99EI_c}\begin{bmatrix}5 & -\frac{1}{2} \\ -\frac{1}{2} & 5\end{bmatrix} \cdot \frac{EI_c}{h^3}\begin{bmatrix}6h \\ 6h\end{bmatrix}$$

$$= \frac{EI_c}{h^3}\left(24 - \frac{144}{11}\right)$$

$$= \frac{120}{11}\frac{EI_c}{h^3}$$

This result can be checked against Eq. 1.3.5:

$$k = \frac{24EI_c}{h^3}\left(\frac{12\rho+1}{12\rho+4}\right)$$

Substituting $\rho = I_b/4I_c = 1/8$ gives

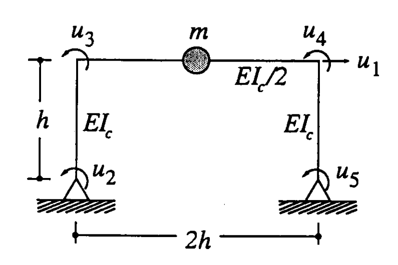
$$k = \frac{24EI_c}{h^3}\left(\frac{12\frac{1}{8}+1}{12\frac{1}{8}+4}\right) = \frac{24EI_c}{h^3}\left(\frac{5}{11}\right) = \frac{120}{11}\frac{EI_c}{h^3}$$

Equation of motion:

$$m\ddot{u} + \left(\frac{120}{11}\frac{EI_c}{h^3}\right)u = p(t)$$

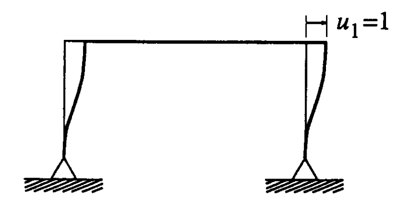
Problem 1.16

1. *Define degrees of freedom (DOF).*



2. *Form the structural stiffness matrix.*

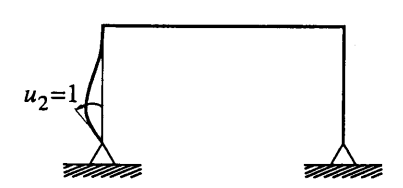
$u_1 = 1\ ,\ u_2 = u_3 = u_4 = u_5 = 0$



$$k_{11} = 2\frac{12EI_c}{h^3} = \frac{24EI_c}{h^3}$$

$$k_{21} = k_{31} = k_{41} = k_{51} = \frac{6EI_c}{h^2}$$

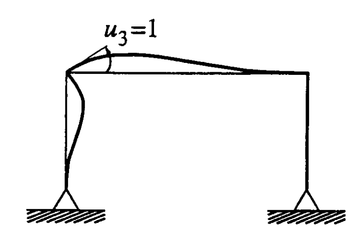
$u_2 = 1\ ,\ u_1 = u_3 = u_4 = u_5 = 0$



$$k_{22} = \frac{4EI_c}{h} \qquad k_{12} = \frac{6EI_c}{h^2}$$

$$k_{32} = \frac{2EI_c}{h} \qquad k_{42} = k_{52} = 0$$

$u_3 = 1\ ,\ u_1 = u_2 = u_4 = u_5 = 0$

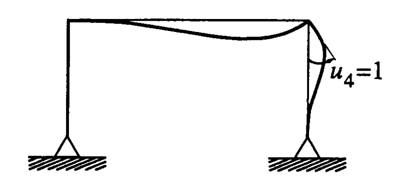


$$k_{33} = \frac{4EI_c}{h} + \frac{4EI_c}{2(2h)} = \frac{5EI_c}{h}$$

$$k_{13} = \frac{6EI_c}{h^2} \qquad k_{23} = \frac{2EI_c}{h}$$

$$k_{43} = \frac{2EI_c}{2(2h)} = \frac{EI_c}{2h} \qquad k_{53} = 0$$

$u_4 = 1\ ,\ u_1 = u_2 = u_3 = u_5 = 0$

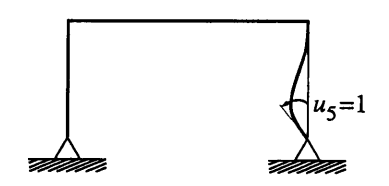


$$k_{44} = \frac{4EI_c}{h} + \frac{4EI_c}{2(2h)} = \frac{5EI_c}{h}$$

$$k_{14} = \frac{6EI_c}{h^2} \qquad k_{24} = 0$$

$$k_{34} = \frac{2EI_c}{2(2h)} = \frac{EI_c}{2h} \qquad k_{54} = \frac{2EI_c}{h}$$

$u_5 = 1\ ,\ u_1 = u_2 = u_3 = u_4 = 0$



$$k_{55} = \frac{4EI_c}{h} \qquad k_{15} = \frac{6EI_c}{h^2}$$

$$k_{45} = \frac{2EI_c}{h} \qquad k_{25} = k_{35} = 0$$

Assemble the stiffness coefficients:

$$k = \frac{EI_c}{h^3}\begin{bmatrix} 24 & 6h & 6h & 6h & 6h \\ 6h & 4h^2 & 2h^2 & 0 & 0 \\ 6h & 2h^2 & 5h^2 & \frac{1}{2}h^2 & 0 \\ 6h & 0 & \frac{1}{2}h^2 & 5h^2 & 2h^2 \\ 6h & 0 & 0 & 2h^2 & 4h^2 \end{bmatrix}$$

3. *Determine the lateral stiffness of the frame.*

First partition **k**.

$$k = \frac{EI_c}{h^3}\left[\begin{array}{c|cccc} 24 & 6h & 6h & 6h & 6h \\ \hline 6h & 4h^2 & 2h^2 & 0 & 0 \\ 6h & 2h^2 & 5h^2 & \frac{1}{2}h^2 & 0 \\ 6h & 0 & \frac{1}{2}h^2 & 5h^2 & 2h^2 \\ 6h & 0 & 0 & 2h^2 & 4h^2 \end{array}\right] = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{t0}^T & \mathbf{k}_{00} \end{bmatrix}$$

Compute the lateral stiffness.

$$k = \mathbf{k}_{tt} - \mathbf{k}_{t0}\mathbf{k}_{00}^{-1}\mathbf{k}_{t0}^T$$

$$k = \frac{24EI_c}{h^3} - \frac{22EI_c}{h^3} = \frac{2EI_c}{h^3}$$

4. *Write the equation of motion.*

$$m\ddot{u} + ku = p(t)$$

$$m\ddot{u} + \left(\frac{2EI_c}{h^3}\right)u = p(t)$$

Problem 1.17

(a) *Equation of motion in the x-direction.*

The lateral stiffness of each wire is the same as the lateral stiffness of a brace derived in Eq. (c) of Example 1.2:

$$k_w = \left(\frac{AE}{L}\right)\cos^2\theta$$

$$= \left(\frac{AE}{h\sqrt{2}}\right)\cos^2 45^\circ = \frac{1}{2\sqrt{2}}\frac{AE}{h}$$

Each of the four sides of the structure includes two wires. If they were not pretensioned, under lateral displacement, only the wire in tension will provide lateral resistance and the one in compression will go slack and will not contribute to the lateral stiffness. However, the wires are pretensioned to a high stress; therefore, under lateral displacement the tension will increase in one wire, but decrease in the other; and both wires will contribute to the lateral direction. Consequently, four wires contribute to the stiffness in the x-direction:

$$k_x = 4k_w = \sqrt{2}\,\frac{AE}{h}$$

Then the equation of motion in the x-direction is

$$m\ddot{u}_x + k_x u_x = 0$$

(b) *Equation of motion in the y-direction.*

The lateral stiffness in the y direction, $k_y = k_x$, and the same equation applies for motion in the y-direction:

$$m\ddot{u}_y + k_y u_y = 0$$

Problem 1.18

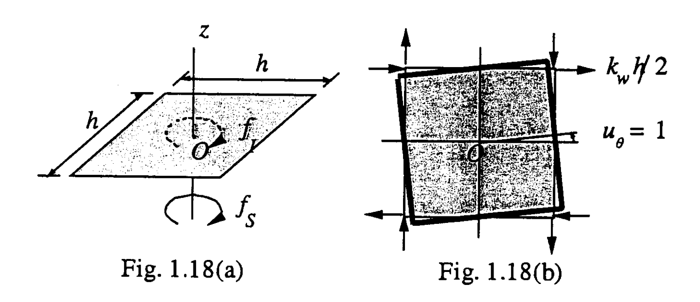


Fig. 1.18(a)   Fig. 1.18(b)

1. *Set up equation of motion.*

The elastic resisting torque f_S and inertia force f_I are shown in Fig. 1.18(a). The equation of dynamic equilibrium is

$$f_I + f_S = 0 \quad \text{or} \quad I_O \ddot{u}_\theta + f_S = 0 \qquad \text{(a)}$$

where

$$I_O = m\frac{h^2 + h^2}{12} = \frac{mh^2}{6} \qquad \text{(b)}$$

2. *Determine torsional stiffness, k_θ.*

$$f_S = k_\theta u_\theta \qquad \text{(c)}$$

Introduce $u_\theta = 1$ in Fig. 1.18(b) and identify the resisting forces due to each wire. All the eight forces are the same; each is $k_w h/2$, where, from Problem 1.17,

$$k_w = \frac{1}{2\sqrt{2}}\frac{AE}{h}$$

The torque required to equilibrate these resisting forces is

$$k_\theta = 8k_w \frac{h}{2}\frac{h}{2} = 2k_w h^2 = \frac{2}{2\sqrt{2}}\left(\frac{AE}{h}\right)h^2$$

$$= \frac{AEh}{\sqrt{2}} \qquad \text{(d)}$$

3. *Set up equation of motion.*

Substituting Eq. (d) in (c) and then Eqs. (c) and (b) in (a) gives the equation of motion:

$$\frac{mh^2}{6}\ddot{u}_\theta + \frac{AEh}{\sqrt{2}}u_\theta = 0$$

Problem 1.19

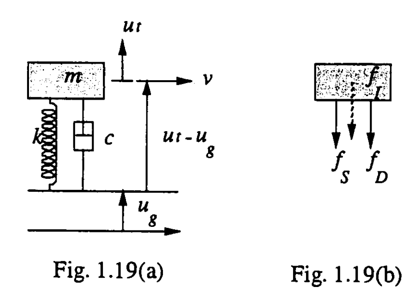


Fig. 1.19(a) Fig. 1.19(b)

Displacement u^t is measured from the static equilibrium position under the weight mg.

From the free-body diagram in Fig. 1.19(b)

$$f_I + f_D + f_S = 0 \qquad \text{(a)}$$

where

$$f_I = m\ddot{u}^t$$
$$f_D = c(\dot{u}^t - \dot{u}_g) \qquad \text{(b)}$$
$$f_S = k(u^t - u_g)$$

Substituting Eqs. (b) in Eq. (a) gives

$$m\ddot{u}^t + c(\dot{u}^t - \dot{u}_g) + k(u^t - u_g) = 0$$

Noting that $x = vt$ and transferring the excitation terms to the right side gives the equation of motion:

$$m\ddot{u}^t + c\dot{u}^t + ku^t = c\dot{u}_g(vt) + ku_g(vt)$$

Problem 2.1

Given:

$$T_n = 2\pi\sqrt{\frac{m}{k}} = 0.5 \text{ sec} \qquad \text{(a)}$$

$$T_n' = 2\pi\sqrt{\frac{m + 50/g}{k}} = 0.75 \text{ sec} \qquad \text{(b)}$$

1. *Determine the weight of the table.*

Taking the ratio of Eq. (b) to Eq. (a) and squaring the result gives

$$\left(\frac{T_n'}{T_n}\right)^2 = \frac{m+50/g}{m} \qquad \Rightarrow \; 1+\frac{50}{mg} = \left(\frac{0.75}{0.5}\right)^2 = 2.25$$

or

$$mg = \frac{50}{1.25} = 40 \text{ lbs}$$

2. *Determine the lateral stiffness of the table.*

Substitute for m in Eq. (a) and solve for k:

$$k = 16\pi^2 m = 16\pi^2\left(\frac{40}{386}\right) = 16.4 \text{lbs/in.}$$

Problem 2.2

1. *Determine the natural frequency.*

$$k = 100 \text{ lb/in.} \qquad m = \frac{400}{386} \text{ lb}-\text{sec}^2/\text{in.}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{400/386}} = 9.82 \text{ rads/sec}$$

2. *Determine initial deflection.*

Static deflection due to weight of the iron scrap

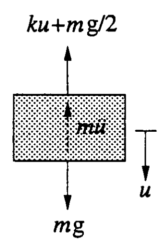
$$u(0) = \frac{200}{100} = 2 \text{ in.}$$

3. *Determine free vibration.*

$$u(t) = u(0)\cos\omega_n t = 2\cos(9.82t)$$

Problem 2.3

1. *Set up equation of motion.*



$$m\ddot{u} + ku = \frac{mg}{2}$$

2. *Solve equation of motion.*

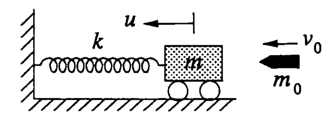
$$u(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{mg}{2k}$$

At $t = 0$, $u(0) = 0$ and $\dot{u}(0) = 0$

$$\therefore A = -\frac{mg}{2k}, \quad B = 0$$

$$u(t) = \frac{mg}{2k}(1 - \cos\omega_n t)$$

Problem 2.4



$$m = \frac{10}{386} = 0.0259 \text{ lb}-\text{sec}^2/\text{in.}$$

$$m_0 = \frac{0.5}{386} = 1.3 \times 10^{-3} \text{ lb}-\text{sec}^2/\text{in.}$$

$$k = 100 \text{ lb/in.}$$

Conservation of momentum implies

$$m_0 v_0 = (m + m_0)\,\dot{u}(0)$$

$$\dot{u}(0) = \frac{m_0 v_0}{m + m_0} = 2.857 \text{ ft/sec} = 34.29 \text{ in./sec}$$

After the impact the system properties and initial conditions are

Mass $= m + m_0 = 0.0272 \text{ lb}-\text{sec}^2/\text{in.}$

Stiffness $= k = 100$ lb/in.

Natural frequency:

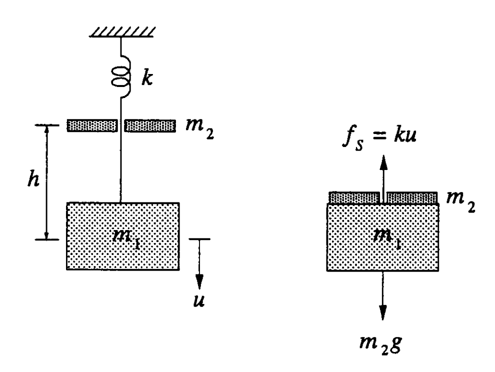
$$\omega_n = \sqrt{\frac{k}{m + m_0}} = 60.63 \text{ rads/sec}$$

Initial conditions: $u(0) = 0, \quad \dot{u}(0) = 34.29$ in./sec

The resulting motion is

$$u(t) = \frac{\dot{u}(0)}{\omega_n} \sin \omega_n t = 0.565 \sin (60.63t) \text{ in.}$$

Problem 2.5



With u measured from the static equilibrium position of m_1 and k, the equation of motion after impact is

$$(m_1 + m_2)\ddot{u} + ku = m_2 g \qquad \text{(a)}$$

The general solution is

$$u(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{m_2 g}{k} \qquad \text{(b)}$$

$$\omega_n = \sqrt{\frac{k}{m_1 + m_2}} \qquad \text{(c)}$$

The initial conditions are

$$u(0) = 0 \qquad \dot{u}(0) = \frac{m_2}{m_1 + m_2}\sqrt{2gh} \qquad \text{(d)}$$

The initial velocity in Eq. (d) was determined by conservation of momentum during impact:

$$m_2\dot{u}_2 = (m_1 + m_2)\dot{u}(0)$$

where

$$\dot{u}_2 = \sqrt{2gh}$$

Impose initial conditions to determine A and B:

$$u(0) = 0 \Rightarrow A = -\frac{m_2 g}{k} \qquad \text{(e)}$$

$$\dot{u}(0) = \omega_n B \Rightarrow B = \frac{m_2}{m_1 + m_2}\frac{\sqrt{2gh}}{\omega_n} \qquad \text{(f)}$$

Substituting Eqs. (e) and (f) in Eq. (b) gives

$$u(t) = \frac{m_2 g}{k}(1 - \cos\omega_n t) + \frac{\sqrt{2gh}}{\omega_n}\frac{m_2}{m_1 + m_2}\sin\omega_n t$$

Problem 2.6

1. *Determine deformation and velocity at impact.*

$$u(0) = \frac{mg}{k} = \frac{10}{50} = 0.2 \text{ in.}$$

$$\dot{u}(0) = -\sqrt{2gh} = -\sqrt{2(386)(36)} = -166.7 \text{ in./sec}$$

2. *Determine the natural frequency.*

$$\omega_n = \sqrt{\frac{kg}{w}} = \sqrt{\frac{(50)(386)}{10}} = 43.93 \text{ rad/sec}$$

3. *Compute the maximum deformation.*

$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$$

$$= (0.2)\cos 316.8t - \left(\frac{166.7}{43.93}\right)\sin 316.8t$$

$$u_o = \sqrt{[u(0)]^2 + \left[\frac{\dot{u}(0)}{\omega_n}\right]^2}$$

$$= \sqrt{0.2^2 + (-3.795)^2} = 3.8 \text{ in.}$$

4. *Compute the maximum acceleration.*

$$\ddot{u}_o = \omega_n^{\,2} u_o = (43.93)^2(3.8)$$

$$= 7334 \text{ in./sec}^2 = 18.98g$$

Problem 2.7

Given:

$$m = \frac{200}{32.2} = 6.211 \text{ lb} - \text{sec}^2/\text{ft}$$

$$f_n = 2 \text{ Hz}$$

Determine EI:

$$k = \frac{3EI}{L^3} = \frac{3EI}{3^3} = \frac{EI}{9} \text{ lb/ft}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \Rightarrow 2 = \frac{1}{2\pi}\sqrt{\frac{EI}{55.90}} \Rightarrow$$

$$EI = (4\pi)^2\, 55.90 = 8827 \text{ lb} - \text{ft}^2$$

Problem 2.8

Equation of motion:

$$m\ddot{u} + c\dot{u} + ku = 0 \qquad \text{(a)}$$

Dividing Eq. (a) through by m gives

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = 0 \qquad \text{(b)}$$

where $\zeta = 1$.

Equation (b) thus reads

$$\ddot{u} + 2\omega_n\dot{u} + \omega_n^2 u = 0 \qquad \text{(c)}$$

Assume a solution of the form $u(t) = e^{st}$. Substituting this solution into Eq. (c) yields

$$(s^2 + 2\omega_n s + \omega_n^2)\, e^{st} = 0$$

Because e^{st} is never zero, the quantity within parentheses must be zero:

$$s^2 + 2\omega_n s + \omega_n^2 = 0$$

or

$$s = \frac{-2\omega_n \pm \sqrt{(2\omega_n)^2 - 4\omega_n^2}}{2} = -\omega_n$$

(double root)

The general solution has the following form:

$$u(t) = A_1\, e^{-\omega_n t} + A_2\, t\, e^{-\omega_n t} \qquad \text{(d)}$$

where the constants A_1 and A_2 are to be determined from the initial conditions: $u(0)$ and $\dot{u}(0)$.

Evaluate Eq. (d) at $t = 0$:

$$u(0) = A_1 \Rightarrow A_1 = u(0) \qquad \text{(e)}$$

Differentiating Eq. (d) with respect to t gives

$$\dot{u}(t) = -\omega_n A_1\, e^{-\omega_n t} + A_2 (1 - \omega_n t)\, e^{-\omega_n t} \qquad \text{(f)}$$

Evaluate Eq. (f) at $t = 0$:

$$\dot{u}(0) = -\omega_n A_1 + A_2(1 - 0)$$

$$\therefore A_2 = \dot{u}(0) + \omega_n A_1 = \dot{u}(0) + \omega_n u(0) \qquad \text{(g)}$$

Substituting Eqs. (e) and (g) for A_1 and A_2 in Eq. (d) gives

$$u(t) = \left\{u(0) + \left[\dot{u}(0) + \omega_n u(0)\right]t\right\} e^{-\omega_n t} \qquad \text{(h)}$$

Problem 2.9

Equation of motion:

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{a}$$

Dividing Eq. (a) through by m gives

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = 0 \tag{b}$$

where $\zeta > 1$.

Assume a solution of the form $u(t) = e^{st}$. Substituting this solution into Eq. (b) yields

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)\, e^{st} = 0$$

Because e^{st} is never zero, the quantity within parentheses must be zero:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

or

$$s = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

The general solution has the following form:

$$u(t) = A_1 \exp\left[\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n t\right] + A_2 \exp\left[\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n t\right] \tag{c}$$

where the constants A_1 and A_2 are to be determined from the initial conditions: $u(0)$ and $\dot{u}(0)$.

Evaluate Eq. (c) at $t = 0$:

$$u(0) = A_1 + A_2 \Rightarrow A_1 + A_2 = u(0) \tag{d}$$

Differentiating Eq. (c) with respect to t gives

$$\dot{u}(t) = A_1\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n \exp\left[\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n t\right] + A_2\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n \exp\left[\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n t\right] \tag{e}$$

Evaluate Eq. (e) at $t = 0$:

$$\dot{u}(0) = A_1\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n + A_2\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n$$
$$= [u(0) - A_2]\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n + A_2\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n$$

or

$$A_2\omega_n\left[-\zeta + \sqrt{\zeta^2-1} + \zeta + \sqrt{\zeta^2-1}\right] = \dot{u}(0) + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n u(0)$$

or

$$A_2 = \frac{\dot{u}(0) + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n u(0)}{2\sqrt{\zeta^2-1}\,\omega_n} \tag{f}$$

Substituting Eq. (f) in Eq. (d) gives

$$A_1 = u(0) - \frac{\dot{u}(0) + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n u(0)}{2\sqrt{\zeta^2-1}\,\omega_n}$$
$$= \frac{2\sqrt{\zeta^2-1}\,\omega_n u(0) - \dot{u}(0) - \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n u(0)}{2\sqrt{\zeta^2-1}\,\omega_n}$$
$$= \frac{-\dot{u}(0) + \left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n u(0)}{2\sqrt{\zeta^2-1}\,\omega_n} \tag{g}$$

The solution, Eq. (c), now reads:

$$u(t) = e^{-\zeta\omega_n t}\left(A_1 e^{-\omega_D' t} + A_2 e^{\omega_D' t}\right)$$

where

$$\omega_D' = \sqrt{\zeta^2 - 1}\,\omega_n$$

$$A_1 = \frac{-\dot{u}(0) + \left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n u(0)}{2\omega_D'}$$

$$A_2 = \frac{\dot{u}(0) + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n u(0)}{2\omega_D'}$$

Problem 2.10

Equation of motion:

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = 0 \qquad \text{(a)}$$

Assume a solution of the form

$$u(t) = e^{st}$$

Substituting this solution into Eq. (a) yields:

$$\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)e^{st} = 0$$

Because e^{st} is never zero

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad \text{(b)}$$

The roots of this characteristic equation depend on ζ.

(a) Underdamped Systems, ζ<1

The two roots of Eq. (b) are

$$s_{1,2} = \omega_n\left(-\zeta \pm i\sqrt{1-\zeta^2}\right) \qquad \text{(c)}$$

Hence the general solution is

$$u(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

which after substituting in Eq. (c) becomes

$$u(t) = e^{-\zeta\omega_n t}\left(A_1 e^{i\omega_D t} + A_2 e^{-i\omega_D t}\right) \qquad \text{(d)}$$

where

$$\omega_D = \omega_n\sqrt{1-\zeta^2} \qquad \text{(e)}$$

Rewrite Eq. (d) in terms of trigonometric functions:

$$u(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_D t + B\sin\omega_D t\right) \qquad \text{(f)}$$

Determine A and B from initial conditions $u(0) = 0$ and $\dot{u}(0)$:

$$A = 0 \qquad B = \frac{\dot{u}(0)}{\omega_D}$$

Substituting A and B into Eq. (f) gives

$$u(t) = \frac{\dot{u}(0)}{\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t \qquad \text{(g)}$$

(b) Critically Damped Systems, $\zeta = 1$

The roots of the characteristic equation [Eq. (b)] are:

$$s_1 = -\omega_n \qquad s_2 = -\omega_n \qquad \text{(h)}$$

The general solution is

$$u(t) = A_1 e^{-\omega_n t} + A_2\, t\, e^{-\omega_n t} \qquad \text{(i)}$$

Determined from the initial conditions $u(0) = 0$ and $\dot{u}(0)$:

$$A_1 = 0 \qquad A_2 = \dot{u}(0) \qquad \text{(j)}$$

Substituting in Eq. (i) gives

$$u(t) = \dot{u}(0)\, t\, e^{-\omega_n t} \qquad \text{(k)}$$

(c) Overdamped Systems, ζ>1

The roots of the characteristic equation [Eq. (b)] are:

$$s_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \qquad \text{(l)}$$

The general solution is:

$$u(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(m)}$$

which after substituting Eq. (l) becomes

$$u(t) = A_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + A_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \qquad \text{(n)}$$

Determined from the initial conditions $u(0) = 0$ and $\dot{u}(0)$:

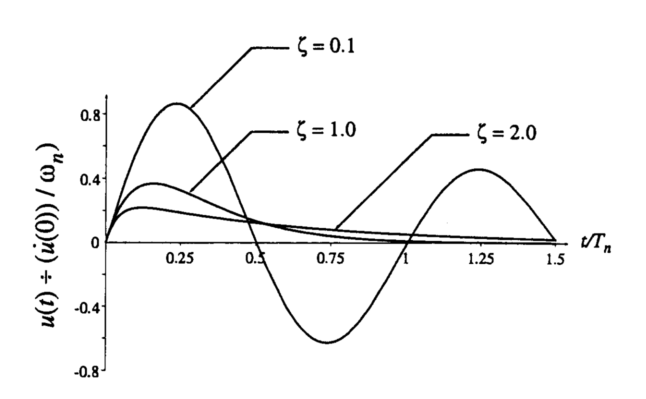
$$-A_1 = A_2 = \frac{\dot{u}(0)}{2\omega_n\sqrt{\zeta^2 - 1}} \qquad \text{(o)}$$

Substituting in Eq. (n) gives

$$u(t) = \frac{\dot{u}(0)\, e^{-\zeta\omega_n t}}{2\omega_n\sqrt{\zeta^2-1}}\left(e^{\omega_n t\sqrt{\zeta^2-1}} - e^{-\omega_n t\sqrt{\zeta^2-1}}\right) \qquad \text{(p)}$$

(d) Response Plots

Plot Eq. (g) with ζ = 0.1; Eq. (k), which is for ζ = 1; and Eq. (p) with ζ = 2.



Problem 2.11

$$\frac{1}{j}\ln\left(\frac{u_1}{u_{j+1}}\right) \approx 2\pi\zeta \Rightarrow \frac{1}{j_{10\%}}\ln\left(\frac{1}{0.1}\right) \approx 2\pi\zeta$$

$\therefore\ j_{10\%} \approx \ln(10)/2\pi\zeta \approx 0.366/\zeta$

Problem 2.12

$$\frac{u_i}{u_{i+1}} = \exp\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

(a) $\zeta = 0.01$: $\quad \dfrac{u_i}{u_{i+1}} = 1.065$

(b) $\zeta = 0.05$: $\quad \dfrac{u_i}{u_{i+1}} = 1.37$

(c) $\zeta = 0.25$: $\quad \dfrac{u_i}{u_{i+1}} = 5.06$

Problem 2.13

Given:

$w = 20.03$ kips (empty); $m = 0.0519$ kip-sec²/in.

$k = 2\,(8.2) = 16.4$ kips/in.

$c = 0.0359$ kip-sec/in.

(a) $T_n = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{0.0519}{16.4}} = 0.353 \text{ sec}$

(b) $\zeta = \dfrac{c}{2\sqrt{km}} = \dfrac{0.0359}{2\sqrt{(16.4)\,(0.0519)}} = 0.0194$

$= 1.94\%$

Problem 2.14

(a) The stiffness coefficient is

$$k = \frac{3000}{2} = 1500 \quad \text{lb/in.}$$

The damping coefficient is

$$c = c_{cr} = 2\sqrt{km}$$

$$c = 2\sqrt{1500\frac{3000}{386}} = 215.9 \quad \text{lb-sec/in.}$$

(b) With passengers the weight is w = 3640 lb. The damping ratio is

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{215.9}{2\sqrt{1500\frac{3640}{386}}} = 0.908$$

(c) The natural vibration frequency for case (b) is

$$\omega_D = \omega_n\sqrt{1-\zeta^2}$$

$$= \sqrt{\frac{1500}{3640/386}}\sqrt{1-(0.908)^2}$$

$$= 12.61 \times 0.419$$

$$= 5.28 \text{ rads/sec}$$

Problem 2.15

1. *Determine ζ and ω_n.*

$$\zeta \approx \frac{1}{2\pi j} \ln\left(\frac{u_1}{u_{j+1}}\right) = \frac{1}{2\pi(20)} \ln\left(\frac{1}{0.2}\right) = 0.0128 = 1.28\%$$

Therefore the assumption of small damping implicit in the above equation is valid.

$$T_D = \frac{3}{20} = 0.15 \text{ sec}; \; T_n \approx T_D = 0.15 \text{ sec};$$

$$\omega_n = \frac{2\pi}{0.15} = 41.89 \text{ rads/sec}$$

2. *Determine stiffness coefficient.*

$$k = \omega_n^2 m = (41.89)^2\, 0.1 = 175.5 \text{ lbs/in.}$$

3. *Determine damping coefficient.*

$$c_{cr} = 2m\omega_n = 2\,(0.1)\,(41.89) = 8.377 \text{ lb-sec/in.}$$

$$c = \zeta c_{cr} = 0.0128\,(8.377) = 0.107 \text{ lb-sec/in.}$$

Problem 2.16

(a) $k = \dfrac{250}{0.8} = 312.5$ lbs/in.

$$m = \frac{w}{g} = \frac{250}{386} = 0.647 \text{ lb} - \text{sec}^2/\text{in.}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 21.98 \text{ rads/sec}$$

(b) Assuming small damping,

$$\ln\left(\frac{u_1}{u_{j+1}}\right) \approx 2j\pi\zeta \Rightarrow$$

$$\ln\left(\frac{u_0}{u_0/8}\right) = \ln(8) \approx 2(2)\pi\zeta \Rightarrow \zeta = 0.165$$

This value of ζ may be too large for small damping assumption; therefore we use the exact equation:

$$\ln\left(\frac{u_1}{u_{j+1}}\right) = \frac{2j\pi\zeta}{\sqrt{1-\zeta^2}}$$

or,

$$\ln(8) = \frac{2(2)\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} = 0.165 \Rightarrow$$

$$\zeta^2 = 0.027(1-\zeta^2) \Rightarrow$$

$$\zeta = \sqrt{0.0267} = 0.163$$

(c) $\omega_D = \omega_n\sqrt{1-\zeta^2} = 21.69$ rads/sec

Damping decreases the natural frequency.

Problem 2.17

Reading values directly from Fig. 1.1.4b:

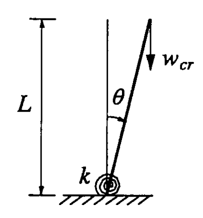
Peak	Time, t_i (sec)	Peak, $\ddot{u}_i$ (g)
1	0.80	0.78
31	7.84	0.50

$$T_D = \frac{7.84 - 0.80}{30} = 0.235 \text{ sec}$$

$$\zeta = \frac{1}{2\pi(30)} \ln\left(\frac{0.78g}{0.50g}\right) = 0.00236 = 0.236\%$$

Problem 2.18

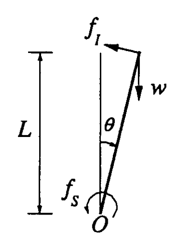
1. *Determine buckling load.*



$$w_{cr}\,(L\theta) = k\theta$$

$$w_{cr} = \frac{k}{L}$$

2. *Draw free-body diagram and set up equilibrium equation.*



$$\sum M_O = 0 \Rightarrow f_I\,L + f_S = w\,L\theta$$
(a)

where

$$f_I = \frac{w}{g} L^2 \ddot{\theta} \qquad f_S = k\theta \qquad \text{(b)}$$

Substituting Eq. (b) in Eq. (a) gives

$$\frac{w}{g} L^2 \ddot{\theta} + (k - wL)\,\theta = 0 \qquad \text{(c)}$$

3. *Compute natural frequency.*

$$\omega_n' = \sqrt{\frac{k - wL}{(w/g)\,L^2}} = \sqrt{\frac{k}{(w/g)\,L^2}\left(1 - \frac{wL}{k}\right)}$$

or

$$\omega_n' = \omega_n \sqrt{1 - \frac{w}{w_{cr}}} \qquad \text{(d)}$$

Problem 2.19

For motion of the building from left to right, the governing equation is

$$m\ddot{u} + ku = -F \qquad \text{(a)}$$

for which the solution is

$$u(t) = A_2 \cos \omega_n t + B_2 \sin \omega_n t - u_F \qquad \text{(b)}$$

With initial velocity of $\dot{u}(0)$ and initial displacement $u(0) = 0$, the solution of Eq. (b) is

$$u(t) = \frac{\dot{u}(0)}{\omega_n} \sin \omega_n t + u_F (\cos \omega_n t - 1) \qquad \text{(c)}$$

$$\dot{u}(t) = \dot{u}(0) \cos \omega_n t - u_F \omega_n \sin \omega_n t \qquad \text{(d)}$$

At the extreme right, $\dot{u}(t) = 0$; hence from Eq. (d)

$$\tan \omega_n t = \frac{\dot{u}(0)}{\omega_n} \frac{1}{u_F} \qquad \text{(e)}$$

Substituting $\omega_n = 4\pi$, $u_F = 0.15$ in. and $\dot{u}(0) =$ 20 in./sec in Eq. (e) gives

$$\tan \omega_n t = \frac{20}{4\pi} \frac{1}{0.15} = 10.61$$

or

$$\sin \omega_n t = 0.9956; \quad \cos \omega_n t = 0.0938$$

Substituting in Eq. (c) gives the displacement to the right:

$$u = \frac{20}{4\pi}(0.9956) + 0.15(0.0938 - 1) = 1.449 \text{ in.}$$

After half a cycle of motion the amplitude decreases by

$$2u_F = 2 \times 0.15 = 0.3 \text{ in.}$$

Maximum displacement on the return swing is

$$u = 1.449 - 0.3 = 1.149 \text{ in.}$$

Problem 2.20

Given:

$$F = 0.1w\,,\ T_n = 0.25\text{ sec}$$

$$u_F = \frac{F}{k} = \frac{0.1w}{k} = \frac{0.1mg}{k} = \frac{0.1g}{\omega_n^2} = \frac{0.1g}{(2\pi/T_n)^2}$$

$$= \frac{0.1g}{(8\pi)^2} = 0.061\text{ in.}$$

The reduction in displacement amplitude per cycle is

$$4u_F = 0.244\text{ in.}$$

The displacement amplitude after 6 cycles is

$$2.0 - 6\,(0.244) = 2.0 - 1.464 = 0.536\text{ in.}$$

Motion stops at the end of the half cycle for which the displacement amplitude is less than u_F. Displacement amplitude at the end of the 7th cycle is 0.536 – 0.244 = 0.292 in.; at the end of the 8th cycle it is 0.292 – 0.244 = 0.048 in.; which is less than u_F. Therefore, the motion stops after 8 cycles.

Problem 3.1

From the given data,

$$\frac{k}{m} = \omega_n^2 = (2\pi f_n)^2 = (8\pi)^2 = 64\pi^2 \quad \text{(a)}$$

$$\frac{k}{m + \Delta m} = (\omega_n')^2 = (2\pi f_n')^2 = (6\pi)^2 = 36\pi^2 \quad \text{(b)}$$

Dividing Eq. (a) by Eq. (b) gives

$$1 + \frac{\Delta m}{m} = \frac{16}{9}$$

$$m = \frac{9}{7}\Delta m = \frac{9}{7}\frac{5}{g} = \frac{6.43}{g}\ \text{lbs/g} \quad \text{(c)}$$

From Eq. (a),

$$k = 64\pi^2 m = 64\pi^2 \frac{6.43}{g} = 10.52\ \text{lbs/in.} \quad \text{(d)}$$

Problem 3.2

At $\omega = \omega_n$, from Eq. (3.2.15),

$$u_o = (u_{st})_o \frac{1}{2\zeta} = 2 \qquad \text{(a)}$$

At $\omega = 0.1\omega_n$, from Eq. (3.2.13),

$$u_o \approx (u_{st})_o = 0.2$$

Substituting $(u_{st})_o = 0.2$ in Eq. (a) gives

$$\zeta = 0.05$$

Problem 3.3

Assuming that damping is small enough to justify the approximation that the resonant frequency is ω_n and the resonant amplitude of R_d is $1/2\zeta$, then the given data implies:

$$(u_o)_{\omega = \omega_n} = (u_{st})_o \frac{1}{2\zeta} \qquad \text{(a)}$$

$$(u_o)_{\omega = 1.2\omega_n} = (u_{st})_o \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}}$$

$$= (u_{st})_o \frac{1}{\sqrt{[1 - (1.2)^2]^2 + [2\zeta(1.2)]^2}} \qquad \text{(b)}$$

Combining Eq. (a) and Eq. (b):

$$\frac{1}{(2\zeta)^2}\left(\frac{(u_o)^2_{\omega=1.2\omega_n}}{(u_o)^2_{\omega=\omega_n}}\right)^2 = \frac{1}{(-0.44)^2 + (2.4\zeta)^2} \qquad \text{(c)}$$

For

$$\frac{(u_o)_{\omega = 1.2\omega_n}}{(u_o)_{\omega = \omega_n}} = \frac{1}{4}$$

Eq. (c) gives

$$64\zeta^2 = 0.1935 + 5.76\zeta^2 \Rightarrow \zeta = 0.0576$$

Assumption of small damping implied in Eq. (a) is reasonable; otherwise we would have to use the exact resonant frequency $= \omega_n\sqrt{1 - 2\zeta^2}$ and exact resonant amplitude $= (u_{st})_o \Big/ \left[2\zeta\sqrt{1-2\zeta^2}\right]$.

Problem 3.4

(a) Machine running at 20 rpm.

$$\frac{\omega}{\omega_n} = \frac{20}{200} = 0.1$$

$$u_o = \frac{(u_{st})_o}{\left|1-(\omega/\omega_n)^2\right|} \qquad \text{(a)}$$

or

$$0.2 = \frac{(u_{st})_o}{\left|1-(0.1)^2\right|} \Rightarrow (u_{st})_o = 0.1980 \text{ in.}$$

For $\zeta = 0.25$ and $\omega/\omega_n = 0.1$,

$$u_o = (u_{st})_o \frac{1}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(b)}$$

or

$$u_o = 0.1980 \frac{1}{\sqrt{[1-(0.1)^2]^2 + [2(0.25)(0.1)]^2}}$$
$$= 0.1997 \text{ in.}$$

(b) Machine running at 180 rpm.

$$\frac{\omega}{\omega_n} = \frac{180}{200} = 0.9$$

From Eq. (a),

$$1.042 = \frac{(u_{st})_o}{\left|1-(0.9)^2\right|} \Rightarrow (u_{st})_o = 0.1980 \text{ in.}$$

For $\zeta = 0.25$ and $\omega/\omega_n = 0.9$, Eq. (b) reads

$$u_o = 0.1980 \frac{1}{\sqrt{\left[1-(0.9)^2\right]^2 + \left[2(0.25)(0.9)\right]^2}}$$
$$= 0.4053 \text{ in.}$$

(c) Machine running at 600 rpm.

$$\frac{\omega}{\omega_n} = \frac{600}{200} = 3$$

From Eq. (a),

$$0.0248 = \frac{(u_{st})_o}{\left|1-(3)^2\right|} \Rightarrow (u_{st})_o = 0.1980 \text{ in.}$$

For $\zeta = 0.25$ and $\omega/\omega_n = 3$, Eq. (b) reads

$$u_o = 0.1980 \frac{1}{\sqrt{[1-(3)^2]^2 + [2(0.25)(3)]^2}}$$
$$= 0.0243 \text{ in.}$$

(d) Summarizing these results together with given data:

ω/ω_n	$(u_o)_{\zeta=0}$	$(u_o)_{\zeta=0.25}$
0.1	0.2	0.1997
0.9	1.042	0.4053
3.0	0.0248	0.0243

The isolator is effective at $\omega/\omega_n = 0.9$; it reduces the deformation amplitude to 39% of the response without isolators. At $\omega/\omega_n = 0.1$ or 3, the isolator has essentially no influence on reducing the deformation.

Problem 3.5

Given:

$$w = 1200 \text{ lbs}, \quad E = 30 \times 10^6 \text{ psi},$$

$$I = 10 \text{ in.}^4, \quad L = 8 \text{ ft}; \quad \zeta = 1\%$$

$$p_o = 60 \text{ lbs}; \quad \omega = \left(\frac{300}{60}\right) 2\pi = 10\pi \text{rads/sec}$$

Stiffness of two beams:

$$k = 2\left(\frac{48EI}{L^3}\right) = 32{,}552 \text{ lbs/in.}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{w/\mathrm{g}}} = \sqrt{\frac{32{,}552}{1200/386}} = 102.3 \text{ rads/sec}$$

Steady state response:

$$R_d = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}}$$

where $\omega/\omega_n = 10\pi/102.3 = 0.3071$. Therefore,

$$R_d = \frac{1}{\sqrt{[1 - 0.0943]^2 + [2 \times 0.01 \times 0.3071]^2}} = 1.104$$

Displacement:

$$u_o = (u_{st})_o R_d = \frac{p_o}{k} R_d$$
$$= \frac{60}{32{,}552} \times 1.104 = 2.035 \times 10^{-3} \text{ in.}$$

Acceleration amplitude:

$$\ddot{u}_o = \omega^2 u_o = (10\pi)^2 \, 2.035 \times 10^{-3}$$
$$= 2.009 \text{ in./sec}^2 = 0.0052\mathrm{g}$$

Problem 3.6

In Eq. (3.2.1) replacing the applied force by $p_o \cos \omega t$ and dividing by m we get

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = \frac{p_o}{m}\cos\omega t \qquad \text{(a)}$$

(a) The particular solution is of the form:

$$u_p(t) = C\sin\omega t + D\cos\omega t \qquad \text{(b)}$$

Differentiating once and then twice gives

$$\dot{u}_p(t) = C\omega\cos\omega t - D\omega\sin\omega t \qquad \text{(c)}$$

$$\ddot{u}_p(t) = -C\omega^2\sin\omega t - D\omega^2\cos\omega t \qquad \text{(d)}$$

Substituting Eqs. (b)-(d) in Eq. (a) and collecting terms:

$$\left[(\omega_n^2 - \omega^2)\,C - 2\zeta\omega_n\omega\,D\right]\sin\omega t$$

$$+ \left[2\zeta\omega_n\omega\,C + (\omega_n^2 - \omega^2)\,D\right]\cos\omega t = \frac{p_o}{m}\cos\omega t$$

Equating coefficients of $\sin\omega t$ and of $\cos\omega t$ on the two sides of the equation:

$$(\omega_n^2 - \omega^2)\,C - (2\zeta\omega_n\omega)\,D = 0 \qquad \text{(e)}$$

$$(2\zeta\omega_n\omega)\,C + (\omega_n^2 - \omega^2)\,D = \frac{p_o}{m} \qquad \text{(f)}$$

Solving Eqs. (e) and (f) for C and D gives

$$C = \frac{p_o}{m}\frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \qquad \text{(g)}$$

$$D = \frac{p_o}{m}\frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \qquad \text{(h)}$$

Substituting Eqs. (g) and (h) in Eq. (b) gives

$$u_p(t) = \frac{p_o}{k}\frac{\left[1 - (\omega/\omega_n)^2\right]\cos\omega t + \left[2\zeta\omega/\omega_n\right]\sin\omega t}{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}$$

(b) Maximum deformation is $u_o = \sqrt{C^2 + D^2}$. Substituting for C and D gives

$$u_o = \frac{p_o}{k}\frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}}$$

This is same as Eq. (3.2.11) for the amplitude of deformation due to sinusoidal force.

Problem 3.7

(a) The displacement amplitude in given by Eq. (3.2.11):

$$u_o = (u_{st})_o \left[(1 - \beta^2)^2 + (2\zeta\beta)^2\right]^{-1/2} \qquad \text{(a)}$$

where $\beta = \omega/\omega_n$. Resonance occurs at β when u_o is maximum, i.e., $du_o/d\beta = 0$. Differentiating Eq. (a) with respect to β gives

$$-\frac{1}{2}\left[(1 - \beta^2)^2 + (2\zeta\beta)^2\right]^{-3/2}$$
$$\times\left[2(1 - \beta^2)(-2\beta) + 2(2\zeta\beta)2\zeta\right] = 0$$

or

$$-4(1 - \beta^2)\beta + 8\zeta^2\beta = 0$$

or

$$1 - \beta^2 = 2\zeta^2 \Rightarrow \beta = \sqrt{1 - 2\zeta^2} \qquad \text{(b)}$$

Resonant frequency:

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

(b) Substituting Eq. (b) in Eq. (a) gives the resonant amplitude:

$$u_o = (u_{st})_o \frac{1}{\sqrt{(2\zeta^2)^2 + 4\zeta^2(1 - 2\zeta^2)}}$$

or

$$u_o = (u_{st})_o \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \qquad \text{(c)}$$

Problem 3.8

(a) From Eq. (3.2.19) the acceleration amplitude is

$$\ddot{u}_o = -\frac{p_o}{m} R_a = -\frac{p_o}{m}\left(\frac{\omega}{\omega_n}\right)^2 R_d$$

$$= -\frac{p_o}{m}\frac{\beta^2}{\left[(1-\beta^2)^2 + (2\zeta\beta)^2\right]^{1/2}} \qquad \text{(a)}$$

where $\beta = \omega/\omega_n$. Resonance occurs at β where $\ddot{u}_o$ is maximum, i.e., $d\ddot{u}_o/d\beta = 0$. Differentiating Eq. (a) with respect to β and setting the result equal to zero gives

$$\frac{1}{(1-\beta^2)^2+(2\zeta\beta)^2}\left\{2\beta\left[(1-\beta^2)^2 + (2\zeta\beta)^2\right]^{1/2} - \left(\frac{\beta^2}{2}\right)\left[(1-\beta^2)^2 + (2\zeta\beta)^2\right]^{-1/2}\left[-4\beta(1-\beta^2) + 8\zeta^2\beta\right]\right\}=0$$

Multiplying the numerator by $[(1-\beta^2)^2 + (2\zeta\beta)^2]^{1/2}$ and dividing it by β gives

$$2\left[(1-\beta^2)^2 + 4\zeta^2\beta^2\right] - \frac{\beta^2}{2}\left[-4(1-\beta^2) + 8\zeta^2\right] = 0$$

or

$$1 - 2\beta^2 + \beta^4 + 4\zeta^2\beta^2 + \beta^2 - \beta^4 - 2\zeta^2\beta^2 = 0$$

or

$$1 = \beta^2(1-2\zeta^2) \Rightarrow \beta = \frac{1}{\sqrt{1-2\zeta^2}} \qquad \text{(b)}$$

Resonant frequency:

$$\omega_r = \frac{\omega_n}{\sqrt{1-2\zeta^2}} \qquad \text{(c)}$$

(b) Dividing both the numerator and denominator of Eq. (a) by β^2 gives:

$$\ddot{u}_o = \frac{p_o}{m}\frac{1}{\left[\left(\frac{1}{\beta^2}-1\right)^2 + \frac{4\zeta^2}{\beta^2}\right]^{1/2}} \qquad \text{(d)}$$

Substituting Eq. (b) in Eq. (d) gives the resonant amplitude:

$$\ddot{u}_o = -\frac{p_o}{m}\frac{1}{\left[(-2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)\right]^{1/2}}$$

$$\ddot{u}_o = -\frac{p_o}{m}\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Problem 3.9

(a) From Eq. (3.2.17), the velocity amplitude is

$$\dot{u}_0 = \frac{p_0}{\sqrt{km}} R_v = \frac{p_0}{\sqrt{km}} \left(\frac{\omega}{\omega_n} \right) R_d$$

$$= \frac{p_0}{\sqrt{km}} \frac{\beta}{[(1-\beta^2)^2 + (2\zeta\beta)^2]^{1/2}} \qquad \text{(a)}$$

where $\beta = \omega / \omega_n$. Resonance occurs at β where $\dot{u}$ is maximum, i.e., $d\dot{u}_0 / d\beta = 0$. Differentiating Eq. (a) with respect to β and setting the derivative equal to zero gives

$$[(1-\beta^2)^2 + (2\zeta\beta)^2]^{-1/2} -$$

$$\frac{\beta}{2}[(1-\beta^2)^2 + (2\zeta\beta)^2]^{-3/2}[2(1-\beta^2)(-2\beta) + 2(2\zeta\beta)2\zeta] = 0$$

Multiplying the numerator by $[(1-\beta^2)^2 + (2\zeta\beta)^2]^{3/2}$ gives

$$[(1-\beta^2)^2 + (2\zeta\beta)^2] - \frac{\beta}{2}[2(1-\beta^2)(-2\beta) + 2(2\zeta\beta)(2\zeta)] = 0$$

or

$$1 - 2\beta^2 + \beta^4 + 4\zeta^2\beta^2 - \beta(-2\beta + 2\beta^3 + 4\zeta^2\beta) = 0$$

or

$$\beta^4 = 1 \Rightarrow \beta = 1 \qquad \text{(b)}$$

Resonant frequency:

$$\omega_r = \omega_n \qquad \text{(c)}$$

(b) Substituding Eq. (b) into Eq. (a) gives

$$\dot{u}_0 = \frac{p_0}{\sqrt{km}} \frac{1}{2\zeta}$$

Problem 3.10

We assume that the structure has no mass other than the roof mass.

From Eq. (3.3.4),

$$\ddot{u}_o = \frac{m_e\, e\, \omega_n^2}{m} \left(\frac{\omega}{\omega_n}\right)^2 R_a \qquad \text{(a)}$$

At $\omega = \omega_n$, $R_a = 1/2\zeta$ and Eq. (a) gives

$$\zeta = \frac{1}{2\ddot{u}_o} \frac{m_e}{m}\, e\, \omega_n^2 \qquad \text{(b)}$$

Given:

$$m_e = 2 \times 50/\text{g} = 100\ \text{lbs/g} = 0.1\ \text{kips/g}$$

$$m = 500\ \text{kips/g}$$

$$e = 12\ \text{in.}$$

$$\omega_n = 2\pi f_n = 2\pi \times 4 = 8\pi$$

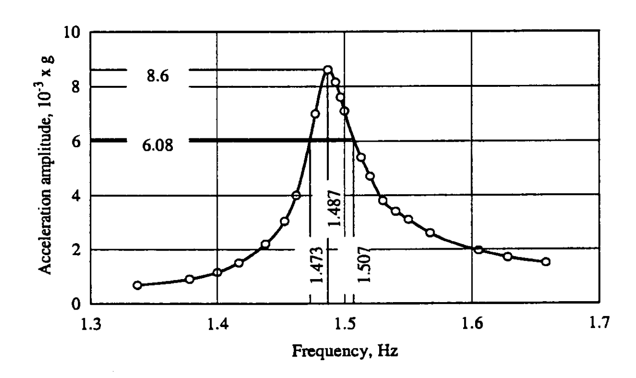
$$\ddot{u}_o = 0.02\text{g} = 7.72\ \text{in./sec}^2$$

Substituting the above data in Eq. (b) gives

$$\zeta = \frac{1}{2\,(7.72)} \frac{0.1}{500}\,(12)\,(8\pi)^2 = 0.0982 = 9.82\%$$

Problem 3.11

The given data is plotted in the form of the frequency response curve shown in the accompanying figure:



(a) Natural frequency

The frequency response curve peaks at

$$f_n = 1.487 \text{ Hz}$$

Assuming small damping, this value is the natural frequency of the system.

(b) Damping ratio

The acceleration at the peak is $r_{peak} = 8.6 \times 10^{-3}$ g. Draw a horizontal line at $r_{peak} \div \sqrt{2} = 6.08 \times 10^{-3}$ g to obtain f_a and f_b in Hz:

$$f_a = 1.473 \text{ Hz} \qquad f_b = 1.507 \text{ Hz}$$

Then,

$$\zeta = \frac{f_b - f_a}{2 f_n} = \frac{1.507 - 1.473}{2\,(1.487)} = 0.0114$$

$$= 1.14\%$$

Problem 3.12

(a) Transmissibility is given by Eq. (3.5.3) with $\zeta = 0$. Thus the force transmitted is

$$(f_T)_o = p_o \frac{1}{\left|1 - (\omega/\omega_n)^2\right|}$$

where $\omega_n = \sqrt{g/\delta_{st}}$ and $\omega = 2\pi f$. Therefore

$$(f_T)_o = p_o \frac{1}{\left|1 - \frac{4\pi^2}{g} f^2 \delta_{st}\right|} \qquad \text{(a)}$$

(b) For $(f_T)_o = 0.1\, p_o$ and $f = 20$, Eq. (a) gives

$$0.1 = \frac{1}{\left|1 - (4\pi^2/386)\,(20)^2\,\delta_{st}\right|} \Rightarrow$$

$$1 - \frac{4\pi^2}{386}(20)^2\,\delta_{st} = \pm 10 \Rightarrow$$

$$\frac{4\pi^2}{386}(20)^2\,\delta_{st} = -9 \text{ or } 11$$

The negative value is invalid. Therefore,

$$\delta_{st} = \frac{11}{(4\pi^2/386)\,(20)^2} = 0.269 \text{ in.}$$

Problem 3.13

The equation governing the deformation $u(t)$ in the suspension system is

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t) \qquad \text{(a)}$$

where $u_g(t) = u_{go}\sin\omega t$ and $\omega = 2\pi v/L$.

Substituting in Eq. (a) gives

$$m\ddot{u} + c\dot{u} + ku = m\omega^2 u_{go}\sin\omega t \qquad \text{(b)}$$

The amplitude of deformation is

$$u_o = (u_{st})_o R_d = \frac{m\omega^2 u_{go}}{k} R_d$$

The amplitude of the spring force is

$$f_{So} = ku_o = m\omega^2 u_{go} R_d \qquad \text{(c)}$$

where

$$R_d = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(d)}$$

Numerical calculations:

$k = 800$ lbs/in.; $m = 4000/386$ lb-sec^2/in.

$u_{go} = 3$ in.

$\omega_n = 8.768$ rads/sec; $\omega = 3.686$ rads/sec

$\omega/\omega_n = 0.420$

Thus Eq. (d) gives

$$R_d = \frac{1}{\sqrt{[1 - (0.42)^2]^2 + [2(0.4)(0.42)]^2}} = 1.124$$

Substituting ω_n and R_d in Eq. (c) gives

$$f_{So} = \frac{4000}{386}(3.686)^2\, 3\,(1.124) = 474.8 \text{ lbs}$$

Problem 3.14

$$u_g(t) = u_{go} \sin \omega t \Rightarrow \ddot{u}_g(t) = -\omega^2 u_{go} \sin \omega t$$

$$p_{eff}(t) = -m\ddot{u}_g(t) = m\omega^2 u_{go} \sin \omega t \equiv p_o \sin \omega t$$

$$u_0 = (u_{st})_o R_d = \frac{m\omega^2 u_{go}}{k} R_d = u_{go} \left(\frac{\omega}{\omega_n} \right)^2 R_d = u_{go} R_a$$

Spring force: $f_S = ku_o$

The resonant frequency for f_S is the same as that for u_o, which is the resonsnat frequency for R_a (from section 3.2.5):

$$\omega_{res} = \frac{\omega_n}{\sqrt{1-2\zeta^2}}$$

$$= \frac{8.786}{\sqrt{1-2(0.4)^2}} = 10.655 \text{ rads/sec}$$

Resonance occurs at this forcing frequency, which implies a speed of

$$v = \frac{\omega L}{2\pi} = \frac{(10.655)100}{2\pi} = 169.6 \text{ ft/sec} = 116 \text{ mph}$$

Problem 3.15

Given: $w = 2000$ lbs, $f = 1500$ cpm $= 25$ Hz, and $TR = 0.10$

For an undamped system,

$$TR = \frac{1}{\left|1 - (\omega/\omega_n)^2\right|} = 0.1$$

For $TR < 1$, $\omega/\omega_n > \sqrt{2}$ and the equation becomes

$$\frac{1}{(\omega/\omega_n)^2 - 1} = 0.1 \Rightarrow$$

$$(\omega/\omega_n)^2 = 11 \Rightarrow \omega/\omega_n = 3.32$$

$$\omega_n = \frac{\omega}{3.32} = \frac{2\pi f}{3.32} = \frac{2\pi(25)}{3.32} = 47.31 \text{ rads/sec}$$

$$k = \omega_n^2 m = \omega_n^2 w/\text{g} = (47.31)^2\, 2000/386$$
$$= 11.6 \text{ kips/in.}$$

Problem 3.16

The excitation is

$$u_g(t) = u_{go} \sin \omega t \quad \text{or} \quad \ddot{u}_g(t) = -\omega^2 u_{go} \sin \omega t \qquad \text{(a)}$$

The equation of motion is

$$m\ddot{u} + c\dot{u} + ku = p_{\text{eff}}(t) \qquad \text{(b)}$$

where

$$p_{\text{eff}}(t) = -m\ddot{u}_g(t) = \omega^2 m u_{go} \sin \omega t \qquad \text{(c)}$$

The deformation response is given by Eq. (3.2.10) with p_o replaced by $\omega^2 m u_{go}$:

$$u(t) = \frac{\omega^2 m u_{go}}{k} R_d \sin(\omega t - \phi) \qquad \text{(d)}$$

where R_d and ϕ are defined by Eqs. (3.2.11) and (3.2.12), respectively.

The total displacement is

$$u^t(t) = u(t) + u_g(t)$$

Substituting Eqs. (a), (d), (3.2.11) and (3.2.12) gives

$$\begin{aligned} u^t(t) &= u_{go} \sin \omega_n t + \frac{(\omega/\omega_n)^2 u_{go}}{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2} \\ &\times \left\{\left[1-(\omega/\omega_n)^2\right] \sin \omega_n t - \left[2\zeta\omega/\omega_n\right] \cos \omega t\right\} \\ &= \frac{u_{go}}{\left[1-(\omega/\omega_n)^2\right]^2 + (2\zeta\omega/\omega_n)^2} \\ &\times \left\{\left[1-(\omega/\omega_n)^2 + 4\zeta^2(\omega/\omega_n)^2\right] \sin \omega_n t - 2\zeta(\omega/\omega_n)^3 \cos \omega t\right\} \end{aligned} \qquad \text{(e)}$$

Equation (e) is of the form $u^t(t) = C \sin \omega t + D \cos \omega t$; hence $u_o^t = \sqrt{C^2 + D^2}$. Substituting for C and D gives

$$u_o^t = u_{go} \left\{ \frac{1 + (2\zeta\omega/\omega_n)^2}{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2} \right\}^{1/2} \qquad \text{(f)}$$

Equation (f) is the same as Eq. (3.6.5).

Problem 3.17

The accelerometer properties are $f_n = 50$ cps and $\zeta = 0.7$; and the excitation is

$$\ddot{u}_g(t) = 0.1\text{g}\sin(2\pi f t);\ f = 10, 20 \text{ and } 40 \text{ Hz} \qquad \text{(a)}$$

Compare the excitation with the measured relative displacement

$$u(t) = \left[(-1/\omega_n^2)\, R_d\right] \ddot{u}_g(t - \phi/\omega) \qquad \text{(b)}$$

where

$$R_d = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(c)}$$

and

$$\phi = \tan^{-1}\left[\frac{2\zeta\,\omega/\omega_n}{1 - (\omega/\omega_n)^2}\right] \qquad \text{(d)}$$

Because the instrument has been calibrated at low excitation frequencies, after separating the instrument constant $-1/\omega_n^2$, the recorded acceleration is

$$\ddot{u}_g(t) = R_d\, \ddot{u}_g(t - \phi/\omega) \qquad \text{(e)}$$

For a given f (or ω), ω/ω_n is computed and substituted in Eqs. (c) and (d) to calculate R_d and ϕ. With R_d and ϕ known for a given excitation frequency, the recorded acceleration is given by Eq. (e).

The computed amplitude R_d and time lag ϕ/ω agrees with Fig. 3.7.3. The difference between R_d and 1 is the error in measured acceleration (Table P3.17).

Table P3.17

f (Hz)	ω/ω_n	R_d	ϕ/ω, sec	% error in R_d
10	0.2	1	0.0045	0
20	0.4	0.991	0.0047	0.9
40	0.8	0.850	0.0050	15

Problem 3.18

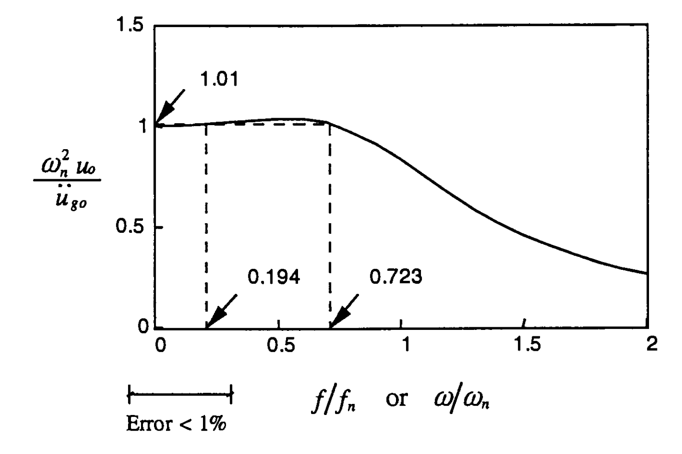
From Eq. (3.7.1),

$$\omega_n^2 u(t) = -R_d \ddot{u}_g(t - \phi/\omega)$$
$$= -R_d \ddot{u}_{go} \sin(2\pi f t - \phi)$$

and therefore,

$$\frac{\omega_n^2 u_o}{\ddot{u}_{go}} = R_d = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \quad \text{(a)}$$

Equation (a) is plotted for $\zeta = 0.6$, the accelerometer damping ratio:



We want to bound R_d as follows:

$$0.99 \le R_d \le 1.01 \quad \text{(b)}$$

The relevant condition is $R_d \le 1.01$ because we are interested in a continuous range of frequencies over which the error is less than 1%. Therefore, impose

$$\frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} = 1.01 \quad \text{(c)}$$

Defining $\beta \equiv \omega/\omega_n$, Eq. (c) can be rewritten as

$$(1-\beta^2)^2 + (2\zeta\beta)^2 = \left(\frac{1}{1.01}\right)^2 \Rightarrow$$

$$\beta^4 - 0.56\beta^2 + 1 = 0.9803 \Rightarrow$$

$$\beta^4 - 0.56\beta^2 + 0.0197 = 0 \Rightarrow$$

$$\beta^2 = 0.0377, \ 0.5223 \Rightarrow \beta = 0.194, \ 0.723$$

Choose $\beta = 0.194$ (see figure) which gives the desired frequency range:

$$f \le 0.194 f_n = 0.194(25) \Rightarrow$$

$$f \le 4.86 \text{ Hz}$$

Problem 3.19

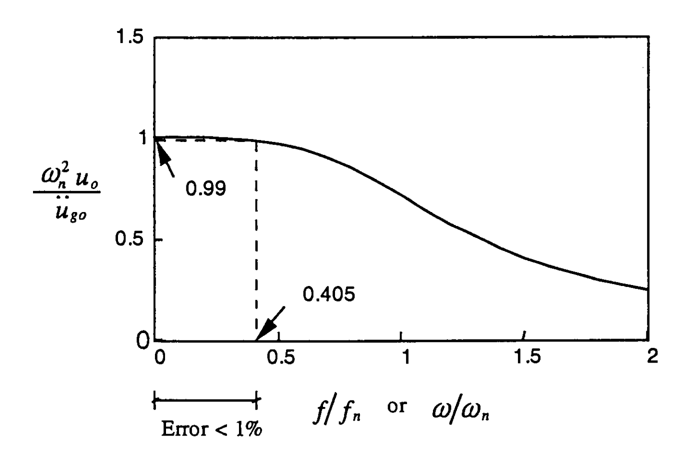
From Eq. (3.7.1),

$$\omega_n^2 u(t) = -R_d \ddot{u}_g(t - \phi/\omega)$$
$$= -R_d \ddot{u}_{go} \sin(2\pi f t - \phi)$$

and therefore,

$$\frac{\omega_n^2 u_o}{\ddot{u}_{go}} = R_d = \frac{1}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(a)}$$

Equation (a) is plotted for $\zeta = 0.7$, the accelerometer damping ratio:



We want to bound R_d as follows

$$0.99 \le R_d \le 1.01 \qquad \text{(b)}$$

The relevant condition is $0.99 \le R_d$ because R_d is always smaller than 1.01 in this case. Therefore, impose

$$\frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} = 0.99 \qquad \text{(c)}$$

Defining $\beta \equiv \omega/\omega_n$, Eq. (c) can be rewritten as

$$(1-\beta^2)^2 + (2\zeta\beta)^2 = \left(\frac{1}{0.99}\right)^2 \Rightarrow$$

$$\beta^4 - 0.04\beta^2 + 1 = 1.0203 \Rightarrow$$

$$\beta^4 - 0.04\beta^2 - 0.0203 = 0 \Rightarrow$$

$$\beta^2 = -0.1239, \; 0.1639$$

Take only the positive value:

$$\beta^2 = 0.1639 \Rightarrow \beta = 0.405$$

which gives the desired frequency range:

$$f \le 0.405 f_n = 0.405(50) \Rightarrow$$

$$f \le 20.25 \text{ Hz}$$

Problem 3.20

From Eq. (3.7.3),

$$\frac{u_o}{u_{go}} = R_a \qquad \text{(a)}$$

For maximum accuracy, $u_o/u_{go} = 1$; this condition after using Eqs. (3.2.20) and (3.2.11) for R_a gives

$$\frac{(\omega/\omega_n)^2}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} = 1 \qquad \text{(b)}$$

Squaring, simplifying and defining $\beta = \omega/\omega_n$:

$$(1 - \beta^2)^2 + (2\zeta\beta)^2 = \beta^4 \Rightarrow$$

$$1 - 2\beta^2 + 4\zeta^2\beta^2 = 0 \Rightarrow$$

$$\zeta^2 = \frac{1}{2} - \frac{1}{4\beta^2}$$

For $\omega/\omega_n >> 1$ $(\beta >> 1)$,

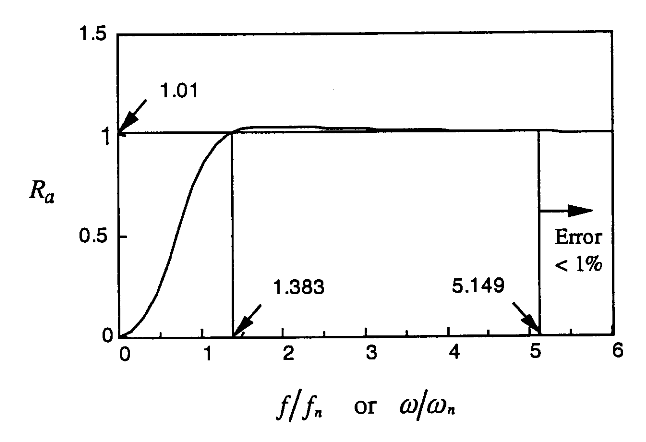
$$\zeta^2 = \frac{1}{2} \Rightarrow \zeta = 0.707$$

Problem 3.21

From Eq. (3.7.3),

$$\frac{u_o}{u_{go}} = R_a = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(a)}$$

Equation (a) is plotted for $\zeta = 0.6$, the instrument damping ratio:



We want to bound R_a as follows (see figure)

$$R_a \le 1.01 \qquad \text{(b)}$$

Therefore imposing $R_a = 1.01$ which, after defining $\beta \equiv \omega/\omega_n$, gives

$$\frac{\beta^2}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} = 1.01$$

Squaring this equation gives

$$(1-\beta^2)^2 + (2\zeta\beta)^2 = \left(\frac{\beta^2}{1.01}\right)^2 \Rightarrow$$

$$1 - 2\beta^2 + \beta^4 + 1.44\beta^2 = 0.9803\beta^4 \Rightarrow$$

$$0.0197\beta^4 - 0.56\beta^2 + 1 = 0 \Rightarrow$$

$$\beta^2 = 1.914,\ 26.51 \Rightarrow \beta = 1.383,\ 5.149$$

Choose $\beta = 5.149$ (see figure) which gives the desired frequency range:

$$f \ge 5.149 f_n = 5.149\,(0.5) \Rightarrow$$

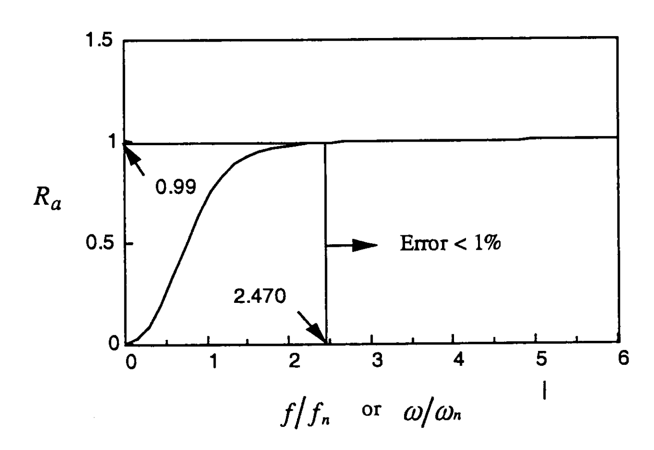
$$f \ge 2.575 \text{ Hz}$$

Problem 3.22

From Eq. (3.7.3),

$$\frac{u_o}{u_{go}} = R_a = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(a)}$$

Equation (a) is plotted for $\zeta = 0.7$, the instrument damping ratio:



Since R_a is always smaller than 1.01, we want to bound R_a as follows

$$R_a \geq 0.99 \qquad \text{(b)}$$

Therefore imposing $R_a = 0.99$ which, after defining $\beta \equiv \omega/\omega_n$, gives

$$\frac{\beta^2}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} = 0.99$$

Squaring this equation gives

$$(1 - \beta^2)^2 + (2\zeta\beta)^2 = \left(\frac{\beta^2}{0.99}\right)^2 \Rightarrow$$

$$1 - 2\beta^2 + \beta^4 + 1.96\beta^2 = 1.0203\beta^4 \Rightarrow$$

$$0.0203\beta^4 + 0.04\beta^2 - 1 = 0 \Rightarrow$$

$$\beta^2 = -8.073, \quad 6.102$$

Take only the positive value:

$$\beta^2 = 6.102 \Rightarrow \beta = 2.470$$

which gives the desired frequency range:

$$f \geq 2.470 f_n = 2.470\,(0.5) \Rightarrow$$

$$f \geq 1.235 \text{ Hz}$$

Problem 3.23

From Eq. (3.8.1),

$$E_D = 2\pi\zeta(\omega/\omega_n)\, ku_o^2 \qquad \text{(a)}$$

where

$$u_o = \frac{p_o}{k} R_d \qquad \text{(b)}$$

In Eq. (a) substituting Eq. (b) and Eq. (3.2.11) for R_d gives

$$E_D = \frac{\pi p_o^2}{k} \frac{2\zeta\omega/\omega_n}{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}$$

Problem 3.24

From Eq. (3.8.9) the loss factor is

$$\xi = \frac{1}{2\pi}\frac{E_D}{E_{S_o}} \qquad \text{(a)}$$

where $E_D = \pi c \omega u_o^2$ and $E_{S_o} = k u_o^2/2$. Substituting these in Eq. (a) gives

$$\xi = \frac{c\omega}{k} \qquad \text{(b)}$$

Problem 3.25

Based on equivalent viscous damping, the displacement amplitude is given by

$$\frac{u_o}{(u_{st})_o} = \frac{\left\{1 - \left[(4/\pi)(F/p_o)\right]^2\right\}^{1/2}}{1 - (\omega/\omega_n)^2} \qquad \text{(a)}$$

From the given data

$$\frac{F}{p_o} = \frac{50}{100} = \frac{1}{2}$$

$$\frac{\omega}{\omega_n} = \frac{T_n}{T} = \frac{0.25}{0.3} = 0.833$$

Substituting these data in Eq. (a) gives

$$\frac{u_o}{(u_{st})_o} = \frac{\left\{1 - \left[(4/\pi)(1/2)\right]^2\right\}^{1/2}}{1 - (0.833)^2} = 2.52$$

Now,

$$(u_{st})_o = \frac{p_o}{k}$$

where

$$p_o = 100 \text{ kips}$$

$$k = \omega_n^2 m = \left(\frac{2\pi}{T_n}\right)^2 \frac{w}{g} = (8\pi)^2 \frac{500}{386}$$
$$= 818 \text{kips/in.}$$

Thus

$$(u_{st})_o = \frac{100}{818} = 0.1222 \text{ in.}$$

and

$$u_o = 2.52\,(0.1222) = 0.308 \text{ in.}$$

Problem 3.26

(a) $p(t)$ is an even function:

$$p(t) = p_o\left(1 - \frac{2}{T_0}t\right) \qquad 0 \le t \le T_0/2 \qquad \text{(a)}$$

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} p(t)dt = \frac{2}{T_0}\int_0^{T_0/2} p_o\left(1 - \frac{2}{T_0}t\right)dt$$

$$= \frac{p_o}{2} \qquad \text{(b)}$$

$$a_j = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} p(t)\cos(j\omega_0 t)dt$$

$$= \frac{4p_0}{T_0}\int_0^{T_0/2}\left(1 - \frac{2}{T_0}t\right)\cos(j\omega_0 t)dt$$

$$= \frac{4p_0}{T_0}\left\{\frac{1}{j\omega_0}\left[\sin(j\omega_0 t)\right]_0^{T_0/2} - \frac{2}{T_0}\int_0^{T_0/2} t\cos(j\omega_0 t)dt\right\}$$

$$= -\frac{8p_0}{T_0^2}\int_0^{T_0/2} t\cos\left(\frac{2\pi jt}{T_0}\right)dt$$

$$= -\frac{4p_0}{\pi jT_0}\left\{\left[t\sin\left(\frac{2\pi j}{T_0}t\right)\right]_0^{T_0/2} + \frac{T_0}{2\pi j}\left[\cos\left(\frac{2\pi j}{T_0}t\right)\right]_0^{T_0/2}\right\}$$

$$= -\frac{2p_0}{\pi^2 j^2}\left[\cos\left(\frac{2\pi j}{T_0}t\right)\right]_0^{T_0/2}$$

$$= -\frac{2p_0}{\pi^2 j^2}\left[\cos(\pi j) - 1\right]$$

$$\therefore a_j = \begin{cases} \dfrac{4p_o}{\pi^2 j^2} & j = 1,3,5,\cdots \\ 0 & j = 2,4,6,\cdots \end{cases} \qquad \text{(c)}$$

$b_n = 0$ because $p(t)$ is an even function (d)

Thus the Fourier series representation of $p(t)$ is

$$p(t) = \frac{p_o}{2} + \frac{4p_o}{\pi^2}\sum_{j=1,3,5,\cdots}^{\infty}\frac{1}{j^2}\cos(j\omega_0 t) \qquad \text{(e)}$$

(b) The steady-state response of an undamped system is obtained by substituting Eqs. (b), (c) and (d) in Eq. (3.13.6) to obtain

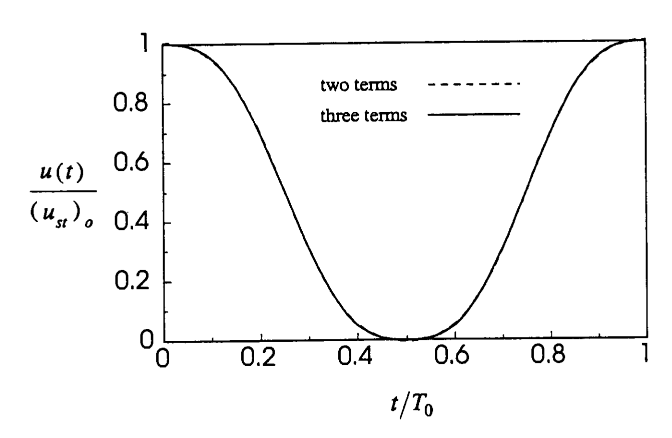
$$\frac{u(t)}{(u_{st})_o} = \frac{1}{2} + \frac{4}{\pi^2}\sum_{j=1,3,5,\cdots}^{\infty}\frac{1}{j^2(1-\beta_j^2)}\cos(j\omega_0 t)$$

(f)

where $(u_{st})_o = p_o/k$ and $\beta_j = j\omega_0/\omega_n$. Equation (f) is indeterminate when $\beta_j = 1$; corresponding values of T_0 are T_n, $3T_n$, $5T_n$, etc.

(c) For $T_0/T_n = 2$, $\beta_j = j\omega_0/\omega_n = jT_n/T_0 = j/2$ and Eq. (f) becomes

$$\frac{u(t)}{(u_{st})_o} = \frac{1}{2} + \frac{16}{\pi^2}\sum_{j=1,3,5,\cdots}^{\infty}\frac{1}{j^2(4-j^2)}\cos\left(\frac{2\pi jt}{T_0}\right)$$



Because of the j^4 in the denominator of the series, two terms are enough to obtain reasonable convergence of the series solution.

Problem 4.1

Equation (4.1.7) with $\tau = 0$ gives the response to $p(t) = \delta(t)$:

$$u(t) = \frac{1}{m\omega_D} e^{-\zeta\omega_n t} \sin \omega_D t \qquad \text{(a)}$$

The condition for $u(t)$ attaining a maximum is $du/dt = 0$:

$$\frac{du}{dt} = \frac{1}{m\omega_D}\left[e^{-\zeta\omega_n t}(-\zeta\omega_n)\sin \omega_D t + e^{-\zeta\omega_n t}\,\omega_D \cos \omega_D t\right]$$

$$= \frac{1}{m\omega_D} e^{-\zeta\omega_n t}\left[-\zeta\omega_n \sin \omega_D t + \omega_D \cos \omega_D t\right] = 0$$

or

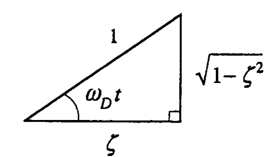
$$\tan \omega_D t = \frac{1}{\zeta}\frac{\omega_D}{\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

or

$$t = \frac{1}{\omega_D}\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \qquad \text{(b)}$$

The maximum displacement is given by Eq. (a) evaluated at t given by Eq. (b). For this t,

$$\sin \omega_D t = \sqrt{1-\zeta^2} \qquad \text{(c)}$$

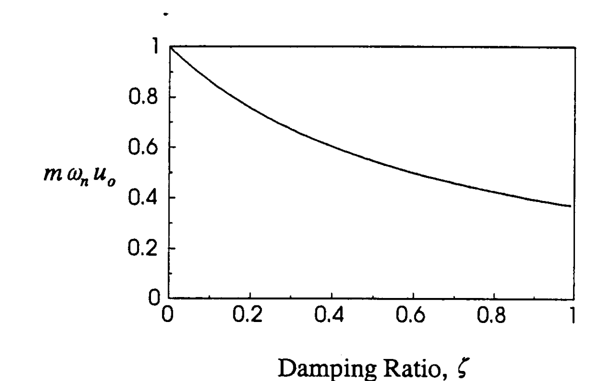


Substituting Eqs. (b) and (c) in Eq. (a) gives

$$u_o = \frac{1}{m\omega_n}\exp\left[-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] \qquad \text{(d)}$$

From Eq. (d) $m\omega_n u_o$ is plotted against ζ.

The effect of damping is small; e.g., 10% damping reduces the response by about 15%.



Problem 4.2

Response to a step function is given by Eq. (4.3.5); for $p_o = 1$ it becomes

$$g(t) = \frac{1}{k}\left\{1 - e^{-\zeta\omega_n t}\left[\cos\omega_D t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D t\right]\right\} \tag{a}$$

The response to a unit impulse force is given by Eq. (4.1.7) with $\tau = 0$:

$$h(t) = \frac{1}{m\omega_D} e^{-\zeta\omega_n t}\sin\omega_D t \tag{b}$$

From Eq. (a),

$$\dot{g}(t) = e^{-\zeta\omega_n t}\frac{1}{k}\left[\zeta\omega_n\cos\omega_D t + \frac{\zeta^2}{\sqrt{1-\zeta^2}}\omega_n\sin\omega_D t + \omega_D\sin\omega_D t - \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_D\cos\omega_D t\right]$$

Canceling the $\cos\omega_D t$ terms gives

$$\begin{aligned}\dot{g}(t) &= e^{-\zeta\omega_n t}\frac{1}{k}\left[\frac{\zeta^2}{\sqrt{1-\zeta^2}}\omega_n + \omega_D\right]\sin\omega_D t \\ &= e^{-\zeta\omega_n t}\frac{1}{k}\omega_n\left[\frac{\zeta^2}{\sqrt{1-\zeta^2}} + \sqrt{1-\zeta^2}\right]\sin\omega_D t \\ &= e^{-\zeta\omega_n t}\frac{1}{m\omega_n}\frac{1}{\sqrt{1-\zeta^2}}\sin\omega_D t \\ &= e^{-\zeta\omega_n t}\frac{1}{m\omega_D}\sin\omega_D t \\ &= h(t) \qquad \text{Q.E.D.}\end{aligned}$$

Problem 4.3

1. *Determine response to the first impulse.*

The response of the system to the first impulse is the unit impulse response of Eq. (4.1.6) times I:

$$u_1(t) = I\left[\frac{1}{m\omega_n}\sin\omega_n t\right] \qquad \text{(a)}$$

2. *Determine response to the second impulse.*

$$u_2(t) = -I\left[\frac{1}{m\omega_n}\sin\omega_n(t-t_d)\right] \quad t \ge t_d \qquad \text{(b)}$$

3. *Determine response to both impulses.*

For $0 \le t \le t_d$:

$$u(t) = I\left[\frac{1}{m\omega_n}\sin\omega_n t\right]$$

$$= \frac{I}{m\omega_n}\sin\frac{2\pi t}{T_n} \qquad \text{(c)}$$

For $t \ge t_d$:

$$u(t) = \frac{I}{m\omega_n}[\sin\omega_n t - \sin\omega_n(t-t_d)]$$

$$= \frac{I}{m\omega_n}2\sin\frac{\omega_n t_d}{2}\cos\frac{\omega_n(2t-t_d)}{2}$$

$$= \frac{2I}{m\omega_n}\left(\sin\frac{\pi t_d}{T_n}\right)\cos\left[2\pi\left(\frac{t}{T_n}-\frac{1}{2}\frac{t_d}{T_n}\right)\right]$$
(d)

4. *Plot displacement response.*

Equations (c) and (d) are plotted for $t_d / T_n = 1/8$, 1/4, 1/2, and 1 in Figs. P4.1a, b, c, and d, respectively.

5. *Determine maximum response during* $0 \le t \le t_d$.

The number of peaks in *u(t)* depend on t_d / T_n; the longer the time t_d between the pulses, more such peaks occur. The first peak occurs at $t_o = T_n / 4$ with the deformation:

$$\frac{u_o}{I / m\omega_n} = 1 \qquad \text{(e)}$$

Thus t_d must be longer than $T_n / 4$ for at least one peak to develop during $0 \le t \le t_d$.

If t_d is shorter than $T_n / 4$ no peak will develop during $0 \le t \le t_d$ and the response simply builds up from zero to $u(t_d)$, where

$$\frac{u(t_d)}{I / m\omega_n} = \sin\frac{2\pi t_d}{T_n} \qquad \text{(f)}$$

The maximum deformation during $0 \le t \le t_d$ is

$$\frac{u_o}{I / m\omega_n} = \begin{cases} \sin 2\pi t_d / T_n & t_d / T_n \le 1/4 \\ 1 & t_d / T_n \ge 1/4 \end{cases} \qquad \text{(g)}$$

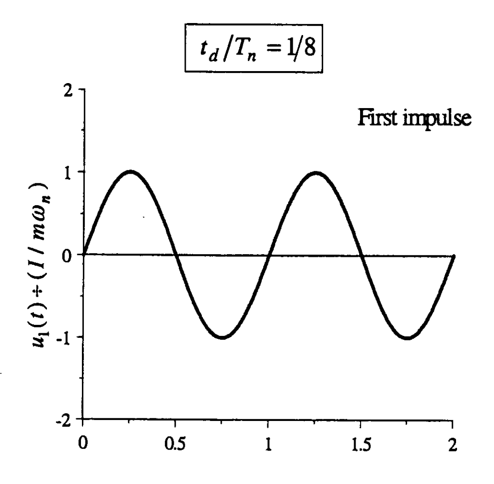
Equation (g) is plotted in Fig. P4.3e.

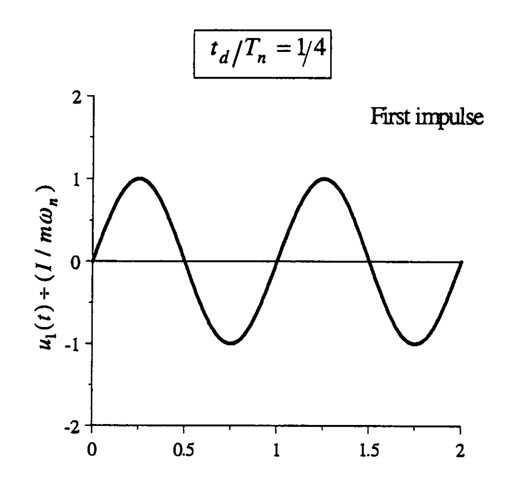
6. *Determine maximum response during* $t \ge t_d$.

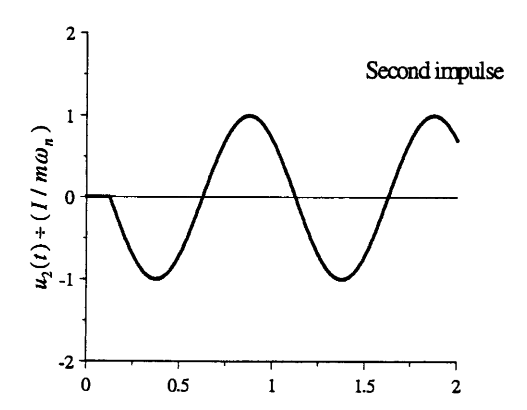
From Eq. (d), the maximum deformation during $t \ge t_d$ is

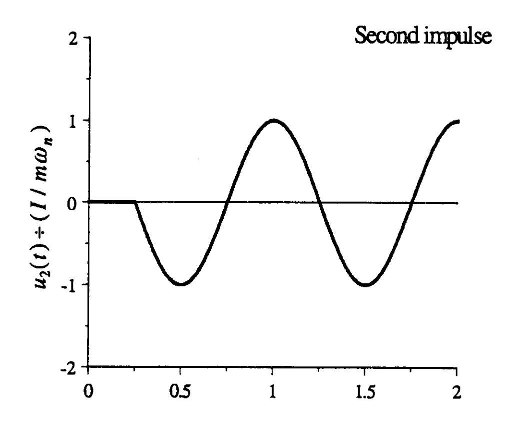
$$\frac{u_o}{I / m\omega_n} = 2\left|\sin(\pi t_d / T_n)\right| \qquad \text{(h)}$$

Equation (h) is plotted in Fig. P4.3e









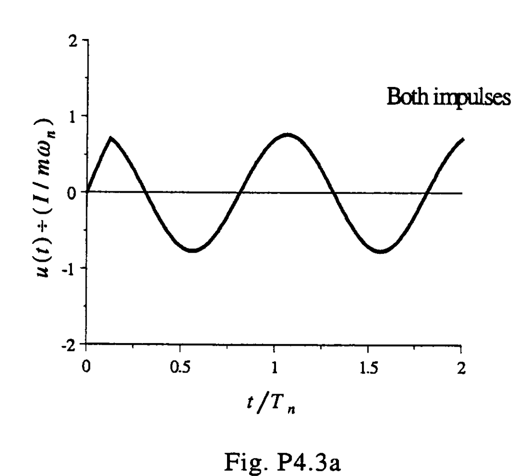


Fig. P4.3a

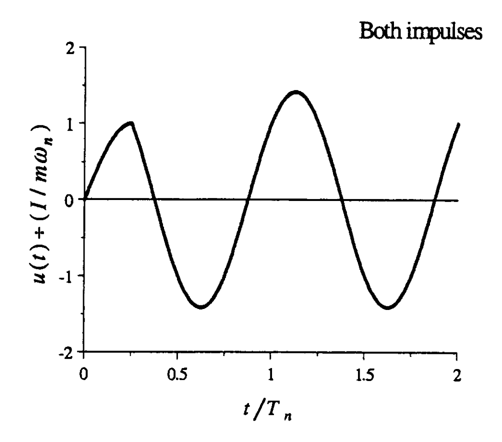
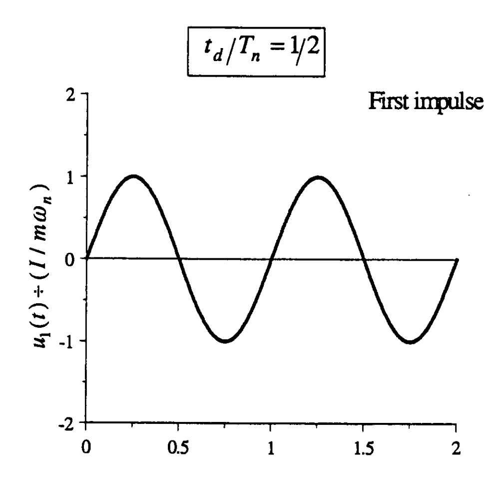
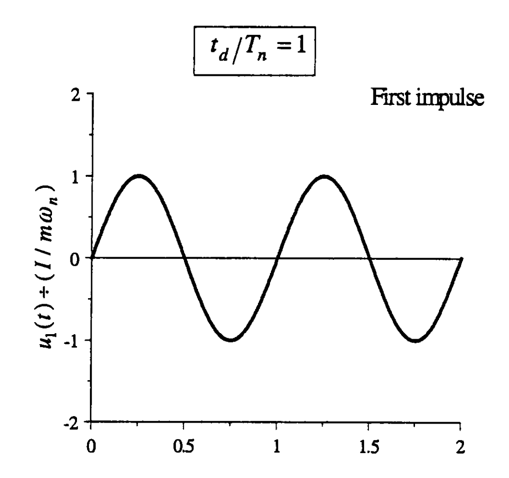
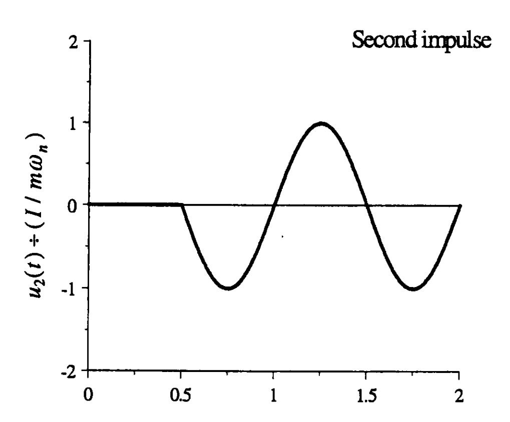
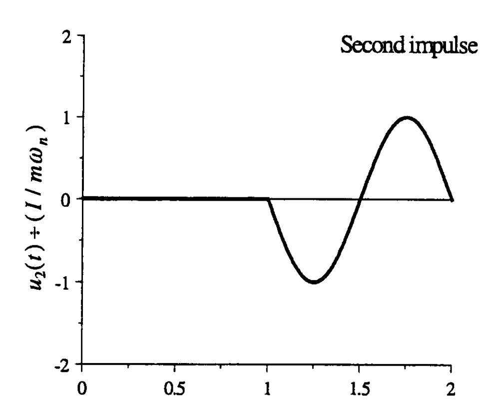


Fig. P4.3 b









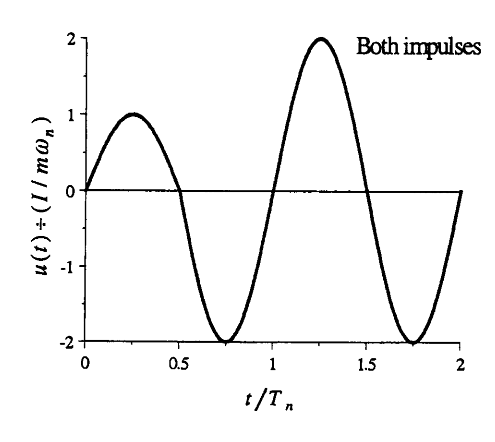


Fig. P4.3c

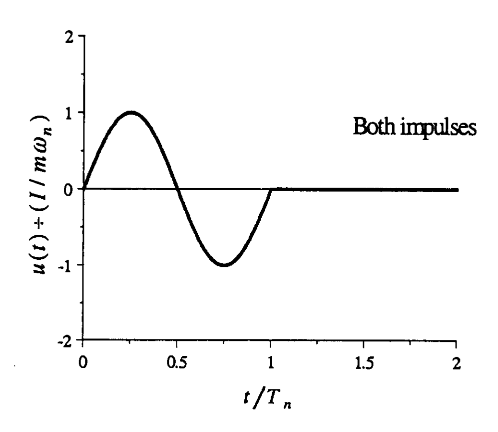


Fig. P4.3d

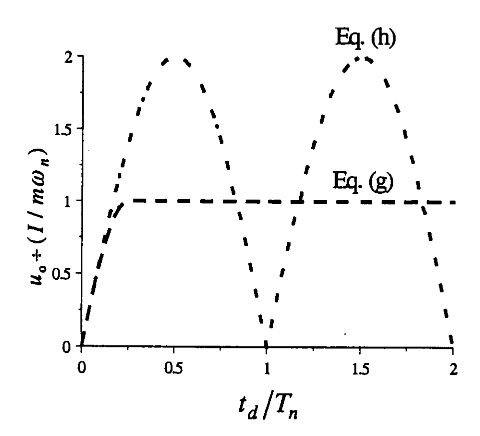


Fig. P4.3e

7. *Determine the overall maximum response.*

From Eqs. (g) and (h), the overall maximum response is given by

$$\frac{u_o}{I/m\omega_n} = \begin{cases} 2\left|\sin(\pi t_d / T_n)\right| & t_d/T_n \le 1/4 \\ \max \text{ of } \begin{cases} 1 \\ 2\left|\sin(\pi t_d / T_n)\right| \end{cases} & t_d/T_n \ge 1/4 \end{cases} \tag{i}$$

Equation (i) is plotted t_d / T_n in Fig. P4.3f to obtain the response spectrum.

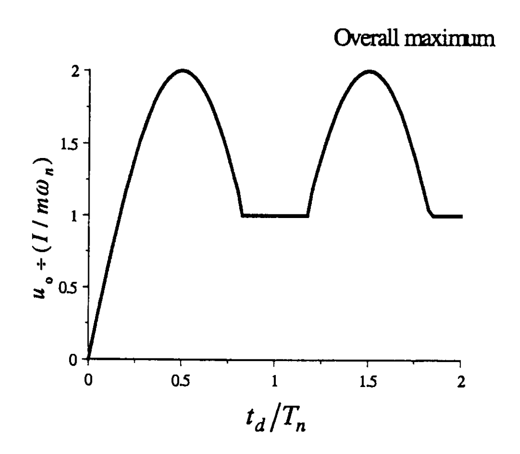


Fig. P4.3f

Problem 4.4

1. *Determine response to the first impulse.*

The response of the system to the first impulse is the unit impulse response of Eq. (4.1.6) times I:

$$u_1(t) = I\left[\frac{1}{m\omega_n}\sin\omega_n t\right] \qquad \text{(a)}$$

2. *Determine response to the second impulse.*

$$u_2(t) = I\left[\frac{1}{m\omega_n}\sin\omega_n (t - t_d)\right] \quad t \ge t_d \qquad \text{(b)}$$

3. *Determine response to the both impulses.*

For $0 \le t \le t_d$:

$$u(t) = I\left[\frac{1}{m\omega_n}\sin\omega_n t\right] = \frac{I}{m\omega_n}\sin\frac{2\pi t}{T_n} \qquad \text{(c)}$$

For $t \ge t_d$:

$$u(t) = \frac{I}{m\omega_n}[\sin\omega_n t + \sin\omega_n (t - t_d)]$$

$$= \frac{I}{m\omega_n} 2\cos\frac{\omega_n t_d}{2}\sin\frac{\omega_n(2t - t_d)}{2}$$

$$= \frac{2I}{m\omega_n}\left(\cos\frac{\pi t_d}{T_n}\right)\sin\left[2\pi\left(\frac{t}{T_n} - \frac{1}{2}\frac{t_d}{T_n}\right)\right] \qquad \text{(d)}$$

4. *Plot displacement response.*

Equations (c) and (d) are plotted for $t_d / T_n = 1/8$, 1/4, 1/2, and 1 in Figs. P4.4a, b, c, and d, respectively.

5. *Determine maximum response during* $0 \le t \le t_d$.

The number of peaks in *u(t)* depend on t_d / T_n; the longer the time t_d between the pulses, more such peaks occur. The first peak occurs at $t_o = T_n / 4$ with the deformation:

$$\frac{u_o}{I / m\omega_n} = 1 \qquad \text{(e)}$$

Thus t_d must be longer than $T_n / 4$ for at least one peak to develop during $0 \le t \le t_d$.

If t_d is shorter than $T_n / 4$ no peak will develop during $0 \le t \le t_d$ and the maximum displacement will occur at t_d :

$$\frac{u(t_d)}{I / m\omega_n} = \sin\frac{2\pi t_d}{T_n} \qquad \text{(f)}$$

The maximum deformation during $0 \le t \le t_d$ is

$$\frac{u_o}{I / m\omega_n} = \begin{cases} \sin 2\pi t_d / T_n & t_d / T_n \le 1/4 \\ 1 & t_d / T_n \ge 1/4 \end{cases} \qquad \text{(g)}$$

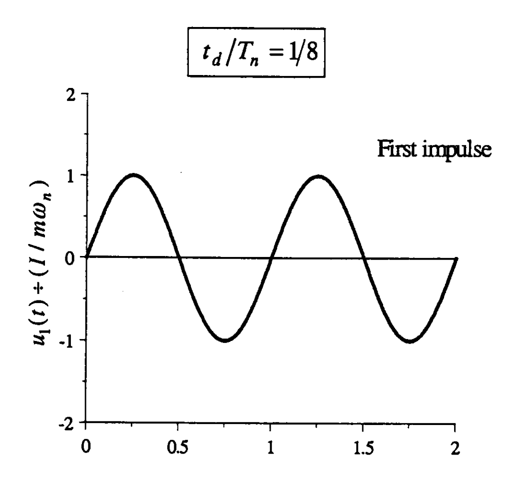
Equation (g) is plotted in Fig. P4.4e.

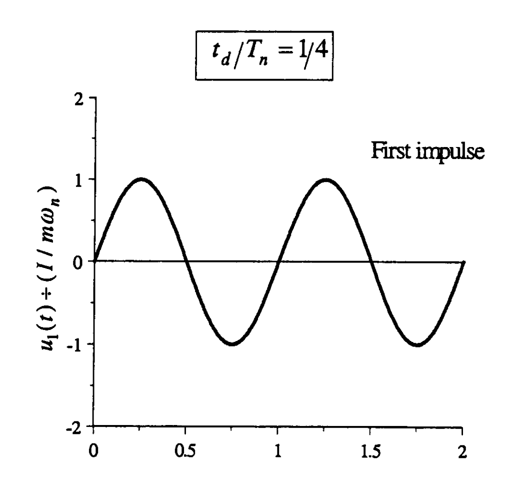
6. *Determine maximum response during* $t \ge t_d$.

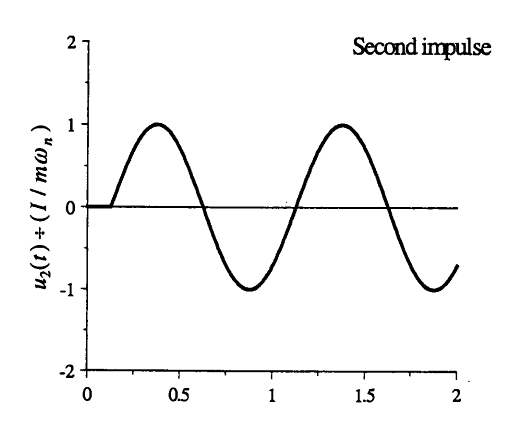
From Eq. (d), the maximum deformation during $t \ge t_d$ is

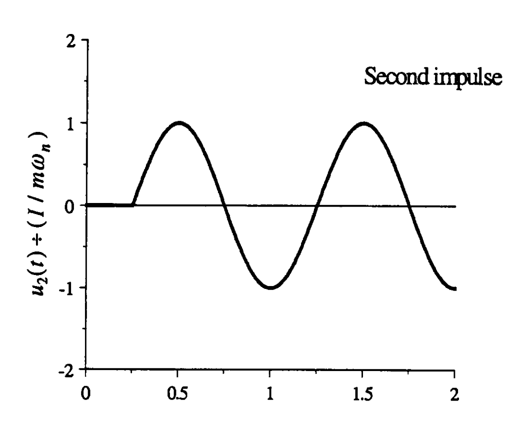
$$\frac{u_o}{I / m\omega_n} = 2\left|\cos(\pi t_d / T_n)\right| \qquad \text{(h)}$$

Equation (h) is plotted in Fig. P4.4e









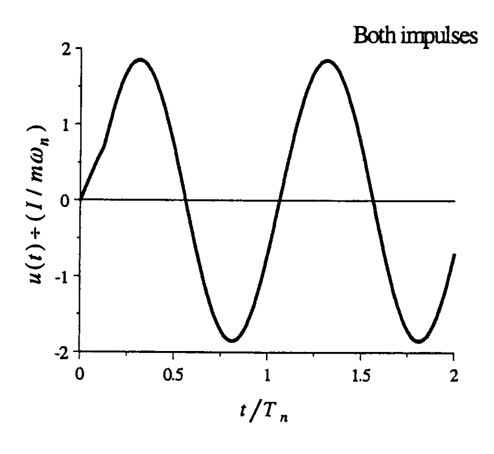


Fig. P4.4a

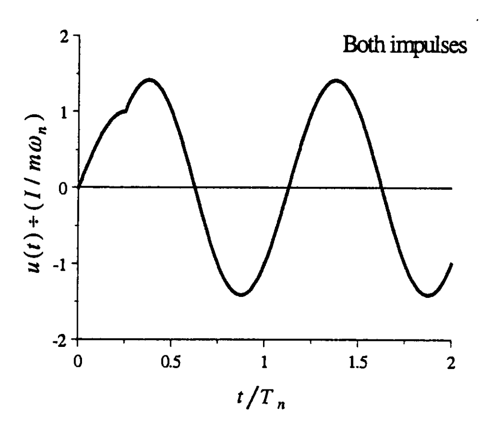
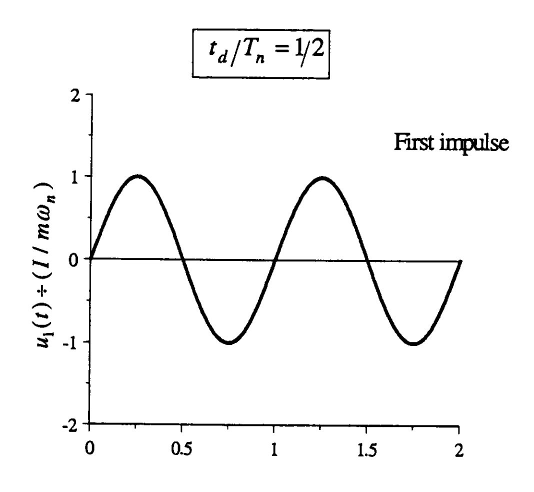
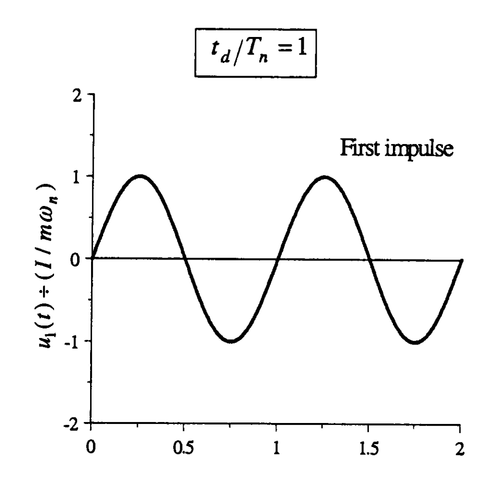
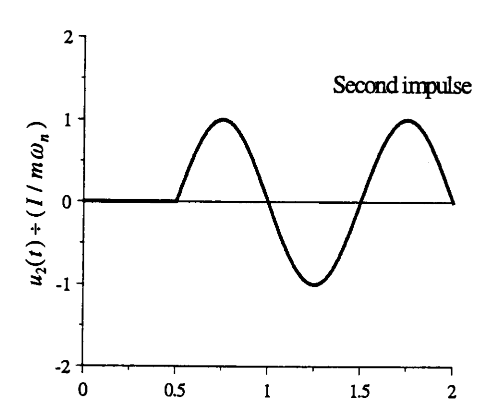
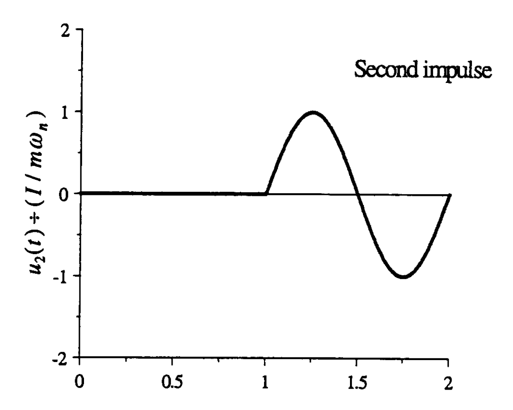


Fig. P4.4 b









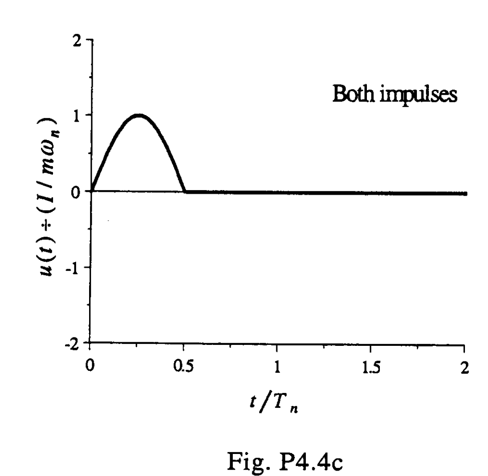


Fig. P4.4c

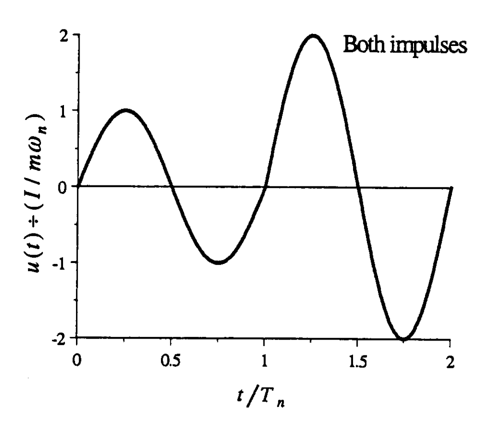


Fig. P4.4d

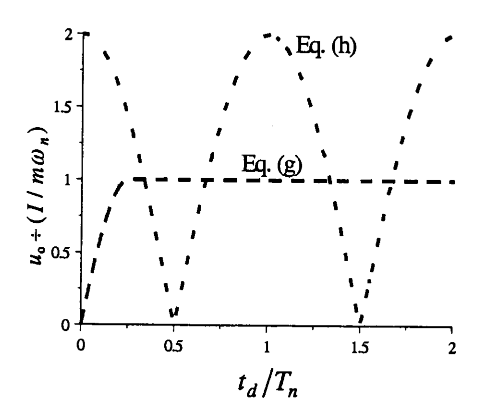


Fig. P4.4e

7. *Determine the overall maximum response.*

From Eqs. (g) and (h), the overall maximum response is given by

$$\frac{u_o}{I/m\omega_n} = \begin{cases} 2\left|\cos\left(\pi t_d/T_n\right)\right| & t_d/T_n \le 1/4 \\ \max.\ \text{of} \begin{cases} 1 \\ 2\left|\cos\left(\pi t_d/T_n\right)\right| \end{cases} & t_d/T_n \ge 1/4 \end{cases} \tag{i}$$

Equation (i) is plotted t_d / T_n in Fig. P4.4f to obtain the response spectrum.

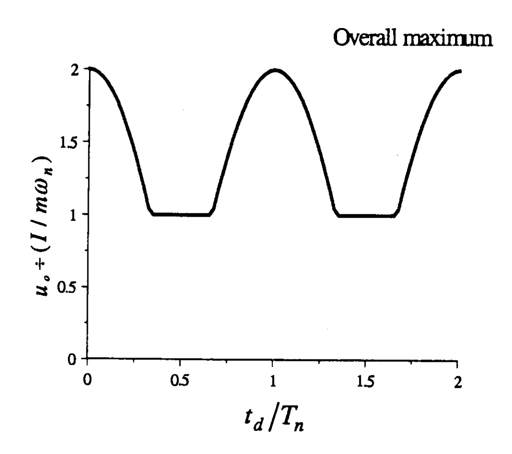


Fig. P4.4f

Problem 4.5

(a) The equation of motion is

$$m\ddot{u} + ku = p_o e^{-at} \qquad \text{(a)}$$

Using Duhamel's integral the solution is

$$u(t) = \frac{p_o}{m\omega_n}\int_0^t e^{-a\tau}\sin\left[\omega_n(t-\tau)\right]d\tau \qquad \text{(b)}$$

Integrate by parts letting $v = \sin\left[\omega_n(t-\tau)\right]$ and $dy = e^{-a\tau}d\tau$:

$$u(t) = vy - \int y dv$$

or

$$u(t) = \frac{p_o}{m\omega_n a}\left\{\left[-e^{-a\tau}\sin\left[\omega_n(t-\tau)\right]\right]_0^t - \int_0^t \omega_n e^{-a\tau}\cos\left[\omega_n(t-\tau)\right]d\tau\right\}$$

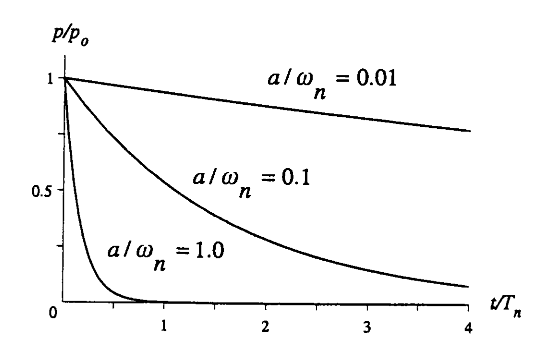
Integrating again by parts, this time with $v = \cos\left[\omega_n(t-\tau)\right]$ and $dy = e^{-a\tau}d\tau$ gives

$$u(t) = \frac{p_o}{m\omega_n(a^2+\omega_n^2)}\left[a\sin\omega_n t - \omega_n\cos\omega_n t + \omega_n e^{-at}\right]$$

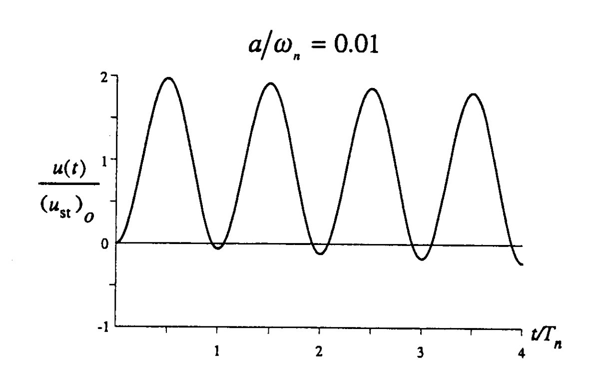
Written in terms of $(u_{st})_o$, the displacement response is

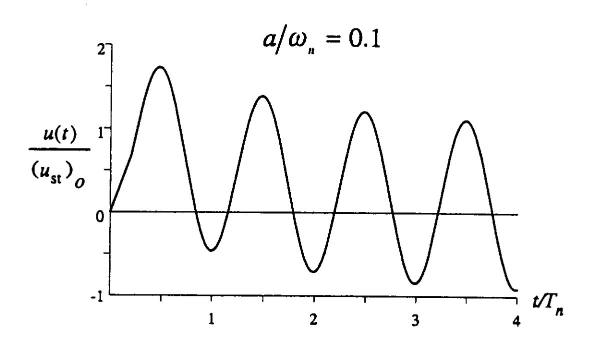
$$\frac{u(t)}{(u_{st})_o} = \frac{1}{1+\frac{a^2}{\omega_n^2}}\left[\frac{a}{\omega_n}\sin\omega_n t - \cos\omega_n t + e^{-at}\right] \qquad \text{(c)}$$

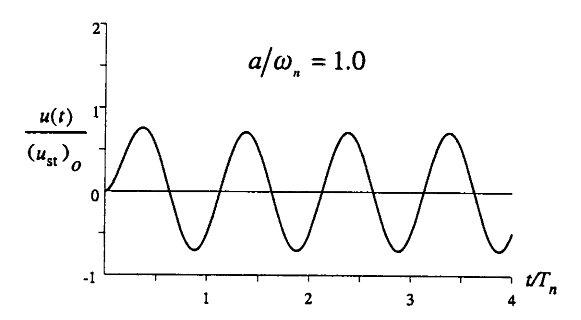
(b) The force $p(t) = p_o e^{-at}$ is plotted for three values of a/ω_n:



The motion given by Eq. (c) is plotted next.







As t increases, term e^{-at} becomes very small and the system attains a steady state harmonic motion whose amplitude is given by

$$\frac{u_{steady}}{(u_{st})_o} = \frac{1}{\sqrt{1+a^2/\omega_n^2}}$$

Problem 4.6

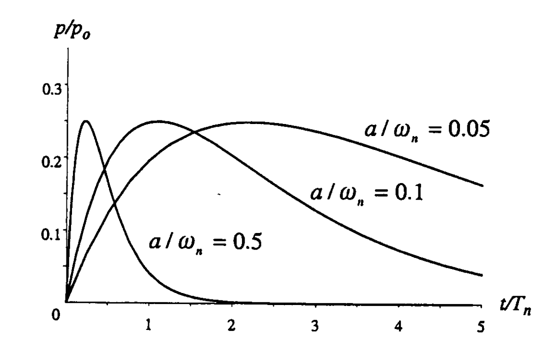
(a) The equation of motion is

$$m\ddot{u} + ku = p_o\left(e^{-at} - e^{-bt}\right) \qquad \text{(a)}$$

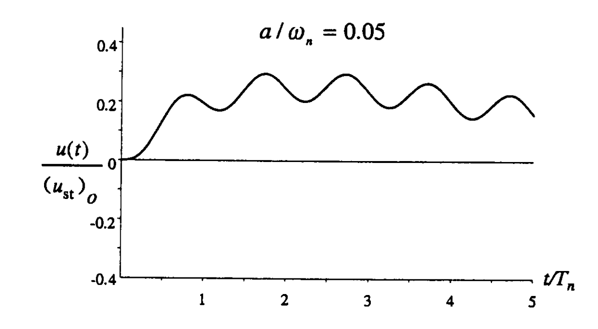
The response to each exponential function is given by an expression of the form in Eq. (c) of Problem 4.5. Combine the responses to the two components of $p(t)$ to obtain the total response:

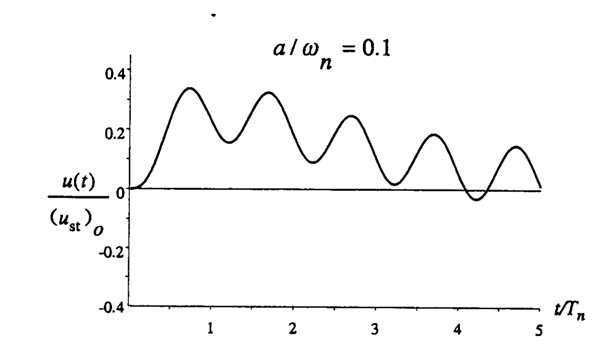
$$\frac{u(t)}{(u_{st})_o} = \frac{1}{1+a^2/\omega_n^2}\left[\frac{a}{\omega_n}\sin\omega_n t - \cos\omega_n t + e^{-at}\right] - \frac{1}{1+b^2/\omega_n^2}\left[\frac{b}{\omega_n}\sin\omega_n t - \cos\omega_n t + e^{-bt}\right] \qquad \text{(b)}$$

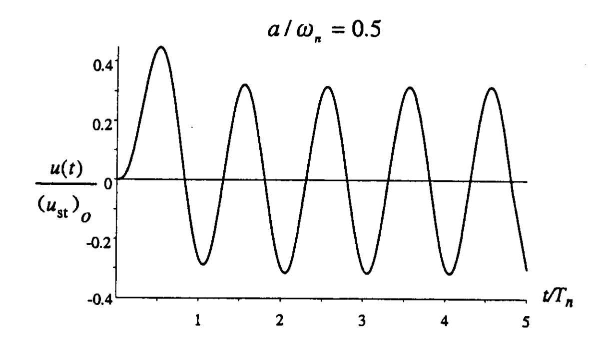
(b) The force $p(t)$ is plotted for $b = 2a$ and three values of a/ω_n .



The motion given by Eq. (b) is plotted next for $b = 2a$ and three values of $a/\omega_n = 0.05, 0.1, 0.5$.







Problem 4.7

The differential equation to be solved is

$$m\ddot{u} + ku = p(t) = p_o \frac{t}{t_r} \qquad \text{(a)}$$

The complimentary and particular solutions are

$$u_c(t) = A\cos\omega_n t + B\sin\omega_n t$$

$$u_p(t) = \frac{p_o}{k}\frac{t}{t_r} = (u_{st})_o \frac{t}{t_r}$$

The complete solution is

$$u(t) = A\cos\omega_n t + B\sin\omega_n t + (u_{st})_o \frac{t}{t_r} \qquad \text{(b)}$$

Differentiating Eq. (b) gives the velocity

$$\dot{u}(t) = -\omega_n A\sin\omega_n t + \omega_n B\cos\omega_n t + \frac{(u_{st})_o}{t_r} \qquad \text{(c)}$$

The constants A and B are determined from the initial conditions

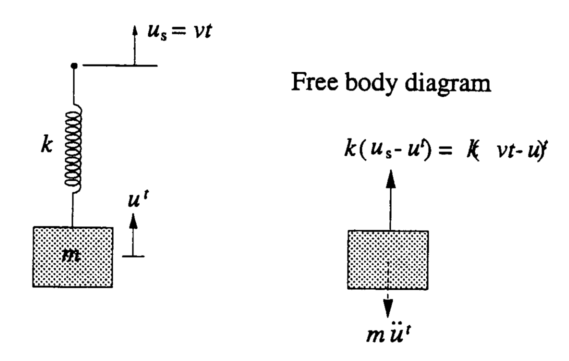
$$u(0) = 0 \Rightarrow A = 0$$

$$\dot{u}(0) = 0 \Rightarrow \omega_n B + \frac{(u_{st})_o}{t_r} = 0; \quad B = -\frac{(u_{st})_o}{\omega_n t_r}$$

Substituting A and B in Eq. (b) gives

$$u(t) = (u_{st})_o\left(\frac{t}{t_r} - \frac{\sin\omega_n t}{\omega_n t_r}\right)$$

Problem 4.8



Dynamic equilibrium gives

$$m\ddot{u}^t - k(vt - u^t) = 0$$

or

$$\ddot{u}^t + \omega_n^2 u^t = \omega_n^2 vt \qquad \text{(a)}$$

The general solution of the differential equation is

$$u^t(t) = u_c(t) + u_p(t) = A\cos\omega_n t + B\sin\omega_n t + vt \qquad \text{(b)}$$

Impose initial conditions: $u^t(0) = 0$ and $\dot{u}^t(0) = 0$:

$$u^t(0) = 0 \Rightarrow A = 0$$

$$\dot{u}^t(0) = B\omega_n + v = 0 \Rightarrow B = -\frac{v}{\omega_n}$$

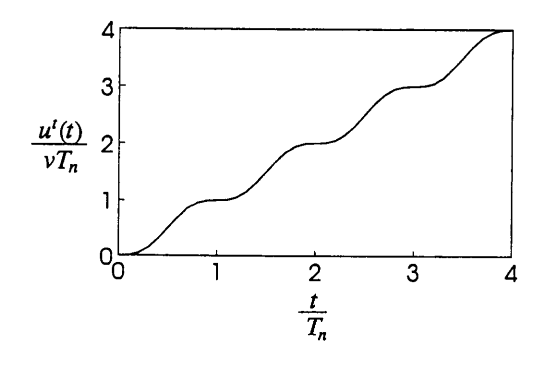
Substituting for A and B in Eq. (b) gives

$$u^t(t) = vt - \left(\frac{v}{\omega_n}\right)\sin\omega_n t \qquad \text{(c)}$$

Equation (c) is written as

$$\frac{u^t(t)}{vT_n} = \frac{t}{T_n} - \frac{1}{2\pi}\sin 2\pi\left(\frac{t}{T_n}\right) \qquad \text{(d)}$$

and plotted in the accompanying diagram.



Problem 4.9

The equation of motion is

$$m\ddot{u} + c\dot{u} + ku = p(t) \qquad \text{(a)}$$

The complementary solution is given by Eq. (f) in Derivation 2.2 in the book and the particular solution is $u_p = p_o/k$. Then the general solution is

$$u(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_D t + B\sin\omega_D t\right) + \frac{p_o}{k} \qquad \text{(b)}$$

When the system starts from rest, i.e. $u(0) = \dot{u}(0) = 0$, the constants A and B are

$$A = -\frac{p_o}{k} \quad , \quad B = -\frac{p_o}{k}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \qquad \text{(c)}$$

Then Eq. (b) becomes

$$\frac{u(t)}{(u_{\text{st}})_o} = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_D t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D t\right) \qquad \text{(d)}$$

The time t_p when *u(t)* attains its maximum (or peak) value is determined by setting the velocity equal to zero:

$$\frac{\dot{u}(t)}{(u_{\text{st}})_o} = e^{-\zeta\omega_n t}\left(\frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}} + \omega_D\right)\sin\omega_D t = 0$$

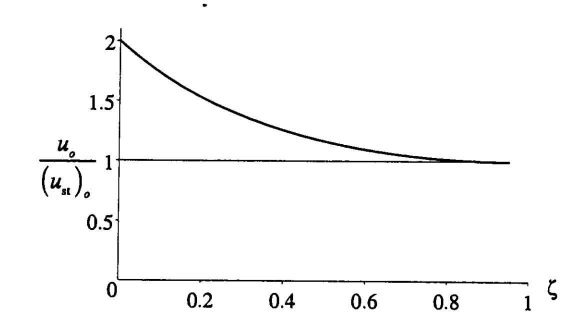
or

$$t_p = \frac{n\pi}{\omega_D} \qquad n = 0,1,2,3.... \qquad \text{(e)}$$

The maximum response occurs at t_p given by Eq. (e) with n=1, i.e. $t_p = \pi/\omega_D$. Substituting this t_p in Eq. (d) gives

$$\frac{u_o}{(u_{\text{st}})_o} = 1 + \exp\left(-\pi\zeta/\sqrt{1-\zeta^2}\right)$$

The relation between u_o and ζ is shown in the following plot.



Problem 4.10

Equation of motion:

$$m\ddot{u} + ku = p(t) = \begin{cases} p_o \dfrac{t}{t_r} & 0 \le t \le t_r \\ p_o & t \ge t_r \end{cases} \qquad \text{(a)}$$

(a) *Solution for* $0 \le t \le t_r$

Substituting $p(\tau)$ in Eq. (4.2.4) gives

$$u(t) = \frac{1}{m\omega_n} \int_0^t \frac{p_o}{t_r} \tau \sin \omega_n (t - \tau)\, d\tau$$

$$= \frac{1}{m\omega_n} \frac{p_o}{t_r} \int_0^t [-(t - \tau) \sin \omega_n (t - \tau) + t \sin \omega_n (t - \tau)]\, d\tau$$

Introducing $\chi \equiv t - \tau$ and integrating the first term by parts gives

$$u(t) = \frac{1}{m\omega_n} \frac{p_o}{t_r} \left[\int_t^0 \chi \sin \omega_n \chi\, d\chi - t \int_t^0 \sin \omega_n \chi\, d\chi \right]$$

$$= \frac{1}{m\omega_n} \frac{p_o}{t_r} \left\{ \frac{1}{\omega_n} \left[-\chi \cos \omega_n \chi \right]_t^0 + \frac{1}{\omega_n^2} \left[\sin \omega_n \chi \right]_t^0 + \frac{t}{\omega_n} \left[\cos \omega_n \chi \right]_t^0 \right\}$$

$$= \frac{p_o}{m\omega_n^2} \frac{1}{t_r} \left(t \cos \omega_n t - \frac{\sin \omega_n t}{\omega_n} + t - t \cos \omega_n t \right)$$

$$= (u_{st})_o \left(\frac{t}{t_r} - \frac{\sin \omega_n t}{\omega_n t_r} \right) \qquad \text{(b)}$$

(b) *Solution for* $t \ge t_r$

Substituting $p(\tau)$ in Eq. (4.2.4) and separating into two integrals gives

$$u(t) = \frac{1}{m\omega_n} \left[\int_0^{t_r} \frac{p_o}{t_r} \tau \sin \omega_n (t - \tau)\, d\tau + \int_{t_r}^t p_o \sin \omega_n (t - \tau)\, d\tau \right] \qquad \text{(c)}$$

The first integral in Eq. (c) is

$$\int_0^{t_r} \frac{p_o}{t_r} \tau \sin \omega_n (t - \tau)\, d\tau = \frac{p_o}{t_r} \int_t^{t - t_r} (\chi - t) \sin \omega_n \chi\, d\chi$$

$$= \frac{p_o}{t_r} \left[\int_t^{t - t_r} \chi \sin \omega_n \chi\, d\chi - t \int_t^{t - t_r} \sin \omega_n \chi\, d\chi \right]$$

$$= \frac{p_o}{t_r} \left\{ \left[\frac{-\chi \cos \omega_n \chi}{\omega_n} + \frac{\sin \omega_n \chi}{\omega_n^2} \right]_t^{t - t_r} + t \left[\frac{\cos \omega_n \chi}{\omega_n} \right]_t^{t - t_r} \right\}$$

$$= \frac{p_o}{t_r} \left[-\frac{(t - t_r) \cos \omega_n (t - t_r)}{\omega_n} + \frac{t \cos \omega_n t}{\omega_n} + \frac{\sin \omega_n (t - t_r)}{\omega_n^2} - \frac{\sin \omega_n t}{\omega_n^2} + \frac{t \cos \omega_n (t - t_r)}{\omega_n} - \frac{t \cos \omega_n t}{\omega_n} \right]$$

$$= \frac{p_o}{t_r} \left[-\frac{t_r \cos \omega_n (t - t_r)}{\omega_n} + \frac{\sin \omega_n (t - t_r)}{\omega_n^2} - \frac{\sin \omega_n t}{\omega_n^2} \right]$$

$$= \frac{p_o}{\omega_n} \left[\cos \omega_n (t - t_r) + \frac{\sin \omega_n (t - t_r) - \sin \omega_n t}{\omega_n t_r} \right] \qquad \text{(d)}$$

The second integral in Eq. (c) is

$$\int_{t_r}^t p_o \sin \omega_n (t - \tau)\, d\tau = \frac{p_o}{\omega_n} \left[\cos \omega_n \chi \right]_{t - t_r}^0$$

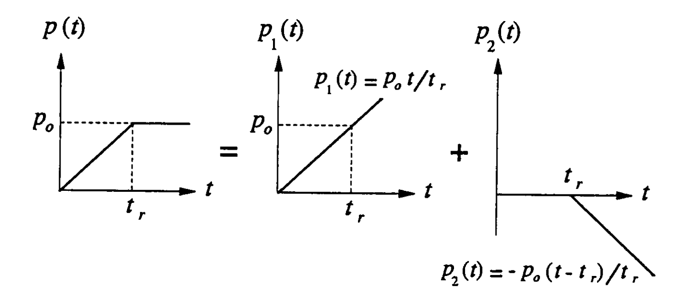
$$= \frac{p_o}{\omega_n} \left[1 - \cos \omega_n (t - t_r) \right] \qquad \text{(e)}$$

Substituting Eqs. (d) and (e) in Eq. (c) gives

$$u(t) = \frac{p_o}{m\omega_n^2} \left[\cos \omega_n (t - t_r) + \frac{\sin \omega_n (t - t_r) - \sin \omega_n t}{\omega_n t_r} + 1 - \cos \omega_n (t - t_r) \right]$$

$$= (u_{st})_o \left\{ 1 - \frac{1}{\omega_n t_r} \left[\sin \omega_n t - \sin \omega_n (t - t_r) \right] \right\} \qquad \text{(f)}$$

Problem 4.11



Equation (4.5.2) gives the response to a ramp function $p(t) = p_o\,(t/t_r)$:

$$u(t) = (u_{st})_o \left(\frac{t}{t_r} - \frac{\sin \omega_n t}{\omega_n t_r} \right) \qquad \text{(a)}$$

(a) *Response for* $0 \le t \le t_r$

Because $p_2(t) = 0$, the response is only due to $p_1(t)$, which is given by Eq. (a):

$$u_1(t) = (u_{st})_o \left(\frac{t}{t_r} - \frac{\sin \omega_n t}{\omega_n t_r} \right) \qquad \text{(b)}$$

(b) *Response for* $t \ge t_r$

The response to $p_1(t)$ is given by Eq. (b) and that due to $p_2(t)$ is obtained by replacing t in Eq. (a) by $t - t_r$ and p_o by $-p_o$:

$$u_2(t) = -(u_{st})_o \left(\frac{t - t_r}{t_r} - \frac{\sin \omega_n (t - t_r)}{\omega_n t_r} \right) \qquad \text{(c)}$$

The total response is the sum of Eqs. (b) and (c):

$$\begin{aligned} u(t) &= u_1(t) + u_2(t) \\ &= (u_{st})_o \left(\frac{t}{t_r} - \frac{\sin \omega_n t}{\omega_n t_r} - \frac{t}{t_r} + 1 + \frac{\sin \omega_n (t - t_r)}{\omega_n t_r} \right) \\ &= (u_{st})_o \left(1 - \frac{1}{\omega_n t_r} [\sin \omega_n t - \sin \omega_n (t - t_r)] \right) \end{aligned}$$

Problem 4.12

System properties:

$$m = w/g = 100.03/386 = 0.2591 \text{ kip}-\text{sec}^2/\text{in.}$$

$$k = 8.2 \text{ kips/in.}$$

$$T_n = 2\pi\sqrt{\frac{m}{k}} = 1.12 \text{ sec}$$

Applied force:

$$p_o = 50 \text{ kips}; \text{(a) } t_r = 0.2 \text{ sec}, \text{(b) } t_r = 4 \text{ sec}$$

(a) $t_r/T_n = 0.2/1.12 = 0.179$

The rise time of the force is relatively short, and the structure will "see" this excitation as a suddenly applied force (Fig. 4.5.3); therefore

$$u_o \approx 2(u_{st})_o = 2\left(\frac{p_o}{k}\right) = 2\left(\frac{50}{8.2}\right) = 2(6.1) = 12.2\text{in.}$$

(b) $t_r/T_n = 4/1.12 = 3.57$

The rise time of the force is relatively long, and it will affect the structure like a static force (Fig. 4.5.3); therefore

$$u_o \approx (u_{st})_o = 6.1 \text{ in.}$$

Problem 4.13

(a) Response results.

We have from Eq. (4.3.2)

$$\left.\begin{array}{l} u(t) = (u_{st})_o(1-\cos\omega_n t) \\ \dot{u}(t) = (u_{st})_o\omega_n \sin\omega_n t \end{array}\right\} \quad 0 \le t \le T_n/2 \qquad \text{(a)}$$

At $t = T_n/2$, $u = 2(u_{st})_o$ and $\dot{u} = 0$.

For $T_n/2 \le t \le T_n$,

$$u(\bar{t}) = A_1 \cos\omega_n\bar{t} + A_2 \sin\omega_n\bar{t} - \frac{p_o}{k}$$

where $\bar{t} = t - T_n/2$. Substituting $u(0) = 2(u_{st})_o$ and $\dot{u}(0) = 0$ gives $A_1 = 3(u_{st})_o$ and $A_2 = 0$. Hence

$$u(t) = (u_{st})_o[-1 + 3\cos\omega_n(t - T_n/2)], \quad T_n/2 \le t \le T_n \quad \text{(b)}$$

At $t = T_n$, i.e. at $\bar{t} = T_n/2$, $u = -4(u_{st})_o$ and $\dot{u} = 0$.

For $T_n \le t \le 3T_n/2$

$$u(\bar{t}) = A_1 \cos\omega_n\bar{t} + A_2 \sin\omega_n\bar{t} + \frac{p_o}{k}$$

where $\bar{t} = t - T_n$. Substituting $u(0) = -4(u_{st})_o$ and $\dot{u}(0) = 0$ gives $A_1 = 5(u_{st})_o$ and $A_2 = 0$. Hence

$$u(t) = (u_{st})_o[1 - 5\cos\omega_n(t - T_n)], \quad T_n \le t \le 3T_n/2 \quad \text{(c)}$$

At $t = 3T_n/2$, i.e. at $\bar{t} = T_n/2$, $u = 6(u_{st})_o$ and $\dot{u} = 0$.

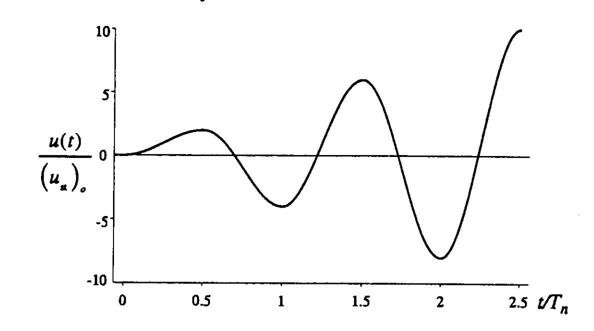
In a similar manner the following results can be obtained:

$$u(t) = (u_{st})_o[-1 + 7\cos\omega_n(t - 3T_n/2)], \quad 3T_n/2 \le t \le 2T_n \quad \text{(d)}$$

$$u(t) = (u_{st})_o[1 - 9\cos\omega_n(t - 2T_n)], \quad 2T_n < t < 5T_n/2 \quad \text{(e)}$$

(b) Response plot.

From Eqs. (a), (b), (c), (d), and (e), $u(t)/(u_{st})_o$ is plotted against t/T_n. Note that $\omega_n(t - T_n/2) = 2\pi(t/T_n - 1/2)$, $\omega_n(t - T_n) = 2\pi(t/T_n - 1)$, etc.



(c) Peak values.

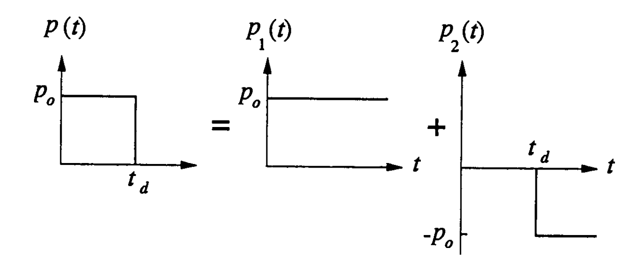
The displacement peaks u_n at the end of n half cycles of applied force are

n	1	2	3	4	5
$u_n/(u_{st})_o$	2	-4	6	-8	10

In general,

$$\frac{u_n}{(u_{st})_o} = (-1)^{n-1}2n$$

Problem 4.14



The response to $p_1(t)$ is given by Eq. (4.7.2):

$$u_1(t) = \frac{p_o}{k}(1 - \cos \omega_n t) \qquad t \geq 0 \qquad \text{(a)}$$

To obtain the response to $p_2(t)$, this equation can be modified as follows: replace p_o by $-p_o$ and t by $t - t_d$:

$$u_2(t) = -\frac{p_o}{k}[1 - \cos \omega_n (t - t_d)] \qquad t \geq t_d \quad \text{(b)}$$

During $0 \leq t \leq t_d$, $p_2(t) = 0$ and the response is given by Eq. (a):

$$u(t) = \frac{p_o}{k}(1 - \cos \omega_n t) \qquad 0 \leq t \leq t_d \qquad \text{(c)}$$

For $t \geq t_d$, the response is the sum of Eqs. (a) and (b):

$$u(t) = \frac{p_o}{k}[\cos \omega_n (t - t_d) - \cos \omega_n t] \qquad t \geq t_d \qquad \text{(d)}$$

Problem 4.15

$$p(t) = \begin{cases} p_o & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(a)}$$

$$u(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n (t - \tau)\, d\tau \qquad \text{(b)}$$

For $0 \le t \le t_d$, Eq. (b) after substituting for $p(t)$ gives

$$u(t) = \frac{1}{m\omega_n} \int_0^t p_o \sin \omega_n (t - \tau)\, d\tau$$

$$= \frac{p_o}{m\omega_n} \left[\frac{1}{\omega_n} \cos \omega_n (t - \tau) \right]_0^t$$

$$= \frac{p_o}{m\omega_n^2} (1 - \cos \omega_n t) = \frac{p_o}{k} (1 - \cos \omega_n t) \qquad \text{(c)}$$

For $t \ge t_d$, Eq. (b) after substituting for $p(t)$ gives

$$u(t) = \frac{1}{m\omega_n} \left[\int_0^{t_d} p_o \sin \omega_n (t - \tau)\, d\tau + \int_{t_d}^t 0 \cdot \sin \omega_n (t - \tau)\, d\tau \right]$$

$$= \frac{p_o}{m\omega_n} \left[\frac{1}{\omega_n} \cos \omega_n (t - \tau) \right]_0^{t_d}$$

$$= \frac{p_o}{k} [\cos \omega_n (t - t_d) - \cos \omega_n t] \qquad \text{(d)}$$

The complete response is:

$$u(t) = \begin{cases} \dfrac{p_o}{k} (1 - \cos \omega_n t) & 0 \le t \le t_d \\ \dfrac{p_o}{k} [\cos \omega_n (t - t_d) - \cos \omega_n t] & t \ge t_d \end{cases}$$

Problem 4.16

The half-cycle sine pulse $p(t)$ is the sum of two sinusoidal excitations: $p_1(t) = p_o \sin \omega t$ starting at $t = 0$ and $p_2(t) = p_o \sin \omega t$ starting at $t = t_d$, where $t_d = \pi/t_d$ (Fig. 4.6.2b).

The response to $p_1(t)$ is given by Eq. (3.1.6b):

$$u_1(t) = \frac{p_o}{k} \frac{1}{1 - (\omega/\omega_n)^2} \left[\sin \omega t - (\omega/\omega_n) \sin \omega_n t \right] \tag{a}$$

To obtain the response to $p_2(t)$, in Eq. (a) replace t by $t - t_d$:

$$u_2(t) = \frac{p_o}{k} \frac{1}{1 - (\omega/\omega_n)^2} \left[\sin \omega(t - t_d) - (\omega/\omega_n) \sin \omega_n (t - t_d) \right] \tag{b}$$

The response to the half cycle sine pulse is

$$u(t) = \begin{cases} u_1(t) & 0 \le t \le t_d \\ u_1(t) + u_2(t) & t \ge t_d \end{cases}$$

These two equations can be shown to be equivalent to Eqs. (4.8.2) and (4.8.3), respectively.

Problem 4.17

1. *Determine the natural vibration period.*

With the base of the columns clamped, the lateral stiffness will be four times of the value computed in Example 4.1:

$$k = 14.93 \text{ kips/in.}$$

The natural period will be halved, i.e.,

$$T_n = 0.25 \text{ sec}$$

2. *Determine R_d.*

$$\frac{t_d}{T_n} = \frac{0.2}{0.25} = 0.8 > \frac{1}{2}$$

Equation (4.7.12) gives

$$R_d = \frac{u_o}{(u_{st})_o} = 2$$

3. *Determine $(u_{st})_o$.*

$$(u_{st})_o = \frac{p_o}{k} = \frac{4}{14.92} = 0.268 \text{ in.}$$

4. *Determine the maximum dynamic deformation.*

$$u_o = (u_{st})_o R_d = (0.268)(2) = 0.536 \text{ in.}$$

5. *Determine the maximum bending stress.*

$$M = \frac{6EI}{L^2} u_o = \left[\frac{6\,(30{,}000)\,(61.9)}{(12 \times 12)^2}\right] 0.536$$
$$= 287.9 \text{ kip} - \text{in.}$$

The bending stress is largest at the outside of the flange at the top and bottom of columns:

$$\sigma = \frac{M}{S} = \frac{287.9}{15.2} = 18.9 \text{ ksi}$$

6. *Effect of base fixity.*

For this excitation, the deformation as well as bending stress is reduced by clamping the columns at their base.

Problem 4.18

1. *Determine* R_d.

$$\frac{t_d}{T_n} = \frac{0.25}{0.5} = \frac{1}{2}$$

For this t_d/t_n, Eq. (4.8.12) gives

$$R_d = \frac{\pi}{2}$$

2. *Determine the maximum dynamic deformation.*

$$u_o = (u_{st})_o R_d = \frac{p_o}{k} R_d = \frac{5}{3.73}\frac{\pi}{2} = 2.105 \text{ in.}$$

3. *Determine the maximumbending stress.*

u_o M

$$M = \frac{3EI}{L^2} u_o = \left[\frac{3\,(30{,}000)\,(61.9)}{(12 \times 12)^2}\right] 2.105$$
$$= 565.7 \text{ kip} - \text{in.}$$

$$\sigma = \frac{M}{S} = \frac{565.7}{15.2} = 37.2 \text{ ksi}$$

Problem 4.19

The equation of motion to be solved is

$$m\ddot{u} + ku = p(t) = \begin{cases} p_o \sin(2\pi t / t_d) & t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(a)}$$

with at rest initial conditions.

1. *Determine response u(t).*

Case 1: $t_d / T_n \neq 1$

Forced Vibration Phase.

The response is given by Eq. (3.1.6b). Substituting $\omega = 2\pi / t_d$,

$\omega_n = 2\pi / T_n$, and $(u_{st})_o = p_o / k$, Eq. (3.1.6b) becomes

$$\frac{u(t)}{(u_{st})_o} = \frac{1}{1-(T_n/t_d)^2}\left[\sin\left(2\pi\frac{t}{t_d}\right) - \frac{T_n}{t_d}\sin\left(2\pi\frac{t}{T_n}\right)\right] \quad t \le t_d \qquad \text{(b)}$$

Free Vibration Phase.

The motion is described by Eq. (4.7.3) with $u(t_d)$ and $\dot{u}(t_d)$, determined from Eq. (b) :

$$u(t_d) = (u_{st})_o \frac{-1}{1-(T_n/t_d)^2}\frac{T_n}{t_d}\sin\left(2\pi\frac{t_d}{T_n}\right) \qquad \text{(c.1)}$$

$$\dot{u}(t_d) = (u_{st})_o \frac{1}{1-(T_n/t_d)^2}\frac{2\pi}{t_d}\left[1-\cos\left(2\pi\frac{t_d}{T_n}\right)\right] \qquad \text{(c.2)}$$

Substituting Eq. (c) in Eq. (4.7.3) gives

$$\frac{u(t)}{(u_{st})_o} = \frac{1}{1-(T_n/t_d)^2}\frac{T_n}{t_d}\left[-\sin\left(\frac{2\pi}{T_n}t_d\right)\cos\frac{2\pi}{T_n}(t-t_d) + \sin\frac{2\pi}{T_n}(t-t_d) - \cos\left(\frac{2\pi}{T_n}t_d\right)\sin\frac{2\pi}{T_n}(t-t_d)\right]$$

$$= \frac{(T_n/t_d)}{1-(T_n/t_d)^2}\left[\sin\frac{2\pi}{T_n}(t-t_d) - \sin\left(\frac{2\pi}{T_n}t\right)\right]$$

$$= \frac{2(T_n/t_d)\sin(\pi t_d/T_n)}{(T_n/t_d)^2-1}\cos\left[2\pi\left(\frac{t}{T_n}-\frac{1}{2}\frac{t_d}{T_n}\right)\right]$$

$$t \ge t_d \qquad \text{(d)}$$

Case 2: $t_d / T_n = 1$

Forced Vibration Phase.

The forced response is now given by Eq. (3.1.13b)

$$\frac{u(t)}{(u_{st})_o} = \frac{1}{2}\left[\sin\frac{2\pi t}{T_n} - \frac{2\pi t}{T_n}\cos\frac{2\pi t}{T_n}\right] \quad t \le t_d \qquad \text{(e)}$$

Free Vibration Phase

From Eq. (e) determine

$$u(t_d) = -\pi(u_{st})_o \quad \text{and} \quad \dot{u}(t_d) = 0 \qquad \text{(f)}$$

The second equation implies that the displacement in the forced vibration phase reaches its maximum at the end of this phase. Substituting Eq. (f) in Eq. (4.7.3) gives

$$\frac{u(t)}{(u_{st})_o} = -\pi\cos 2\pi\left(\frac{t}{T_n}-1\right) = -\pi\cos 2\pi\frac{t}{T_n} \quad t \ge t_d \qquad \text{(g)}$$

2. *Plot response history.*

The time variation of the normalized deformation, $u(t)/(u_{st})_o$, given by Eqs. (b) and (d), is plotted in Fig. P4.19a for several values of t_d / T_n. For the special case of $t_d / T_n = 1$, Eqs. (e) and (g) describe the response of the system and these are also plotted in Fig. P4.19a. The static solution is included in these figures.

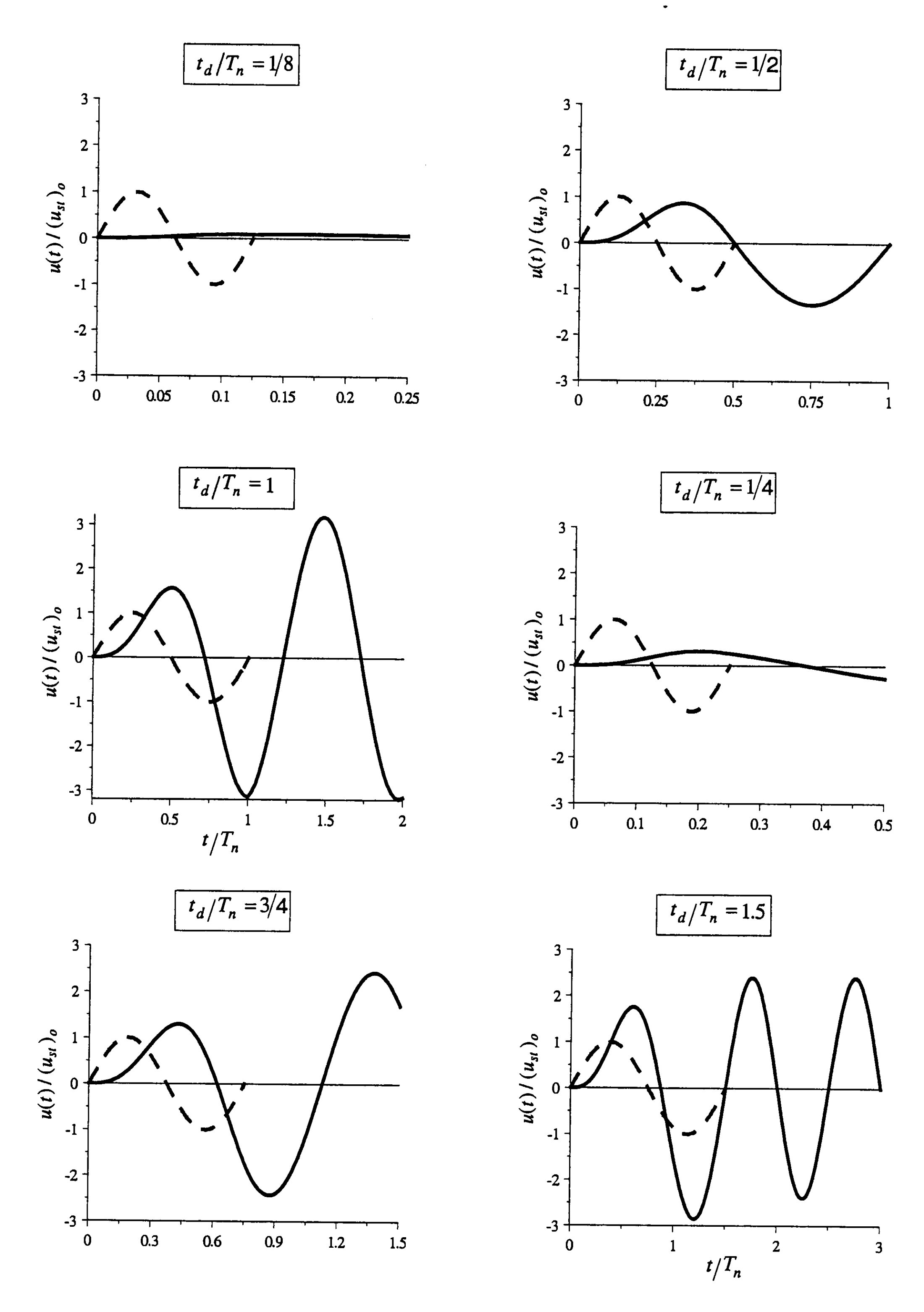


Fig. P4.19a

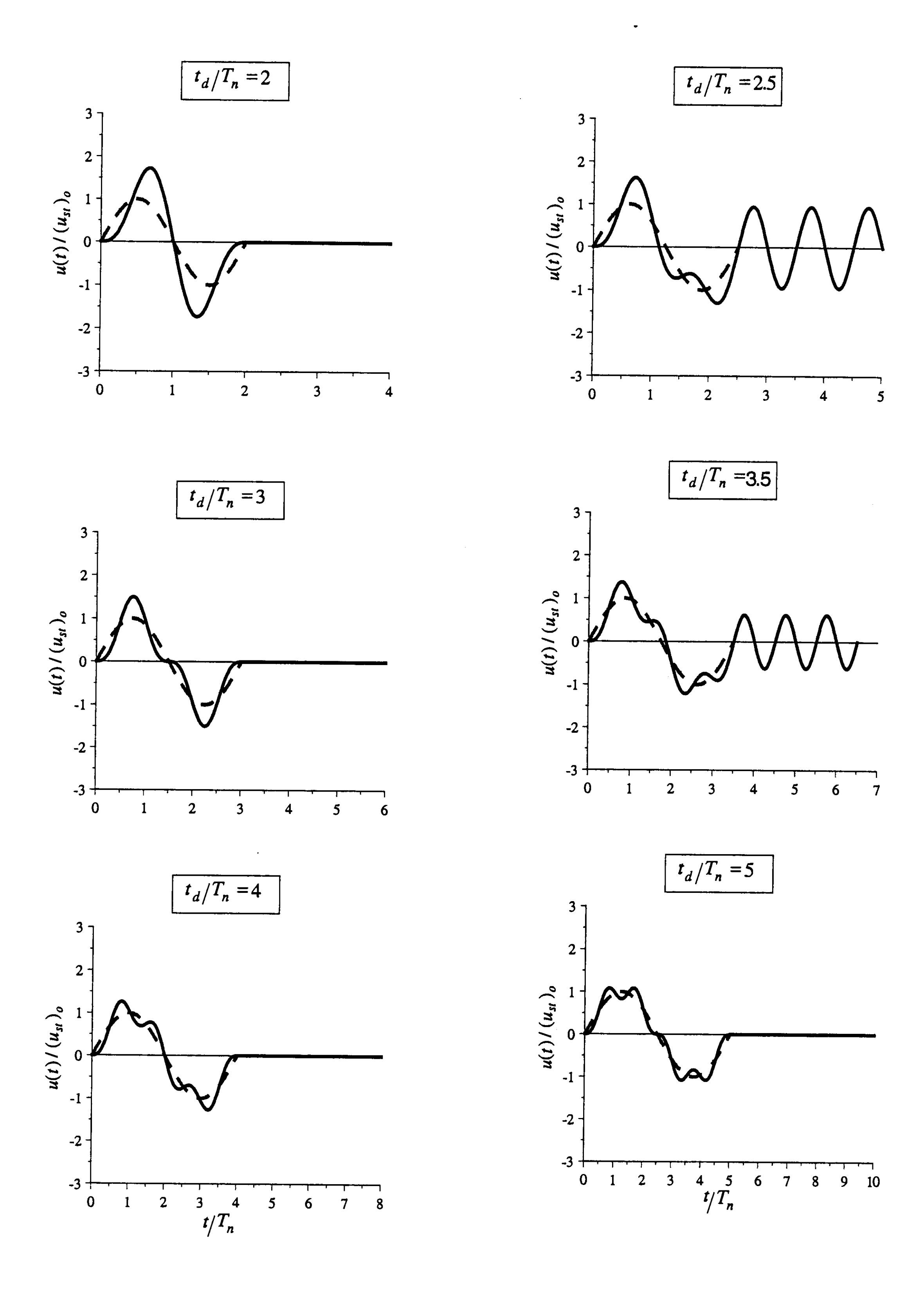


Fig. P4.19a (continued)

3. *Determine maximum response.*

During the forced vibration phase, the number of local maxima and minima depends on t_d / T_n; the longer the pulse duration, more such peaks occur. These peaks occur at time instant t_0 when $\dot{u}(t) = 0$. This condition applied to Eq. (b) gives

$$\cos\frac{2\pi t_0}{t_d} = \cos\frac{2\pi t_0}{T_n}$$

$$(t_0)_l = \frac{l}{1+(t_d/T_n)} t_d \qquad l=1, 2, 3...... \qquad \text{(h)}$$

Only those l for which $(t_0)_l < t_d$ are relevant. Substituting Eq. (h) into Eq. (b) gives

$$R_d = \frac{u_o}{(u_{st})_o} = \left| \frac{1}{1-(T_n/t_d)^2} * \left[\sin\frac{2\pi l}{1+(t_d/T_n)} - \frac{T_n}{t_d}\sin\frac{2\pi l}{1+(T_n/t_d)} \right] \right| \qquad \text{(i)}$$

At least one local maximum occurs during the force pulse, irrespective of the t_d / T_n value. If $t_d / T_n > 1/2$ the displacement reverses in sign during the excitation and has a negative value at the end of the excitation. If $t_d / T_n > 1$ a local minimum develops during the force pulse. If $t_d / T_n > 2$ more than one local maximum and/or more than one local minimum may develop. We define

$$u_{max} = \max_t\ u(t) \qquad u_{min} = \min_t u(t)$$

Figure P4.19b shows $u_{max}/(u_{st})_o$ and $-u_{min}/(u_{st})_o$ plotted as a function of t_d / T_n. The response spectrum for the larger of the two values during the force pulse is shown as 'Forced Response' in Fig. P4.19c.

During the free vibration phase, the response is given by Eq. (d) and its amplitude is

$$R_d = \frac{u_o}{(u_{st})_o} = \left| \frac{2(T_n/t_d)\sin(\pi t_d/T_n)}{(T_n/t_d)^2 - 1} \right| \qquad \text{(j)}$$

This equation is plotted in Fig. P4.19c.

For the special case of $t_d / T_n = 1$, the maximum response during the forced vibration can be determined from Eq. (e):

$$\dot{u}(t) = 0 \Rightarrow t_0 = T_n \text{ and}$$

$$\frac{u(t_0)}{(u_{st})_o} = -\pi \Rightarrow R_d = \pi \qquad \text{(k)}$$

Similarly, the maximum response during free vibration can be determined from Eq. (g):

$$R_d = \pi \qquad \text{(l)}$$

The overall maximum response is the larger of the two maxima determined separately for the forced and free vibration phases. Fig. P4.19c shows that if $t_d > T_n$, the overall maximum is the largest peak that develops during the force pulse. On the other hand, if $t_d < T_n$, the overall maximum is given by the peak response during the free vibration phase. For the special case of $t_d = T_n$, as mentioned earlier, the two individual maxima are equal. The overall maximum response is plotted against t_d / T_n in Fig. P4.19d; for each t_d / T_n it is the larger of the two plots in Fig. P4.19c. This is the shock spectrum for the full-cycle sine pulse force.

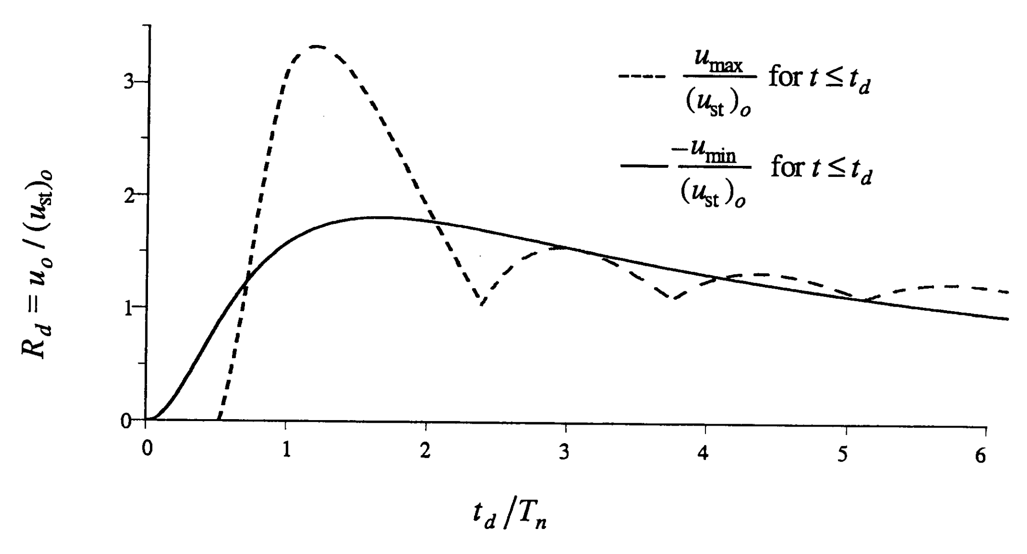


Fig. 4.19b

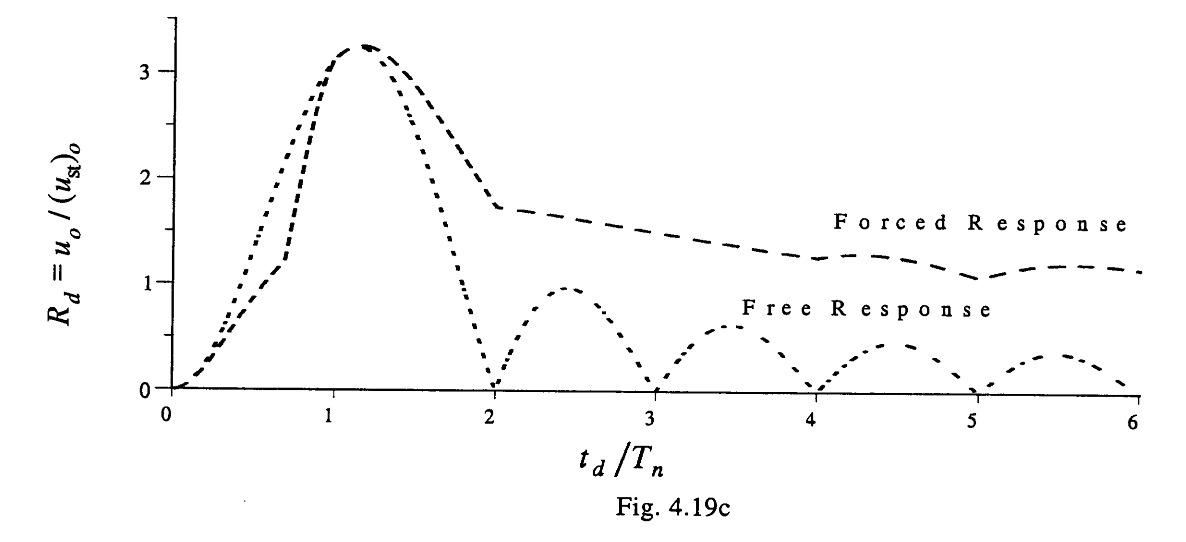


Fig. 4.19c

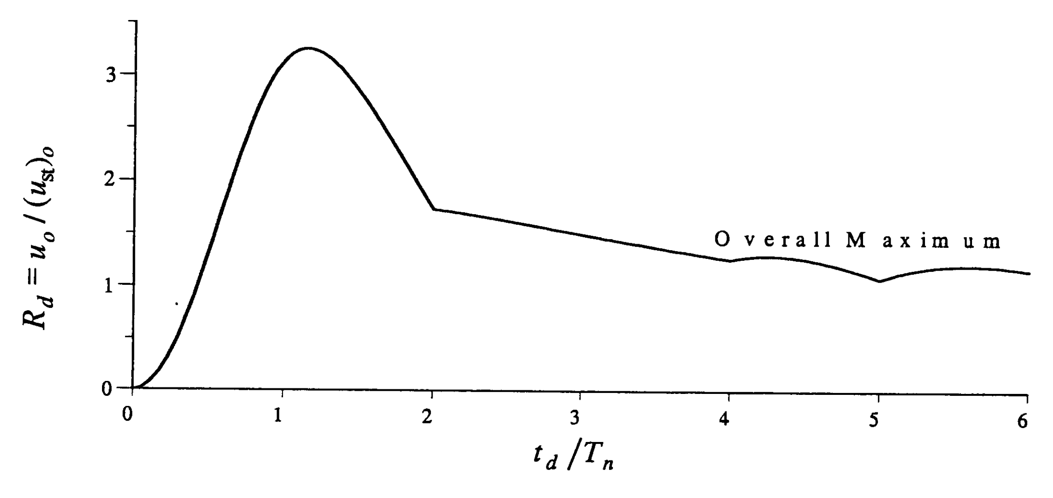
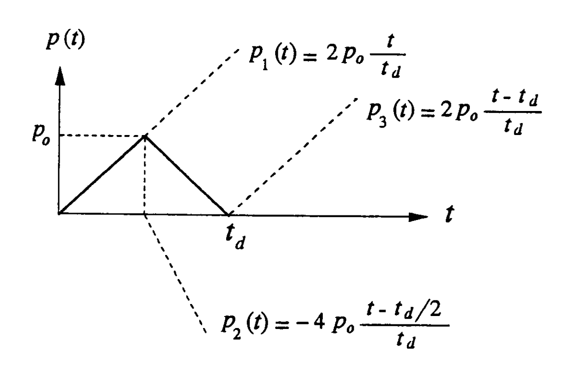


Fig. 4.19d

Problem 4.20

A symmetrical triangular pulse can be expressed as the superposition of three linear functions as shown.



The response to $p_1(t)$ is given by Eq. (4.4.2) with t_r replaced by $t_d/2$:

$$u_1(t) = \frac{p_o}{k}\left(\frac{2t}{t_d} - \frac{2\sin\omega_n t}{\omega_n t_d}\right) \qquad t \ge 0 \quad \text{(a)}$$

The response to $p_2(t)$ is given by Eq. (a) with t replaced by $t - (t_d/2)$ and p_o replaced by $-2p_o$:

$$u_2(t) = -\frac{2p_o}{k}\left[\frac{2(t - t_d/2)}{t_d} - \frac{2\sin\omega_n(t - t_d/2)}{\omega_n t_d}\right] \qquad t \ge \frac{t_d}{2} \quad \text{(b)}$$

The response to $p_3(t)$ is obtained by replacing t by $t - t_d$ in Eq. (a):

$$u_3(t) = \frac{p_o}{k}\left[\frac{2(t - t_d)}{t_d} - \frac{2\sin\omega_n(t - t_d)}{\omega_n t_d}\right] \quad t \ge t_d \qquad \text{(c)}$$

For $0 \le t \le t_d/2$, the total response is given by Eq. (a).

For $t_d/2 \le t \le t_d$, the total response is the sum of Eqs. (a) and (b):

$$\begin{aligned} u(t) &= u_1(t) + u_2(t) \\ &= \frac{p_o}{k}\left[\frac{2t}{t_d} - \frac{2\sin\omega_n t}{\omega_n t_d}\right] \\ &\quad - \frac{2p_o}{k}\left[\frac{2(t - t_d/2)}{t_d} - \frac{2\sin\omega_n(t - t_d/2)}{\omega_n t_d}\right] \\ &= \frac{2p_o}{k}\left\{1 - \frac{t}{t_d} + \frac{1}{\omega_n t_d}[2\sin\omega_n(t - t_d/2) - \sin\omega_n t]\right\} \end{aligned} \qquad \text{(d)}$$

For $t \ge t_d$, the total response is the sum of either Eqs. (a), (b) and (c) or Eqs. (c) and (d):

$$\begin{aligned} u(t) &= \frac{2p_o}{k}\left[1 - \frac{t}{t_d} + \frac{1}{\omega_n t_d}(2\sin\omega_n(t - t_d/2) - \sin\omega_n t)\right] \\ &\quad + \frac{p_o}{k}\left[\frac{2(t - t_d)}{t_d} - \frac{2\sin\omega_n(t - t_d)}{\omega_n t_d}\right] \\ &= \frac{2p_o}{k}\left\{\frac{1}{\omega_n t_d}[2\sin\omega_n(t - t_d/2) - \sin\omega_n(t - t_d) - \sin\omega_n t]\right\} \end{aligned} \qquad \text{(e)}$$

The total response can be summarized as

$$u(t) = \frac{2p_o}{k}\begin{cases} \frac{t}{t_d} - \frac{1}{\omega_n t_d}\sin\omega_n t & 0 \le t \le t_d/2 \\ 1 - \frac{t}{t_d} + \frac{1}{\omega_n t_d}\left[2\sin\omega_n\left(t - \frac{t_d}{2}\right) - \sin\omega_n t\right] & t_d/2 \le t \le t_d \\ \frac{1}{\omega_n t_d}\left[2\sin\omega_n\left(t - \frac{t_d}{2}\right) - \sin\omega_n(t - t_d) - \sin\omega_n t\right] & t \ge t_d \end{cases} \qquad \text{(f)}$$

which is the same as Eq. (4.9.1) because $p_o/k = (u_{st})_o$ and $\omega_n = 2\pi/T_n$.

Problem 4.21

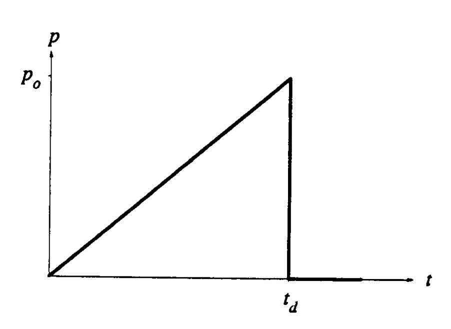


Fig. P.4.21a

Equation of motion:

$$m\ddot{u} + ku = \begin{cases} p_o(t/t_d) & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(a)}$$

1. *Forced Vibration Phase.*

The response is given by Eq. (4.4.2) with t_r replaced by t_d:

$$\frac{u(t)}{(u_{st})_o} = \frac{t}{t_d} - \frac{1}{t_d}\frac{\sin\omega_n t}{\omega_n} \qquad t \le t_d$$

Rewritting in terms of t_d / T_n gives

$$\frac{u(t)}{(u_{st})_o} = \frac{t}{t_d} - \frac{1}{2\pi}\left(\frac{T_n}{t_d}\right)\sin\frac{2\pi t}{T_n} \qquad 0 \le t \le t_d \qquad \text{(b)}$$

2. *Free Vibration Phase.*

The free vibration resulting from $u(t_d)$ and $\dot{u}(t_d)$ is

$$u(t) = u(t_d)\cos[\omega_n(t - t_d)] + \frac{\dot{u}(t_d)}{\omega_n}\sin[\omega_n(t - t_d)] \qquad \text{(c)}$$

From Eq. (b), $u(t_d)$ and $\dot{u}(t_d)$ are determined:

$$u(t_d) = (u_{st})_o\left(1 - \frac{\sin\omega_n t_d}{\omega_n t_d}\right)$$
$$\dot{u}(t_d) = (u_{st})_o\frac{1}{t_d}(1 - \cos\omega_n t_d) \qquad \text{(d)}$$

Substituting Eq. (d) in Eq. (c) gives

$$\frac{u(t)}{(u_{st})_o} = \cos\omega_n(t - t_d) - \frac{\sin\omega_n t}{\omega_n t_d} + \frac{\sin\omega_n(t - t_d)}{\omega_n t_d} \qquad \text{(e)}$$
$$t \ge t_d$$

Rewriting in terms of t_d/T_n gives

$$\frac{u(t)}{(u_{st})_o} = \cos\frac{2\pi}{T_n}(t - t_d) + \frac{1}{2\pi}\left(\frac{T_n}{t_d}\right)\sin\frac{2\pi}{T_n}(t - t_d) - \frac{1}{2\pi}\left(\frac{T_n}{t_d}\right)\sin\frac{2\pi t}{T_n} \qquad t \ge t_d \qquad \text{(f)}$$

3. *Response Plots.*

The normalized deformation $u(t)/(u_{st})_o$ given by Eqs. (b) and (f) is plotted as a function of t/T_n for t_d/T_n = 1/2 and 2 (Fig. P4.21 b-c).

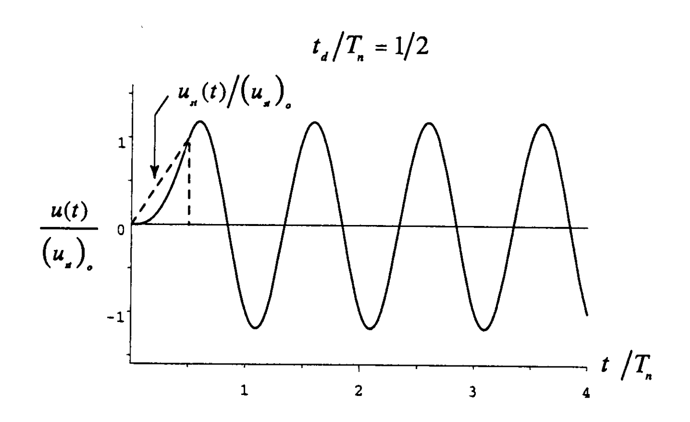


Fig. P4.21b

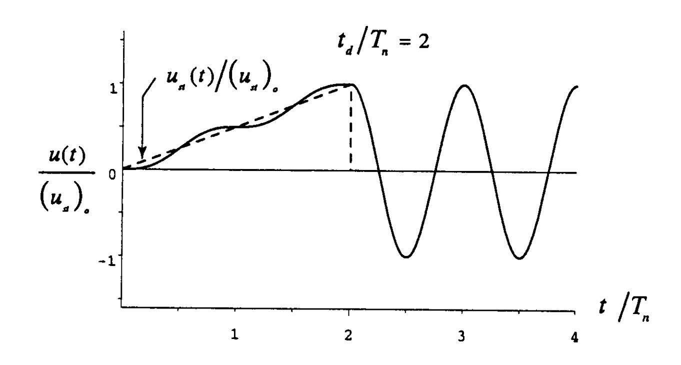


Fig. P4.21c

4. *Response Spectrum.*

During the forced vibration phase u is a non-decreasing function of t. Thus, the maximum value of u during this phase can be found by evaluating Eq. (b) at $t = t_d$:

$$R_d = \frac{u(t_d)}{(u_{st})_o} = 1 - \frac{T_n}{2\pi t_d}\sin\frac{2\pi t_d}{T_n} \qquad \text{(g)}$$

This equation is plotted in Fig. P4.21d.

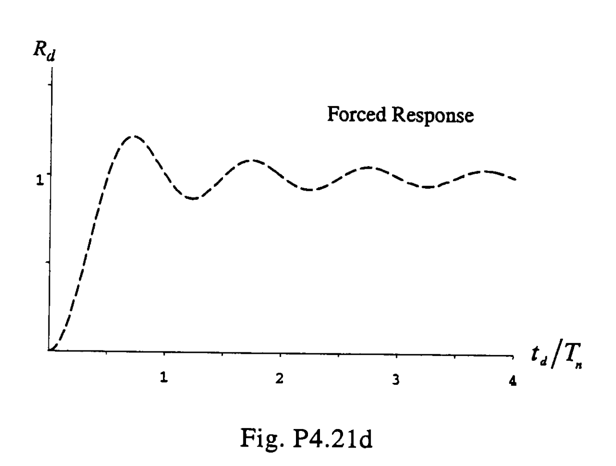
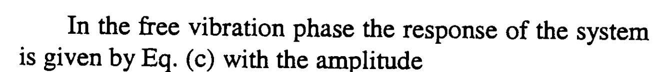


Fig. P4.21d



In the free vibration phase the response of the system is given by Eq. (c) with the amplitude

$$u_o = \sqrt{[u(t_d)]^2 + \left[\frac{\dot{u}(t_d)}{\omega_n}\right]^2} \qquad \text{(h)}$$

Substituting Eq. (d) and manipulating gives

$$R_d = \sqrt{\left[1 - \frac{T_n}{2\pi t_d}\sin\left(\frac{2\pi t_d}{T_n}\right)\right]^2 + \frac{T_n^2}{\pi^2 t_d^2}\left[\sin\left(\frac{\pi t_d}{T_n}\right)\right]^4} \qquad \text{(i)}$$

This equation is plotted in Fig. P4.21e.

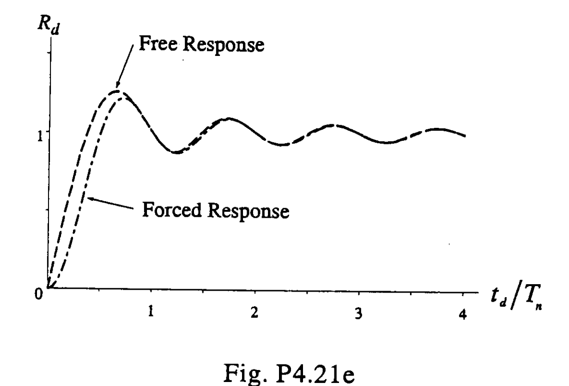


Fig. P4.21e

The overall maximum response is the larger of the maxima shown in Figs. P4.21d and e. This maximum is always given by Eq. (i) and is plotted in Fig. P4.21f to obtain the shock spectrum.

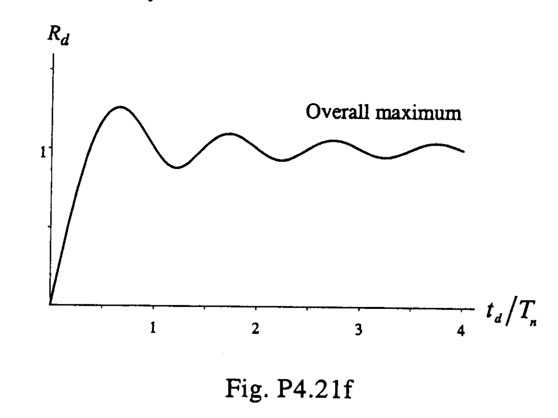


Fig. P4.21f

Problem 4.22

The given excitation is expressed as the superposition of four linear functions shown.

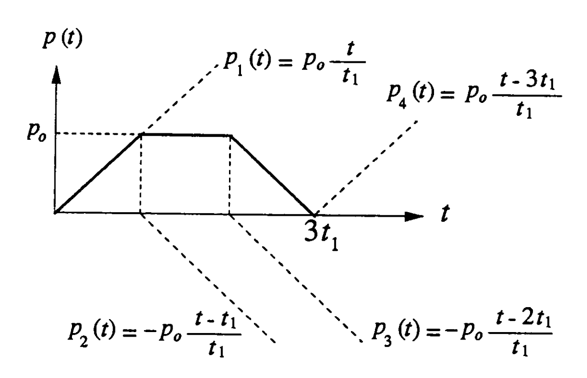


Fig. P4.22

The response to $p_1(t)$ is given by Eq. (4.2.2) with t_r replaced by t_1:

$$u_1(t) = \frac{p_o}{k}\left(\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right) \qquad t \ge 0 \qquad \text{(a)}$$

This equation can be adapted to write the responses $u_2(t)$, $u_3(t)$ and $u_4(t)$ to $p_2(t)$, $p_3(t)$ and $p_4(t)$, respectively:

$$u_2(t) = -\frac{p_o}{k}\left(\frac{t - t_1}{t_1} - \frac{\sin\omega_n(t - t_1)}{\omega_n t_1}\right) \qquad t \ge t_1 \qquad \text{(b)}$$

$$u_3(t) = -\frac{p_o}{k}\left(\frac{t - 2t_1}{t_1} - \frac{\sin\omega_n(t - 2t_1)}{\omega_n t_1}\right) \qquad t \ge 2t_1 \qquad \text{(c)}$$

$$u_4(t) = \frac{p_o}{k}\left(\frac{t - 3t_1}{t_1} - \frac{\sin\,\omega_n(t - 3t_1)}{\omega_n t_1}\right) \qquad t \ge 3t_1 \qquad \text{(d)}$$

For $t \le t_1$, the response is given by Eq. (a):

$$u(t) = \frac{p_o}{k}\left(\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right) \qquad t \le t_1 \qquad \text{(e)}$$

For $t_1 \le t \le 2t_1$, the response is the sum of Eqs. (a) and (b):

$$u(t) = \frac{p_o}{k}\left(\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right) - \frac{p_o}{k}\left(\frac{t - t_1}{t_1} - \frac{\sin\omega_n(t - t_1)}{\omega_n t_1}\right)$$

or

$$u(t) = \frac{p_o}{k}\left\{1 - \frac{1}{\omega_n t_1}\left[\sin\omega_n t - \sin\omega_n(t - t_1)\right]\right\} \qquad t_1 \le t \le 2t_1 \qquad \text{(f)}$$

For $2t_1 \le t \le 3t_1$, the response is the sum of Eqs. (a), (b) and (c):

$$u(t) = \frac{p_o}{k}\left(\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right) - \frac{p_o}{k}\left(\frac{t - t_1}{t_1} - \frac{\sin\omega_n(t - t_1)}{\omega_n t_1}\right) - \frac{p_o}{k}\left(\frac{t - 2t_1}{t_1} - \frac{\sin\omega_n(t - 2t_1)}{\omega_n t_1}\right)$$

or

$$u(t) = \frac{p_o}{k}\left\{3 - \frac{t}{t_1} - \frac{1}{\omega_n t_1}\left[\sin\omega_n t - \sin\omega_n(t - t_1) - \sin\omega_n(t - 2t_1)\right]\right\} \qquad 2t_1 \le t \le 3t_1 \qquad \text{(g)}$$

For $t \ge 3t_1$, the response is the sum of Eqs. (a), (b), (c), and (d):

$$u(t) = \frac{p_o}{k}\left(\frac{t}{t_1} - \frac{\sin\omega_n t}{\omega_n t_1}\right) - \frac{p_o}{k}\left(\frac{t - t_1}{t_1} - \frac{\sin\omega_n(t - t_1)}{\omega_n t_1}\right) - \frac{p_o}{k}\left(\frac{t - 2t_1}{t_1} - \frac{\sin\omega_n(t - 2t_1)}{\omega_n t_1}\right) + \frac{p_o}{k}\left(\frac{t - 3t_1}{t_1} - \frac{\sin\omega_n(t - 3t_1)}{\omega_n t_1}\right)$$

or

$$u(t) = \frac{p_o}{k}\left(-\frac{1}{\omega_n t_1}\right)\left[\sin\omega_n t - \sin\omega_n(t - t_1) - \sin\,\omega_n(t - 2t_1) + \sin\omega_n(t - 3t_1)\right] \qquad t \ge 3t_1 \qquad \text{(h)}$$

The desired solution is given by Eqs. (e), (f), (g) and (h).

Problem 4.23

The response for $t \geq 3t_1$ is given by Eq. (h) of Problem 4.22. After some trigonometric and algebraic manipulation this can be rewritten as

$$u(t) = \frac{p_o}{k}\frac{4}{\omega_n t_1}\sin\omega_n t_1 \sin\frac{\omega_n t_1}{2}\sin\omega_n\left(t - \frac{3t_1}{2}\right) \tag{a}$$

The response attains a maximum when

$$\frac{du}{dt} = 0 \Rightarrow \cos\omega_n\left(t - \frac{3t_1}{2}\right) = 0$$

The first peak occurs at $t = t_o$ given by

$$\omega_n\left(t_o - \frac{3t_1}{2}\right) = \frac{\pi}{2} \Rightarrow t_o = \frac{T_n}{4} + \frac{3t_1}{2} \tag{b}$$

where $T_n = 2\pi/\omega_n$.

The maximum response is obtained by evaluating Eq. (a) at t_o given by Eq. (b):

$$u_o = \frac{p_o}{k}\frac{4}{\omega_n t_1}\sin\omega_n t_1 \sin\frac{\omega_n t_1}{2} \tag{c}$$

Problem 4.24

If $t_d \ll T_n$, the maximum deformation can be estimated by assuming that the force of Fig. P4.22 is a pure impulse. Its magnitude is

$$\mathscr{I} = \int_0^{t_d} p(t)\, dt = 2p_o \frac{t_d}{3} \qquad \text{(a)}$$

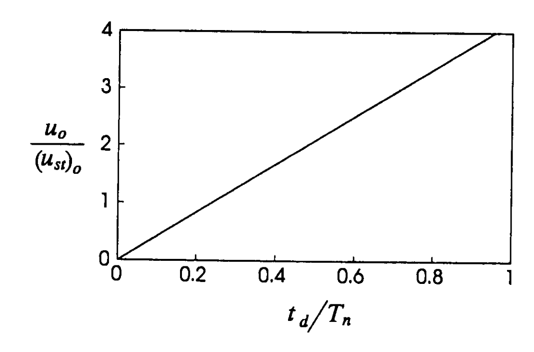
From Eq. (4.10.3) the maximum deformation is

$$u_o = \frac{\mathscr{I}}{m\omega_n} = \frac{2p_o t_d}{3m\omega_n} \qquad \text{(b)}$$

Defining $(u_{st})_o = p_o/k$, Eq. (b) becomes

$$\frac{u_o}{(u_{st})_o} = \frac{4\pi}{3}\left(\frac{t_d}{T_n}\right) \qquad \text{(c)}$$

A plot of Eq. (c) gives the shock spectrum:



For $t_d/T_n = 1/4$, Eq. (c) gives the approximate result:

$$\frac{u_o}{(u_{st})_o} = \frac{\pi}{3} = 1.047 \qquad \text{(d)}$$

The exact value of the maximum response is given by Eq. (c) of Problem 4.23:

$$\frac{u_o}{(u_{st})_o} = \frac{4}{\omega_n t_1} \sin \omega_n t_1 \ \sin \frac{\omega_n t_1}{2}$$

where

$$\omega_n t_1 = \frac{\omega_n t_d}{3} = \frac{2\pi}{3}\frac{t_d}{T_n} = \frac{2\pi}{3}\frac{1}{4} = \frac{\pi}{6}$$

Therefore,

$$\frac{u_o}{(u_{st})_o} = \frac{4}{\pi/6} \sin\frac{\pi}{6} \ \sin\frac{\pi}{12} = 0.9885 \qquad \text{(e)}$$

$$\text{Error} = \frac{1.047 - 0.9885}{0.9885} \times 100\% = 5.9\%$$

Problem 4.25

(a) During the first half of the excitation, $p(t) = p_o$ and the response is given by Eq. (4.3.2) where $(u_{st})_o = p_o/k$:

$$\frac{u(t)}{(u_{st})_o} = 1 - \cos \omega_n t \qquad 0 \le t \le t_d/2 \qquad \text{(a)}$$

For the second half of the excitation, $p(t) = -p_o$ and the general solution is

$$u(t) = A \cos \omega_n \bar{t} + B \sin \omega_n \bar{t} - (u_{st})_o \qquad t_d/2 \le t \le t_d \qquad \text{(b)}$$

where $\bar{t} = t - t_d/2$; A and B are to be determined from $u(t_d/2)$ and $\dot{u}(t_d/2)$. Equation (a) gives

$$u(t_d/2) = (u_{st})_o \left[1 - \cos(\omega_n t_d/2)\right] \qquad \text{(c)}$$

$$\dot{u}(t_d/2) = (u_{st})_o \omega_n \sin(\omega_n t_d/2) \qquad \text{(d)}$$

Differentiating Eq. (b) gives

$$\dot{u}(t) = -A\omega_n \sin \omega_n \bar{t} + B\omega_n \cos \omega_n \bar{t} \qquad \text{(e)}$$

From Eqs. (b) and (c),

$$u(\bar{t} = 0) = A - (u_{st})_o = (u_{st})_o [1 - \cos(\omega_n t_d/2)]$$

$$\Rightarrow A = (u_{st})_o [2 - \cos \omega_n t_d/2] \qquad \text{(f)}$$

From Eqs. (d) and (e),

$$\dot{u}(\bar{t} = 0) = \omega_n B = (u_{st})_o \omega_n \sin(\omega_n t_d/2)$$

$$\Rightarrow B = (u_{st})_o \omega_n \sin(\omega_n t_d/2) \qquad \text{(g)}$$

Substituting Eqs. (f) and (g) in Eq. (b) gives

$$u(t) = (u_{st})_o \{[2 - \cos(\omega_n t_d/2)] \cos \omega_n (t - t_d/2) + \sin(\omega_n t_d/2) \sin \omega_n (t - t_d/2) - 1\} \qquad t_d/2 \le t \le t_d \qquad \text{(h)}$$

After the force ends at t_d, the system vibrates freely and the response is

$$u(t) = u(t_d) \cos \omega_n (t - t_d) + \frac{\dot{u}(t_d)}{\omega_n} \sin \omega_n (t - t_d) \qquad t \ge t_d \qquad \text{(i)}$$

This free vibration is initiated by $u(t_d)$ and $\dot{u}(t_d)$ determined from Eq. (h):

$$u(t_d) = (u_{st})_o [2 \cos(\omega_n t_d/2) - \cos \omega_n t_d - 1] \qquad \text{(j)}$$

$$\dot{u}(t_d) = (u_{st})_o \omega_n [\sin \omega_n t_d - 2 \sin(\omega_n t_d/2)] \qquad \text{(k)}$$

Substituting Eqs. (j) and (k) in Eq. (i) gives

$$u(t) = (u_{st})_o \{ [2 \cos(\omega_n t_d/2) - \cos \omega_n t_d - 1] \cos \omega_n (t - t_d) + [\sin \omega_n t_d - 2 \sin(\omega_n t_d/2)] \sin \omega_n (t - t_d) \} \qquad t \ge t_d \qquad \text{(l)}$$

Equations (a), (h) and (l) give the desired results.

(b) The maximum displacement during the free vibration phase is

$$u_o = \sqrt{u(t_d)^2 + \left[\frac{\dot{u}(t_d)}{\omega_n}\right]^2}$$

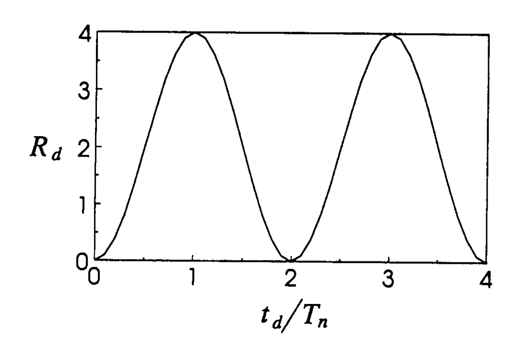
which after substituting Eqs. (j) and (k) and much trigonometric and algebraic manipulation becomes

$$u_o = 4 (u_{st})_o \sin^2\left(\frac{\omega_n t_d}{4}\right) = 4 (u_{st})_o \sin^2\left(\frac{\pi t_d}{2T_n}\right)$$

The corresponding deformation response factor is

$$R_d \equiv \frac{u_o}{(u_{st})_o} = 4 \sin^2\left(\frac{\pi t_d}{2T_n}\right) \qquad \text{(m)}$$

A plot of Eq. (m) gives the shock spectrum:



(c) If the excitation is considered as a pure impulse, its magnitude $\mathcal{I} = 0$, implying zero response, a meaningless result. The pure impulse approximation is valid only for excitations which are a single pulse.

Problem 4.26

(a) From Example 2.7, weight $w = 20.03$ kips (empty tank), $T_n = 0.5$ sec, and $\zeta = 2.75\%$. Therefore,

$$t_d/T_n = 0.08/0.5 = 0.16$$

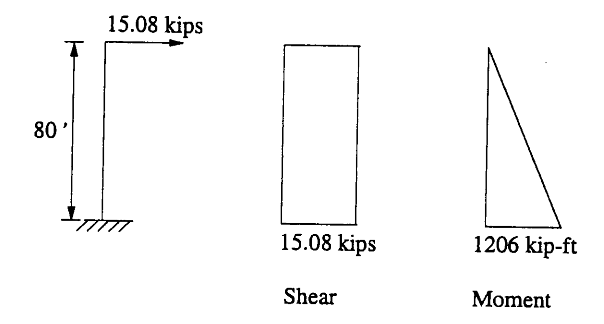
Because $t_d/T_n < 0.25$, the force may be treated as a pure impulse of magnitude $\mathscr{I} = 1.2$ kip−sec (see Example 4.2). Neglecting the effect of damping,

$$u_o = \frac{\mathscr{I}}{k}\frac{2\pi}{T_n} = \frac{(1.2)\,2\pi}{8.2\,(0.5)} = 1.84$$

The equivalent static force is

$$f_{So} = ku_o = (8.2)\,1.83 = 15.08 \text{ kips}$$

The resulting shear and bending moment diagrams for the tower are as shown:



(b) The maximum responses of the tank for empty and full conditions are summarized next:

	Empty	Full
u_o, in.	1.84	0.821
V_b, kips	15.08	6.73
M_b, kip − ft	1206	538

Increasing the mass has the effect of reducing the dynamic response.

This can be explained by observing that a pure impulse of magnitude $\mathscr{I}$ imparts an initial velocity $\dot{u}(0) = \mathscr{I}/m$ which is smaller for the larger mass.

Problem 5.1

Over the time interval $t_i \leq t \leq t_{i+1}$ the excitation function is

$$p(\tau) = \tilde{p}_i \qquad 0 \leq \tau \leq \Delta t_i \qquad \text{(a)}$$

where $\Delta t_i \equiv t_{i+1} - t_i$ and the equation to be solved is

$$m\ddot{u} + ku = \tilde{p}_i \qquad \text{(b)}$$

subject to the initial conditions $u(\tau = 0) = u_i$ and $\dot{u}(\tau = 0) = \dot{u}_i$.

The response $u(\tau)$ over $0 \leq \tau \leq \Delta t_i$ is the sum of two parts: (1) free vibration due to initial displacement u_i and velocity $\dot{u}_i$ at $\tau = 0$; and (2) response to step force $\tilde{p}_i$ with zero initial conditions:

$$u(\tau) = u_i \cos \omega_n \tau + \frac{\dot{u}_i}{\omega_n} \sin \omega_n \tau + \frac{\tilde{p}_i}{k}(1 - \cos \omega_n \tau) \qquad \text{(c.1)}$$

$$\frac{\dot{u}(\tau)}{\omega_n} = -u_i \sin \omega_n \tau + \frac{\dot{u}_i}{\omega_n} \cos \omega_n \tau + \frac{\tilde{p}_i}{k} \sin \omega_n \tau \qquad \text{(c.2)}$$

Evaluate Eqs. (c) at $\tau = \Delta t_i$ or $t = t_i + \Delta t_i$:

$$u_{i+1} = u_i \cos(\omega_n \Delta t_i) + \dot{u}_i \left[\frac{\sin(\omega_n \Delta t_i)}{\omega_n}\right] + \frac{\tilde{p}_i}{k}\left[1 - \cos(\omega_n \Delta t_i)\right] \qquad \text{(d.1)}$$

$$\dot{u}_{i+1} = u_i\left[-\omega_n \sin(\omega_n \Delta t_i)\right] + \dot{u}_i \cos(\omega_n \Delta t_i) + \frac{\tilde{p}_i}{k}\omega_n \sin(\omega_n \Delta t_i) \qquad \text{(d.2)}$$

Substituting $\tilde{p}_i = (p_i + p_{i+1})/2$ gives

$$u_{i+1} = Au_i + B\dot{u}_i + Cp_i + Dp_{i+1} \qquad \text{(e.1)}$$

$$\dot{u}_{i+1} = A'u_i + B'\dot{u}_i + C'p_i + D'p_{i+1} \qquad \text{(e.2)}$$

where

$$A = \cos(\omega_n \Delta t_i) \qquad A' = -\omega_n \sin(\omega_n \Delta t_i) \qquad \text{(f.1)}$$

$$B = \frac{\sin(\omega_n \Delta t_i)}{\omega_n} \qquad B' = \cos(\omega_n \Delta t_i) \qquad \text{(f.2)}$$

$$C = \frac{1 - \cos(\omega_n \Delta t_i)}{2k} \qquad C' = \frac{\omega_n \sin(\omega_n \Delta t_i)}{2k} \qquad \text{(f.3)}$$

$$D = \frac{1 - \cos(\omega_n \Delta t_i)}{2k} \qquad D' = \frac{\omega_n \sin(\omega_n \Delta t_i)}{2k} \qquad \text{(f.4)}$$

Problem 5.2

1. *Initial calculations.*

Substituting $\omega_n = 6.283$, $k = 10$ and $\Delta t_i = 0.1$ in the equations for A, B, C, $\cdots$, D' in Problem 5.1 gives

$A = 0.8090$ $B = 0.09355$ $C = 0.009550$ $D = 0.009550$

$A' = -3.6932$ $B' = 0.8090$ $C' = 0.1847$ $D' = 0.1847$

2. *Apply the recurrence Eqs. (e) of Problem 5.1.*

The resulting computations are summarized in Table P5.2a and P5.2b, wherein the theoretical result is calculated from Eqs. (3.1.6b) and (4.7.3).

Table P5.2a: Numerical solution using piecewise constant interpolation of excitation

t_i	p_i	Cp_i	Dp_{i+1}	$B\dot{u}_i$	$\dot{u}_i$	Au_i	u_i	Theoretical u_i
0.0	0.0000	0.0000	0.0477	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	5.0000	0.0477	0.0827	0.0864	0.9233	0.0386	0.0477	0.0333
0.2	8.6602	0.0827	0.0955	0.2894	3.0931	0.2067	0.2554	0.2405
0.3	10.0000	0.0955	0.0827	0.4682	5.0047	0.5454	0.6742	0.6789
0.4	8.6603	0.0827	0.0477	0.4682	5.0047	0.9642	1.1918	1.2312
0.5	5.0000	0.0477	0.0000	0.2030	2.1698	1.2644	1.5628	1.6364
0.6	0.0000	0.0000	0.0000	– 0.2894	– 3.0931	1.2257	1.5151	1.6031
0.7	0.0000	0.0000	0.0000	– 0.7575	– 8.0978	0.7575	0.9364	0.9907
0.8	0.0000	0.0000	0.0000	– 0.9364	– 10.0095	0.0000	0.0000	0.0000
0.9	0.0000	0.0000	0.0000	– 0.7575	– 8.0979	– 0.7575	– 0.9364	– 0.9907
1.0	0.0000				– 3.0931		– 1.5151	– 1.6031

Table P5.2b: Numerical solution using piecewise constant interpolation of excitation

t_i	p_i	$C'p_i$	$D'p_{i+1}$	$A'u_i$	u_i	$B'\dot{u}_i$	$\dot{u}_i$	Theoretical $\dot{u}_i$
0.0	0.0000	0.0000	0.9233	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	5.0000	0.9233	1.5992	– 0.1763	0.0477	0.7470	0.9233	0.9769
0.2	8.6602	1.5992	1.8466	– 0.9434	0.2554	2.5024	3.0931	3.2727
0.3	10.0000	1.8466	1.5992	– 2.4899	0.6742	4.0489	5.0047	5.2953
0.4	8.6603	1.5992	0.9233	– 4.4016	1.1918	4.0489	5.0047	5.2953
0.5	5.0000	0.9233	0.0000	– 5.7718	1.5628	1.7554	2.1698	2.2958
0.6	0.0000	0.0000	0.0000	– 5.5955	1.5151	– 2.5024	– 3.0931	– 3.2727
0.7	0.0000	0.0000	0.0000	– 3.4582	0.9364	– 6.5513	– 8.0978	– 8.5680
0.8	0.0000	0.0000	0.0000	0.0000	0.0000	– 8.0979	– 10.0095	– 10.5906
0.9	0.0000	0.0000	0.0000	3.4582	– 0.9364	– 6.5513	– 8.0979	– 8.5680
1.0	0.0000				– 1.5151		– 3.0931	– 3.2727

Problem 5.3

Solution to this problem is available as Example 5.2 in the textbook.

Problem 5.4

1.0 *Initial calculations*

$m = 0.2533 \quad k = 10 \quad c = 0.1592$

$u_0 = 0 \quad \dot{u}_0 = 0 \quad \Delta t = 0.05 \text{ sec}$

1.1 $\ddot{u}_0 = (p_0 - c\dot{u}_0 - ku_0)/m = 0$

1.2 $u_{-1} = u_0 - \dot{u}_0 \Delta t + \ddot{u}_0 (\Delta t)^2/2 = 0$

1.3 $\hat{k} = m/(\Delta t)^2 + c/2\Delta t = 102.91$

1.4 $a = m/(\Delta t)^2 - c/2\Delta t = 99.73$

1.5 $b = k - \left[2m/(\Delta t)^2\right] = -192.64$

2.0 *Calculation for each time step*

2.1 $\hat{p}_i = p_i - au_{i-1} - bu_i$

$= p_i - 99.73u_{i-1} + 192.64u_i$

2.2 $u_{i+1} = \hat{p}_i/\hat{k} = \hat{p}_i/102.91$

3.0 *Repetition for next time step.* Computational steps 2.1 and 2.2 are repeated for $i = 0, 1, 2, 3, \ldots$ leading to Table P5.4a, wherein the theoretical result is also included.

Comparison with Problem 5.3 and theoretical solution:

Table P5.4b shows this comparison, where the smaller Δt is seen to give more accurate results.

Table P5.4a: Numerical solution by central difference method

t_i	p_i	u_{i-1}	u_i	$\hat{p}_i$ (Eq. 2.1)	u_{i+1} (Eq. 2.2)	Theoretical u_{i+1}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0042
0.05	2.5882	0.0000	0.0000	2.5882	0.0251	0.0328
0.10	5.0000	0.0000	0.0251	9.8448	0.0957	0.1053
0.15	7.0711	0.0251	0.0957	22.9915	0.2234	0.2332
0.20	8.6602	0.0957	0.2234	42.1576	0.4096	0.4176
0.25	9.6593	0.2234	0.4096	66.2938	0.6442	0.6487
0.30	10.0000	0.4096	0.6442	93.2417	0.9060	0.9060
0.35	9.6593	0.6442	0.9060	119.9551	1.1656	1.1605
0.40	8.6603	0.9060	1.1656	142.8465	1.3880	1.3782
0.45	7.0711	1.1656	1.3880	158.2206	1.5374	1.5241
0.50	5.0000	1.3880	1.5374	162.7449	1.5814	1.5665
0.55	2.5882	1.5374	1.5814	153.9036	1.4955	1.4814
0.60	0.0000	1.5814	1.4955	130.3811	1.2669	1.2602
0.65	0.0000	1.4955	1.2669	94.9169	0.9223	0.9245
0.70	0.0000	1.2669	0.9223	51.3267	0.4987	0.5101
0.75	0.0000	0.9223	0.4987	4.0973	0.0398	0.0593
0.80	0.0000	0.4987	0.0398	− 42.0695	− 0.4088	− 0.3832
0.85	0.0000	0.0398	− 0.4088	− 82.7204	− 0.8038	− 0.7751
0.90	0.0000	− 0.4088	− 0.8038	− 114.0761	− 1.1085	− 1.0802
0.95	0.0000	− 0.8038	− 1.1085	− 133.3771	− 1.2960	− 1.2718
1.00	0.0000	− 1.1085	− 1.2960	− 139.1210	− 1.3518	− 1.3349

Table P5.4b

t_i	u_i ($\Delta t = 0.1$)	u_i ($\Delta t = 0.05$)	u_i (Theoretical)
0.00	0.0000	0.0000	0.0000
0.10	0.0000	0.0251	0.0328
0.20	0.1914	0.2234	0.2332
0.30	0.6293	0.6442	0.6487
0.40	1.1825	1.1656	1.1605
0.50	1.5808	1.5374	1.5241
0.60	1.5412	1.4955	1.4814
0.70	0.9141	0.9223	0.9245
0.80	− 0.0247	0.0398	0.0593
0.90	− 0.8968	− 0.8038	− 0.7751
1.00	− 1.3726	− 1.2960	− 1.2718

Problem 5.5

1.0 *Initial calculations*

$m = 0.2533 \quad k = 10 \quad c = 0.6366$

$u_0 = 0 \quad \dot{u}_0 = 0 \quad \Delta t = 0.05 \text{ sec}$

1.1 $\ddot{u}_0 = (p_0 - c\dot{u}_0 - ku_0)/m = 0$

1.2 $u_{-1} = u_0 - \dot{u}_0\Delta t + \ddot{u}_0(\Delta t)^2/2 = 0$

1.3 $\hat{k} = m/(\Delta t)^2 + c/2\Delta t = 107.69$

1.4 $a = m/(\Delta t)^2 - c/2\Delta t = 94.95$

1.5 $b = k - \left[2m/(\Delta t)^2\right] = -192.64$

2.0 *Calculations for each time step*

2.1 $\hat{p}_i = p_i - au_{i-1} - bu_i$

$= p_i - 94.95u_{i-1} + 192.64u_i$

2.2 $u_{i+1} = \hat{p}_i/\hat{k} = \hat{p}_i/107.69$

3.0 *Repetition for the next time step.* Computational steps 2.1 and 2.2 are repeated for $i = 0, 1, 2, 3, \ldots$ leading to Table P5.5; also included is the theoretical result, calculated from Eq. (3.2.5) — valid for $t \le 0.6$ sec — and from Eq. (2.2.4) modified appropriately — valid for $t \ge 0.6$ sec.

The response of systems with $\zeta = 0.05$ and $\zeta = 0.20$ is compared in the accompanying figure; numerical solution gives the peak displacement $u_o = 1.5814$ in. for $\zeta = 0.05$ and 1.2947 in. for $\zeta = 0.20$.

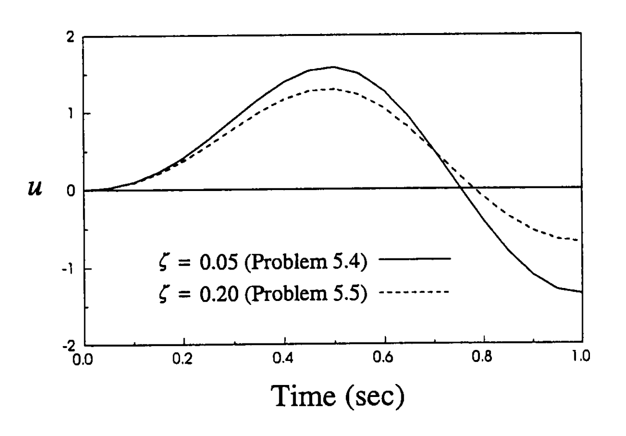


Table P5.5: Numerical solution by central difference method

t_i	p_i	u_{i-1}	u_i	$\hat{p}_i$ (Eq. 2.1)	u_{i+1} (Eq. 2.2)	Theoretical u_{i+1}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0041
0.05	2.5882	0.0000	0.0000	2.5882	0.0240	0.0313
0.10	5.0000	0.0000	0.0240	9.6300	0.0894	0.0984
0.15	7.0711	0.0240	0.0894	22.0161	0.2044	0.2133
0.20	8.6602	0.0894	0.2044	39.5535	0.3673	0.3744
0.25	9.6593	0.2044	0.3673	61.0036	0.5665	0.5704
0.30	10.0000	0.3673	0.5665	84.2527	0.7824	0.7822
0.35	9.6593	0.5665	0.7824	106.5884	0.9898	0.9854
0.40	8.6603	0.7824	0.9898	125.0455	1.1612	1.1530
0.45	7.0711	0.9898	1.1612	136.7796	1.2702	1.2593
0.50	5.0000	1.1612	1.2702	139.4247	1.2947	1.2826
0.55	2.5882	1.2702	1.2947	131.3981	1.2202	1.2087
0.60	0.0000	1.2947	1.2202	112.1188	1.0412	1.0362
0.65	0.0000	1.2202	1.0412	84.7074	0.7866	0.7888
0.70	0.0000	1.0412	0.7866	52.6710	0.4891	0.4980
0.75	0.0000	0.7866	0.4891	19.5312	0.1814	0.1957
0.80	0.0000	0.4891	0.1814	– 11.5039	– 0.1068	– 0.0889
0.85	0.0000	0.1814	– 0.1068	– 37.8014	– 0.3510	– 0.3316
0.90	0.0000	– 0.1068	– 0.3510	– 57.4792	– 0.5338	– 0.5152
0.95	0.0000	– 0.3510	– 0.5338	– 69.4929	– 0.6453	– 0.6298
1.00	0.0000	– 0.5338	– 0.6453	– 73.6329	– 0.6838	– 0.6729

Problem 5.6

1.0 *Initial calculations*

$m = 0.2533 \quad k = 10 \quad c = 0.1592$

$u_0 = 0 \quad \dot{u}_0 = 0 \quad \Delta t = 1/3 \text{ sec}$

1.1 $\ddot{u}_0 = (p_0 - c\dot{u}_0 - ku_0)/m = 0$

1.2 $u_{-1} = u_0 - \dot{u}_0 \Delta t + \ddot{u}_0 (\Delta t)^2/2 = 0$

1.3 $\hat{k} = m/(\Delta t)^2 + c/2\Delta t = 2.518$

1.4 $a = m/(\Delta t)^2 - c/2\Delta t = 2.041$

1.5 $b = k - \left[2m/(\Delta t)^2\right] = 5.441$

2.0 *Calculations for each time step*

2.1 $\hat{p}_i = p_i - au_{i-1} - bu_i$

$= p_i - 2.041u_{i-1} - 5.441u_i$

2.2 $u_{i+1} = \hat{p}_i/\hat{k} = \hat{p}_i/2.518$

3.0 *Repetition for the next time step.* Computational steps 2.1 and 2.2 are repeated for $i = 0, 1, 2, 3, \ldots$ leading to Table P5.6; also included is the theoretical result, calculated from Eq. (3.2.5) — valid for $t \leq 0.6$ sec — and from Eq. (2.2.4) modified appropriately — valid for $t \geq 0.6$ sec.

The central difference method gives meaningless results because $\Delta t/T_n = 1/3$ exceeds the stability limit of $1/\pi$.

Table P5.6: Numerical solution by central difference method

t_i	p_i	u_{i-1}	u_i	$\hat{p}_i$ (Eq. 2.1)	u_{i+1} (Eq. 2.2)	Theoretical u_{i+1}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.8190
$0.3\overline{3}$	9.8481	0.0000	0.0000	9.8481	3.9104	1.1595
$0.6\overline{6}$	0.0000	0.0000	3.9104	– 21.2744	– 8.4474	– 1.2718
1.00	0.0000	3.9104	– 8.4474	37.9773	15.0796	0.1998
$1.3\overline{3}$	0.0000	– 8.4474	15.0796	– 64.7998	– 25.7299	0.8524
$1.6\overline{6}$	0.0000	15.0796	– 25.7299	109.2072	43.3626	– 0.9262
2.00	0.0000	– 25.7299	43.3626	– 183.4013	– 72.8227	0.1390

Problem 5.7

The solution to this problem is available as Example 5.3 in the text. Table P5.7 shows the results obtained by the average acceleration and central difference methods. For this problem, both methods give the peak response with roughly the same accuracy.

Table P5.7

t_i	u_i (Ex. 5.3) (ave. accel.)	u_i (Ex. 5.2) (cent. diff.)	u_i (Theoretical)
0.00	0.0000	0.0000	0.0000
0.10	0.0437	0.0000	0.0328
0.20	0.2326	0.1914	0.2332
0.30	0.6121	0.6293	0.6487
0.40	1.0825	1.1825	1.1605
0.50	1.4309	1.5808	1.5241
0.60	1.4231	1.5412	1.4814
0.70	0.9622	0.9141	0.9245
0.80	0.1908	– 0.0247	0.0593
0.90	– 0.6044	– 0.8968	– 0.7751
1.00	– 1.1442	– 1.3726	– 1.2718

Problem 5.8

1.0 *Initial calculations*

$m = 0.2533 \quad k = 10 \quad c = 0.1592$

$u_0 = 0 \quad \dot{u}_0 = 0 \quad p_0 = 0 \quad \Delta t = 0.05 \text{ sec}$

1.1 $\ddot{u}_0 = (p_0 - c\dot{u}_0 - ku_0)/m = 0$

1.2 $\Delta t = 0.05$

1.3 $\hat{k} = k + (2/\Delta t)c + \left[4/(\Delta t)^2\right]m = 421.65$

1.4 $a = (4/\Delta t)\, m + 2c = 20.58$

$b = 2m = 0.5066$

2.0 *Calculations for each time step*

2.1 $\Delta\hat{p}_i = \Delta p_i + a\dot{u}_i + b\ddot{u}_i$

$= \Delta p_i + 20.58\dot{u}_i + 0.5066\ddot{u}_i$

2.2 $\Delta u_i = \Delta\hat{p}_i / \hat{k} = \Delta\hat{p}_i / 421.65$

2.3 $\Delta\dot{u}_i = (2/\Delta t)\,\Delta u_i - 2\dot{u}_i = 40\Delta u_i - 2\dot{u}_i$

2.4 $\Delta\ddot{u}_i = \left[4/(\Delta t)^2\right](\Delta u_i - \Delta t\,\dot{u}_i) - 2\ddot{u}_i$

$= 1600\,(\Delta u_i - 0.05\dot{u}_i) - 2\ddot{u}_i$

2.5 $u_{i+1} = u_i + \Delta u_i$

$\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i$

$\ddot{u}_{i+1} = \ddot{u}_i + \Delta\ddot{u}_i$

3.0 *Repetition for the next time step.* Steps 2.1 through 2.5 are repeated for successive time steps and are summarized in Table P5.8a, wherein the theoretical result is also included.

Table P5.8b shows that the smaller Δt gives a more accurate value of the peak response.

Table P5.8b

t_i	u_i ($\Delta t = 0.1$)	u_i ($\Delta t = 0.05$)	u_i (Theoretical)
0.00	0.0000	0.0000	0.0000
0.10	0.0437	0.0356	0.0328
0.20	0.2326	0.2329	0.2332
0.30	0.6127	0.6390	0.6487
0.40	1.0825	1.1400	1.1605
0.50	1.4309	1.5002	1.5241
0.60	1.4231	1.4673	1.4814
0.70	0.9622	0.9361	0.9245
0.80	0.1908	0.0950	0.0593
0.90	– 0.6044	– 0.7306	– 0.7751
1.00	– 1.1442	– 1.2407	– 1.2718

Table P5.8a: Numerical solution by average acceleration method

t_i	p_i	$\ddot{u}_i$	Δp_i	$\Delta\hat{p}_i$	Δu_i	$\Delta\dot{u}_i$	$\Delta\ddot{u}_i$	$\dot{u}_i$	u_i	Theoretical u_i
0.00	0.0000	0.0000	2.5882	2.5882	0.0061	0.2455	9.8212	0.0000	0.0000	0.0000
0.05	2.5882	9.8212	2.4118	12.4409	0.0295	0.6891	7.9236	0.2455	0.0061	0.0042
0.10	5.0000	17.7448	2.0711	30.2987	0.0719	1.0049	4.7080	0.9347	0.0356	0.0328
0.15	7.0711	22.4528	1.5892	52.8861	0.1254	1.1378	0.6073	1.9396	0.1075	0.1053
0.20	8.6602	23.0601	0.9990	76.0229	0.1803	1.0571	– 3.8381	3.0774	0.2329	0.2332
0.25	9.6593	19.2220	0.3407	95.1770	0.2257	0.7600	– 8.0436	4.1345	0.4132	0.4176
0.30	10.0000	11.1784	– 0.3407	106.0635	0.2515	0.2727	– 11.4471	4.8945	0.6390	0.6487
0.35	9.6593	– 0.2687	– 0.9990	105.2198	0.2495	– 0.3528	– 13.5738	5.1672	0.8905	0.9060
0.40	8.6603	– 13.8425	– 1.5892	90.4920	0.2146	– 1.0444	– 14.0902	4.8145	1.1400	1.1605
0.45	7.0711	– 27.9327	– 2.0711	61.3759	0.1456	– 1.7177	– 12.8435	3.7701	1.3546	1.3782
0.50	5.0000	– 40.7762	– 2.4118	19.1735	0.0455	– 2.2858	– 9.8804	2.0524	1.5002	1.5241
0.55	2.5882	– 50.6566	– 2.5882	– 33.0564	– 0.0784	– 2.6690	– 5.4459	– 0.2335	1.5457	1.5665
0.60	0.0000	– 56.1025	0.0000	– 88.1614	– 0.2091	– 2.5586	9.8620	– 2.9024	1.4673	1.4814
0.65	0.0000	– 46.2405	0.0000	– 135.8272	– 0.3221	– 1.9633	13.9508	– 5.4610	1.2582	1.2602
0.70	0.0000	– 32.2897	0.0000	– 169.1684	– 0.4012	– 1.1997	16.5927	– 7.4243	0.9361	0.9245
0.75	0.0000	– 15.6969	0.0000	– 185.4546	– 0.4398	– 0.3453	17.5808	– 8.6239	0.5349	0.5101
0.80	0.0000	1.8838	0.0000	– 183.6558	– 0.4356	0.5160	16.8712	– 8.9693	0.0950	0.0593
0.85	0.0000	18.7550	0.0000	– 164.4888	– 0.3901	1.3023	14.5825	– 8.4533	– 0.3405	– 0.3832
0.90	0.0000	33.3375	0.0000	– 130.2963	– 0.3090	1.9414	10.9796	– 7.1510	– 0.7306	– 0.7751
0.95	0.0000	44.3171	0.0000	– 84.7757	– 0.2011	2.3770	6.4439	– 5.2096	– 1.0397	– 1.0802
1.00	0.0000	50.7610						– 2.8327	– 1.2407	– 1.2718

Problem 5.9

1.0 *Initial calculations*

$m = 0.2533 \quad k = 10 \quad c = 0.1592$

$u_0 = 0 \quad \dot{u}_0 = 0 \quad p_0 = 0 \quad \Delta t = 1/3 \text{ sec}$

1.1 $\ddot{u}_0 = (p_0 - c\dot{u}_0 - ku_0)/m = 0$

1.2 $\Delta t = 1/3 \text{ sec}$

1.3 $\hat{k} = k + (2/\Delta t)c + \left[4/(\Delta t)^2\right]m = 20.074$

1.4 $a = (4/\Delta t)\,m + 2c = 3.3580$

$b = 2m = 0.5066$

2.0 Calculations for each time step

2.1 $\Delta\hat{p}_i = \Delta p_i + a\dot{u}_i + b\ddot{u}_i$

$= \Delta p_i + 3.3580\dot{u}_i + 0.5066\ddot{u}_i$

2.2 $\Delta u_i = \Delta\hat{p}_i/\hat{k} = \Delta\hat{p}_i/20.074$

2.3 $\Delta\dot{u}_i = (2/\Delta t)\,\Delta u_i - 2\dot{u}_i = 6\Delta u_i - 2\dot{u}_i$

2.4 $\Delta\ddot{u}_i = \left[4/(\Delta t)^2\right](\Delta u_i - \Delta t\,\dot{u}_i) - 2\ddot{u}_i$

$= 36\left[\Delta u_i - (1/3)\dot{u}_i\right] - 2\ddot{u}_i$

2.5 $u_{i+1} = u_i + \Delta u_i$

$\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i$

$\ddot{u}_{i+1} = \ddot{u}_i + \Delta\ddot{u}_i$

3.0 *Repetition for the next time step.* Steps 2.1 through 2.5 are repeated for successive time steps and are summarized in Table P5.9; also included is the theoretical result, calculated from Eq. (3.2.5) — valid for $t \le 0.6$ sec — and from Eq. (2.2.4) modified appropriately — valid for $t \ge 0.6$ sec.

The numerical results are in large error but the solution is stable, unlike Problem 5.6.

Table P5.9: Numerical solution by average acceleration method

t_i	p_i	$\ddot{u}_i$	Δp_i	$\Delta\hat{p}_i$	Δu_i	$\Delta\dot{u}_i$	$\Delta\ddot{u}_i$	$\dot{u}_i$	u_i	Theoretical u_i
0.00	0.0000	0.0000	9.8481	9.8481	0.4906	2.9436	17.6616	0.0000	0.0000	0.0000
$0.3\overline{3}$	9.8481	17.6616	− 9.8481	8.9838	0.4475	− 3.2019	− 54.5348	2.9436	0.4906	0.8189
$0.6\overline{6}$	0.0000	− 36.8733	0.0000	− 19.5477	− 0.9738	− 5.3261	41.7898	− 0.2583	0.9381	1.1595
1.00	0.0000	4.9166	0.0000	− 16.2615	− 0.8101	6.3083	28.0170	− 5.5844	− 0.0357	− 1.2718
$1.3\overline{3}$	0.0000	32.9336	0.0000	19.1151	0.9522	4.2657	− 40.2728	0.7239	− 0.8457	0.1997
$1.6\overline{6}$	0.0000	− 7.3393	0.0000	13.0367	0.6494	− 6.0825	− 21.8168	4.9896	0.1065	0.8525
2.00	0.0000	− 29.1561						− 1.0929	0.7559	− 0.9262

Problem 5.10

The solution to this problem is available as Example 5.4 in the textbook. Table P5.10 compares the numerical results obtained from the average acceleration method (Example 5.3) and from the linear acceleration method (Example 5.4) with the theoretical results. The linear acceleration method gives more accurate results.

Table P5.10

t_i	u_i (Ex. 5.3) (ave. accel.)	u_i (Ex. 5.4) (lin. accel.)	u_i (Theoretical)
0.00	0.0000	0.0000	0.0000
0.10	0.0437	0.0300	0.0328
0.20	0.2326	0.2193	0.2332
0.30	0.6121	0.6166	0.6487
0.40	1.0825	1.1130	1.1605
0.50	1.4309	1.4782	1.5241
0.60	1.4231	1.4625	1.4814
0.70	0.9622	0.9514	0.9245
0.80	0.1908	0.1273	0.0593
0.90	– 0.6044	– 0.6954	– 0.7751
1.00	– 1.1442	– 1.2208	– 1.2718

Problem 5.11

1.0 Initial calculations

$m = 0.2533 \quad k = 10 \quad c = 0.1592$

$u_0 = 0 \quad \dot{u}_0 = 0 \quad p_0 = 0 \quad \Delta t = 0.05 \text{ sec}$

1.1 $\ddot{u}_0 = (p_0 - c\dot{u}_0 - ku_0)/m = 0$

1.2 $\Delta t = 0.05 \text{ sec}$

1.3 $\hat{k} = k + (3/\Delta t)\,c + \left[6/(\Delta t)^2\right] m = 627.48$

1.4 $a = (6/\Delta t)\,m + 3c = 30.87$

$b = 3m + (\Delta t/2)\,c = 0.7639$

2.0 *Calculations for each time step*

2.1 $\Delta\hat{p}_i = \Delta p_i + a\dot{u}_i + b\ddot{u}_i$

$= \Delta p_i + 30.87\dot{u}_i + 0.7639\ddot{u}_i$

2.2 $\Delta u_i = \Delta\hat{p}_i/\hat{k} = \Delta\hat{p}_i/627.48$

2.3 $\Delta\dot{u}_i = (3/\Delta t)\,\Delta u_i - 3\dot{u}_i - (\Delta t/2)\,\ddot{u}_i$

$= 60\Delta u_i - 3\dot{u}_i - 0.025\ddot{u}_i$

2.4 $\Delta\ddot{u}_i = \left[6/(\Delta t)^2\right](\Delta u_i - \Delta t\,\dot{u}_i) - 3\ddot{u}_i$

$= 2400\,(\Delta u_i - 0.05\dot{u}_i) - 3\ddot{u}_i$

2.5 $u_{i+1} = u_i + \Delta u_i$

$\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i$

$\ddot{u}_{i+1} = \ddot{u}_i + \Delta\ddot{u}_i$

3.0 *Repetition for the next time step.* Steps 2.1 through 2.5 are repeated for successive time steps and are summarized in Table P5.11a, wherein the theoretical result is also included.

Table P5.11b compares the numerical results using $\Delta t = 0.05$ and 0.1 sec, and the theoretical results. The smaller Δt gives more accurate results.

Table P5.11a: Numerical solution by linear acceleration method

t_i	p_i	$\ddot{u}_i$	Δp_i	$\Delta\hat{p}_i$	Δu_i	$\Delta\dot{u}_i$	$\Delta\ddot{u}_i$	$\dot{u}_i$	u_i	Theoretical u_i
0.00	0.0000	0.0000	2.5882	2.5882	0.0041	0.2475	9.8994	0.0000	0.0000	0.0000
0.05	2.5882	9.8994	2.4118	17.6147	0.0281	0.6944	7.9769	0.2475	0.0041	0.0042
0.10	5.0000	17.8763	2.0711	44.8060	0.0714	1.0119	4.7215	0.9419	0.0322	0.0328
0.15	7.0711	22.5978	1.5892	79.1705	0.1262	1.1442	0.5738	1.9537	0.1036	0.1053
0.20	8.6602	23.1716	0.9990	114.3455	0.1822	1.0607	– 3.9167	3.0980	0.2298	0.2332
0.25	9.6593	19.2549	0.3407	143.4420	0.2286	0.7588	– 8.1564	4.1586	0.4120	0.4176
0.30	10.0000	11.0985	– 0.3407	159.9580	0.2549	0.2655	– 11.5760	4.9175	0.6406	0.6487
0.35	9.6593	– 0.4775	– 0.9990	158.6548	0.2528	– 0.3663	– 13.6957	5.1830	0.8955	0.9060
0.40	8.6603	– 14.1732	– 1.5892	136.2945	0.2172	– 1.0632	– 14.1810	4.8167	1.1484	1.1605
0.45	7.0711	– 28.3542	– 2.0711	92.1554	0.1469	– 1.7397	– 12.8812	3.7535	1.3656	1.3782
0.50	5.0000	– 41.2354	– 2.4118	28.2626	0.0450	– 2.3080	– 9.8494	2.0138	1.5125	1.5241
0.55	2.5882	– 51.0848	– 2.5882	– 50.6946	– 0.0808	– 2.6877	– 5.3396	– 0.2942	1.5575	1.5665
0.60	0.0000	– 56.4244	0.0000	– 135.1657	– 0.2154	– 2.5683	10.1178	– 2.9819	1.4767	1.4814
0.65	0.0000	– 46.3066	0.0000	– 206.7292	– 0.3295	– 1.9594	14.2377	– 5.5502	1.2613	1.2602
0.70	0.0000	– 32.0688	0.0000	– 256.3469	– 0.4085	– 1.1817	16.8708	– 7.5096	0.9318	0.9245
0.75	0.0000	– 15.1980	0.0000	– 279.9422	– 0.4461	– 0.3146	17.8106	– 8.6913	0.5233	0.5101
0.80	0.0000	2.6126	0.0000	– 276.0509	– 0.4399	0.5561	17.0187	– 9.0059	0.0772	0.0593
0.85	0.0000	19.6313	0.0000	– 245.8817	– 0.3919	1.3472	14.6235	– 8.4498	– 0.3628	– 0.3832
0.90	0.0000	34.2548	0.0000	– 193.1193	– 0.3078	1.9853	10.9029	– 7.1027	– 0.7546	– 0.7751
0.95	0.0000	45.1577	0.0000	– 123.4966	– 0.1968	2.4142	6.2530	– 5.1173	– 1.0624	– 1.0802
1.00	0.0000	51.4107						– 2.7031	– 1.2592	– 1.2718

Table P5.11b

t_i	$u_i(\Delta t = 0.1)$	$u_i(\Delta t = 0.05)$	u_i (Theoretical)
0.00	0.0000	0.0000	0.0000
0.10	0.0300	0.0322	0.0328
0.20	0.2193	0.2298	0.2332
0.30	0.6166	0.6406	0.6487
0.40	1.1130	1.1484	1.1605
0.50	1.4782	1.5125	1.5241
0.60	1.4625	1.4767	1.4814
0.70	0.9514	0.9318	0.9245
0.80	0.1273	0.0772	0.0593
0.90	– 0.6954	– 0.7546	– 0.7751
1.00	– 1.2208	– 1.2592	– 1.2718

Problem 5.12

The governing equation is

$$\hat{k} u_{i+1} = \hat{p}_i$$

where

$$\hat{k} = \frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t} = 102.91$$

and

$$\hat{p}_i = p_i - \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t}\right] u_{i-1} - (f_S)_i + \frac{2m}{(\Delta t)^2} u_i$$

$$= p_i - 99.730 u_{i-1} - (f_S)_i + 202.64 u_i$$

The solution steps are summarized in Table P5.12.

Table P5.12: Numerical solution by central difference method

t_i	p_i	$(f_S)_i$	u_{i-1}	u_i	$\hat{p}_i$	$\dot{u}_{i+1}$	u_{i+1}
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.05	2.5882	0.2515	0.0000	0.0000	2.5882	0.2515	0.0251
0.10	5.0000	0.9566	0.0000	0.0251	9.8448	0.9566	0.0957
0.15	7.0711	2.2341	0.0251	0.0957	22.9915	1.9826	0.2234
0.20	8.6602	4.0964	0.0957	0.2234	42.1576	3.1398	0.4096
0.25	9.6593	6.4418	0.2234	0.4096	66.2938	4.2077	0.6442
0.30	10.0000	7.5000	0.4096	0.6442	93.2417	4.9638	0.9060
0.35	9.6593	7.5000	0.6442	0.9060	121.5154	5.3659	1.1808
0.40	8.6603	7.5000	0.9060	1.1808	150.0748	5.5225	1.4583
0.45	7.0711	7.5000	1.1808	1.4583	177.3219	5.4227	1.7230
0.50	5.0000	7.5000	1.4583	1.7230	201.2263	4.9704	1.9553
0.55	2.5882	7.5000	1.7230	1.9553	219.4796	4.0964	2.1327
0.60	0.0000	7.5000	1.9553	2.1327	229.6683	2.7637	2.2317
0.65	0.0000	7.5000	2.1327	2.2317	232.0418	1.2207	2.2547
0.70	0.0000	7.5000	2.2317	2.2547	226.8420	– 0.2746	2.2042
0.75	0.0000	6.2816	2.2547	2.2042	214.3029	– 1.7237	2.0824
0.80	0.0000	4.4905	2.2042	2.0824	195.8701	– 3.0095	1.9033
0.85	0.0000	2.3184	2.0824	1.9033	173.5170	– 3.9632	1.6861
0.90	0.0000	– 0.0117	1.9033	1.6861	149.5368	– 4.5022	1.4530

Problem 5.13

The solution to this problem is available as Example 5.5 in the textbook.

Problem 5.14

The solution to this problem is available as Example 5.6 in the textbook.

Problem 6.1

We will solve this problem by the procedure described in Section 5.2 using piece-wise linear interpolation of $\ddot{u}_g(t)$ with $\Delta t = 0.02$ sec.

1. *Initial calculation.*

$T_n = 2$ sec. $\omega_n = \pi$ $\Delta t = 0.02$ sec.

$\zeta = 0$ $\omega_D = \omega_n = \pi$ $e^{-\zeta\omega_n\Delta t} = 1$

$\sin\omega_D\Delta t = 0.062791$ $\cos\omega_D\Delta t = 0.99803$

Substituting these in Table 5.2.1 of the book with $m = 1$ and $k = \omega_n^2$ gives

$A = 0.99803$ $B = 0.019987$

$C = 1.3328\times10^{-4}$ $D = 6.6654\times10^{-5}$

$A' = -0.19726$ $B' = 0.99803$

$C' = 9.9101\times10^{-3}$ $D' = 9.9967\times10^{-3}$

2. *Apply recurrence Eq. (5.2.5).*

In these equations, $p_i = -m\ddot{u}_g(t_i) = -\ddot{u}_g(t_i)$ and the resulting computations are summarized in Table P6.1a and P6.1b.

3. *Plot response history.*

The deformation is plotted as a function of time in Fig. 6.4.1b for $\zeta = 0$.

Table 6.1a: Numerical solution using piece-wise linear interpolation of excitation

t_i	p_i	Cp_i	Dp_{i+1}	$B\dot{u}_i$	$\dot{u}_i$	Au_i	u_i
0.0000	0.0000	0.0000	-0.0002	0.0000	0.0000	0.0000	0.0000
0.0200	-2.4318	-0.0003	-0.0001	-0.0005	-0.0243	-0.0002	-0.0002
0.0400	-1.4050	-0.0002	-0.0000	-0.0013	-0.0626	-0.0011	-0.0011
0.0600	-0.3821	-0.0001	-0.0001	-0.0016	-0.0801	-0.0025	-0.0025
0.0800	-1.6521	-0.0002	-0.0002	-0.0020	-0.0998	-0.0043	-0.0043
0.1000	-2.9259	-0.0004	-0.0003	-0.0029	-0.1445	-0.0067	-0.0067
0.1200	-4.1958	-0.0006	-0.0002	-0.0043	-0.2141	-0.0102	-0.0102
0.1400	-2.6325	-0.0004	-0.0001	-0.0056	-0.2798	-0.0152	-0.0152
0.1600	-1.0692	-0.0001	0.0000	-0.0063	-0.3133	-0.0212	-0.0212
0.1800	0.4941	0.0001	-0.0001	-0.0063	-0.3142	-0.0275	-0.0275
0.2000	-1.4205	-0.0002	-0.0002	-0.0063	-0.3174	-0.0337	-0.0338

Table P6.1b: Numerical solution using piece-wise linear interpolation of excitation

t_i	p_i	$C'p_i$	$D'p_{i+1}$	$B'\dot{u}_i$	$\dot{u}_i$	$A'u_i$	u_i
0.0000	0.0000	0.0000	-0.0243	0.0000	0.0000	0.0000	0.0000
0.0200	-2.4318	-0.0243	-0.0140	-0.0243	-0.0243	0.0000	-0.0002
0.0400	-1.4050	-0.0140	-0.0038	-0.0624	-0.0626	0.0002	-0.0011
0.0600	-0.3821	-0.0038	-0.0165	-0.0799	-0.0801	0.0005	-0.0025
0.0800	-1.6521	-0.0165	-0.0292	-0.0996	-0.0998	0.0008	-0.0043
0.1000	-2.9259	-0.0292	-0.0419	-0.1442	-0.1445	0.0013	-0.0067
0.1200	-4.1958	-0.0419	-0.0263	-0.2136	-0.2141	0.0020	-0.0102
0.1400	-2.6325	-0.0263	-0.0107	-0.2793	-0.2798	0.0030	-0.0152
0.1600	-1.0692	-0.0107	0.0049	-0.3127	-0.3133	0.0042	-0.0212
0.1800	0.4941	0.0049	-0.0142	-0.3136	-0.3142	0.0054	-0.0275
0.2000	-1.4205	-0.0142	-0.0333	-0.3168	-0.3174	0.0067	-0.0338

Problem 6.2

We will solve this problem by the procedure described in Section 5.2 using piece-wise linear interpolation of $\ddot{u}_g(t)$ with $\Delta t = 0.02$ sec.

1. *Initial calculation.*

$T_n = 2$ sec. $\quad \omega_n = \pi \quad \Delta t = 0.02$ sec.

$\zeta = 0.05 \quad \omega_D = \omega_n\sqrt{1-\zeta^2} = 3.1377$

$e^{-\zeta\omega_n\Delta t} = 0.99686 \quad \sin\omega_D\Delta t = 0.062712$

$\cos\omega_D\Delta t = 0.99803$

Substituting these in Table 5.2.1 of the book with $m = 1$ and $k = \omega_n^2$ gives

$A = 0.99803 \quad B = 0.019924$

$C = 1.3297\times10^{-4} \quad D = 6.6549\times10^{-5}$

$A' = -0.19664 \quad B' = 0.99177$

$C' = 9.9484\times10^{-3} \quad D' = 9.9758\times10^{-3}$

2. *Apply recurrence Eq. (5.2.5).*

In these equations, $p_i = -m\ddot{u}_g(t_i) = -\ddot{u}_g(t_i)$ and the resulting computations are summarized in Table P6.2a and P6.2b.

3. *Plot response history.*

The deformation is plotted as a function of time in Fig. 6.4.1b for $\zeta = 5\%$.

Table P6.2a: Numerical solution using piece-wise linear interpolation of excitation

t_i	p_i	Cp_i	Dp_{i+1}	$B\dot{u}_i$	$\dot{u}_i$	Au_i	u_i
0.0000	0.0000	0.0000	-0.0002	0.0000	0.0000	0.0000	0.0000
0.0200	-2.4318	-0.0003	-0.0001	-0.0005	-0.0243	-0.0002	-0.0002
0.0400	-1.4050	-0.0002	-0.0000	-0.0012	-0.0622	-0.0011	-0.0011
0.0600	-0.3821	-0.0001	-0.0001	-0.0016	-0.0793	-0.0025	-0.0025
0.0800	-1.6521	-0.0002	-0.0002	-0.0020	-0.0984	-0.0042	-0.0042
0.1000	-2.9259	-0.0004	-0.0003	-0.0028	-0.1424	-0.0066	-0.0066
0.1200	-4.1958	-0.0006	-0.0002	-0.0042	-0.2109	-0.0101	-0.0101
0.1400	-2.6325	-0.0004	-0.0001	-0.0055	-0.2752	-0.0150	-0.0150
0.1600	-1.0692	-0.0001	0.0000	-0.0061	-0.3068	-0.0209	-0.0209
0.1800	0.4941	0.0001	-0.0001	-0.0061	-0.3059	-0.0270	-0.0271
0.2000	-1.4205	-0.0002	-0.0002	-0.0061	-0.3073	-0.0331	-0.0331

Table P6.2b: Numerical solution using piece-wise linear interpolation of excitation

t_i	p_i	$C'p_i$	$D'p_{i+1}$	$B'\dot{u}_i$	$\dot{u}_i$	$A'u_i$	u_i
0.0000	0.0000	0.0000	-0.0243	0.0000	0.0000	0.0000	0.0000
0.0200	-2.4318	-0.0242	-0.0140	-0.0241	-0.0243	0.0000	-0.0002
0.0400	-1.4050	-0.0140	-0.0038	-0.0617	-0.0622	0.0002	-0.0011
0.0600	-0.3821	-0.0038	-0.0165	-0.0787	-0.0793	0.0005	-0.0025
0.0800	-1.6521	-0.0164	-0.0292	-0.0976	-0.0984	0.0008	-0.0042
0.1000	-2.9259	-0.0291	-0.0419	-0.1412	-0.1424	0.0013	-0.0066
0.1200	-4.1958	-0.0417	-0.0263	-0.2092	-0.2109	0.0020	-0.0101
0.1400	-2.6325	-0.0262	-0.0107	-0.2729	-0.2752	0.0030	-0.0150
0.1600	-1.0692	-0.0106	0.0049	-0.3043	-0.3068	0.0041	-0.0209
0.1800	0.4941	0.0049	-0.0142	-0.3034	-0.3059	0.0053	-0.0271
0.2000	-1.4205	-0.0141	-0.0333	-0.3048	-0.3073	0.0065	-0.0331

Problem 6.3

We will solve this problem by the central difference method described in Section 5.3 with $\Delta t = 0.02$ sec.

1. *Initial calculations.*

$T_n = 2$ sec $\quad \omega_n = \pi \quad \zeta = 0.05 \quad \Delta t = 0.02$ sec

$m = 1 \quad k = \omega_n^2 = 9.8696 \quad c = 2\zeta\omega_n = 0.3142$

$u_0 = 0 \quad \dot{u}_0 = 0$

Substituting these into Eq.s (1.1)-(1.5) in Table 5.3.1 of the book gives

$\ddot{u}_0 = 0 \quad u_{-1} = 0$

$\hat{k} = 2507.9 \quad a = 2942.1 \quad b = -4990.1$

2. *Calculations for each time step.*

We apply the recurrence Eq.s (2.1)-(2.3) in Table 5.3.1 of the book. In these equations, $p_i = -m\ddot{u}_g(t_i) = -\ddot{u}_g(t_i)$ and the resulting computations are summarized in Table P6.3.

3. *Plot response history*

The deformation is plotted as a function of time in Fig. P6.3. Note that it is essentially the same as that shown in Fig. 6.4.1b with $\zeta = 5\%$.

Table P6.3: Numerical solution by central difference method

t_i	p_i	u_{i-1}	u_i	$\hat{p}_i$	u_{i+1}
0	0	0	0	0	0
0.02	-2.4343	0	0	-2.4343	-0.0010
0.04	-1.4065	0	-0.0010	-6.2503	-0.0025
0.06	-0.3825	-0.0010	-0.0025	-10.4003	-0.0041
0.08	-1.6538	-0.0025	-0.0041	-16.1372	-0.0064
0.10	-2.9289	-0.0041	-0.0064	-24.7036	-0.0099
0.12	-4.2002	-0.0064	-0.0099	-37.3192	-0.0149
0.14	-2.6352	-0.0099	-0.0149	-52.3443	-0.0209
0.16	-1.0703	-0.0149	-0.0209	-68.1395	-0.0272
0.18	0.4946	-0.0209	-0.0272	-83.0730	-0.0331
0.20	-1.4220	-0.0272	-0.0331	-99.0081	-0.0395

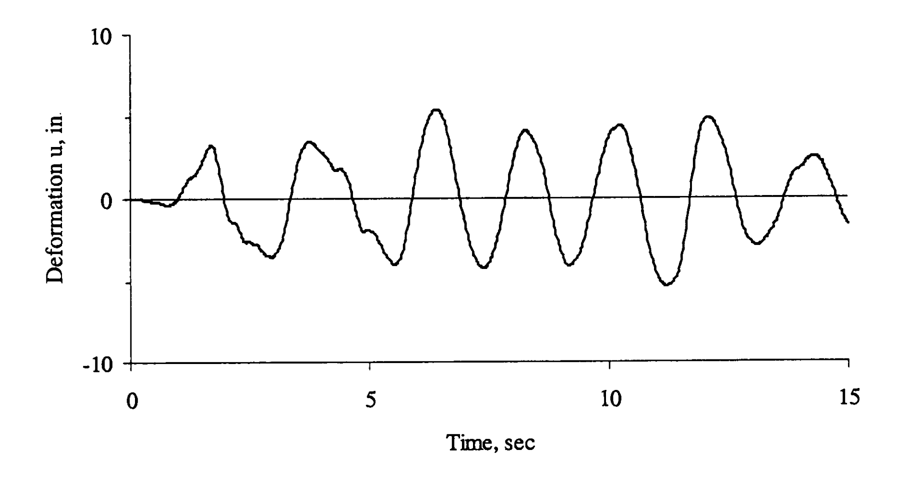


Fig. P6.3

Problem 6.4

The equation of motion to be solved is

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t); \quad \ddot{u}_g(t) = \dot{u}_{go}\delta(t) \qquad \text{(a)}$$

We have solved a related equation:

$$m\ddot{u} + c\dot{u} + ku = p(t); \quad p(t) = \delta(t)$$

and its solution is given by Eq. (4.1.7) specialized for $\tau = 0$:

$$u(t) = h(t) = \frac{1}{m\omega_D} e^{-\zeta\omega_n t} \sin \omega_D t \qquad \text{(b)}$$

Therefore solution to Eq. (a) is Eq. (b) multiplied by $-m\dot{u}_{go}$:

$$u(t) = -\frac{\dot{u}_{go}}{\omega_D} e^{-\zeta\omega_n t} \sin \omega_D t \qquad \text{(c)}$$

For maximum response, $du/dt = 0$:

$$\frac{du}{dt} = -\frac{\dot{u}_{go}}{\omega_D}\left[e^{-\zeta\omega_n t}(-\zeta\omega_n)\sin \omega_D t + e^{-\zeta\omega_n t}\omega_D \cos \omega_D t\right]$$

$$= -\frac{\dot{u}_{go}}{\omega_D} e^{-\zeta\omega_n t}\left[-\zeta\omega_n \sin \omega_D t + \omega_D \cos \omega_D t\right] = 0$$

or

$$\tan \omega_D t = \frac{\omega_D}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

The maximum occurs at

$$t_o = \frac{1}{\omega_D}\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \qquad \text{(d)}$$

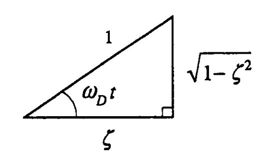
and the maximum response is

$$u_o = u(t_o) = \frac{\dot{u}_{go}}{\omega_D} \exp\left[-\zeta\omega_n t_o\right] \sin \omega_D t_o \qquad \text{(e)}$$

From Eq. (d)

$$\omega_D t_o = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \qquad \text{(f)}$$

$$\therefore \sin \omega_D t_o = \sqrt{1-\zeta^2} \qquad \text{(g)}$$



Substituting Eqs. (f) and (g) in Eq. (e) and using $\omega_D = \omega_n\sqrt{1-\zeta^2}$ gives

$$u_o = \frac{\dot{u}_{go}}{\omega_n} \exp\left[-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] \qquad \text{(h)}$$

Response spectra:

$$D = \frac{\dot{u}_{go}}{2\pi} T_n \exp\left[-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$

$$V = \frac{2\pi}{T_n} D$$

$$A = \left(\frac{2\pi}{T_n}\right)^2 D$$

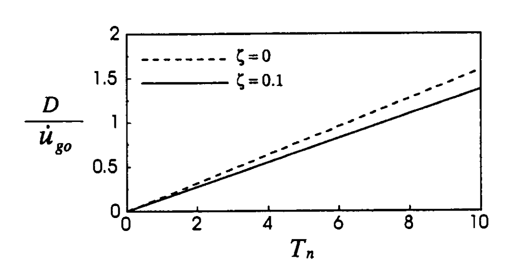
For $\zeta = 0$,

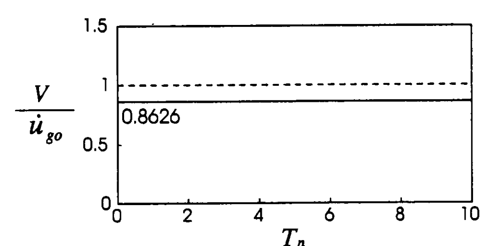
$$D = \frac{\dot{u}_{go}}{2\pi} T_n; V = \dot{u}_{go}; \quad A = \frac{2\pi\dot{u}_{go}}{T_n}$$

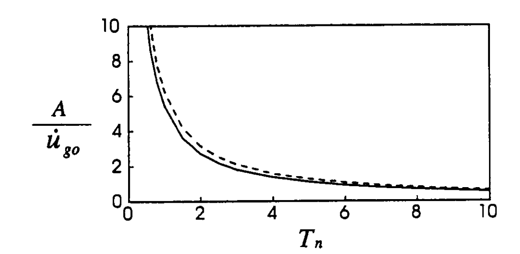
For $\zeta = 0.1$,

$$D = \frac{0.8626}{2\pi}\dot{u}_{go}T_n; \quad V = 0.8626\dot{u}_{go}; \quad A = \frac{2\pi(0.8626)}{T_n}\dot{u}_{go}$$

These spectra are plotted in the accompanying figure.







Problem 6.5

1. *Determine response to the first impulse.*

The ground motion impulse can be represented by the effective earthquake force,

$$p_{\text{eff}}(t) = -m\ddot{u}_g(t) = -m\dot{u}_{go}\delta(t)$$

The response of the system to the first impulse is the unit impulse response of Eq. (4.1.6) times $-m\dot{u}_{go}$:

$$u_1(t) = -\frac{\dot{u}_{go}}{\omega_n}\sin\omega_n t \tag{a}$$

2. *Determine response to second impulse.*

$$u_2(t) = \frac{\dot{u}_{go}}{\omega_n}\sin\omega_n(t - t_d) \qquad t \ge t_d \tag{b}$$

3. *Determine response to both impulses.*

For $0 \le t \le t_d$:

$$u(t) = -\frac{\dot{u}_{go}}{\omega_n}\sin\omega_n t$$

$$= -\frac{\dot{u}_{go}}{\omega_n}\sin\frac{2\pi t}{T_n} \tag{c}$$

For $t \ge t_d$:

$$u(t) = -\frac{\dot{u}_{go}}{\omega_n}[\sin\omega_n t - \sin\omega_n(t - t_d)]$$

$$= -\frac{\dot{u}_{go}}{\omega_n}2\sin\frac{\omega_n t_d}{2}\cos\frac{\omega_n(2t - t_d)}{2}$$

$$= -\frac{2\dot{u}_{go}}{\omega_n}\left(\sin\frac{\pi t_d}{T_n}\right)\cos\left[2\pi\left(\frac{t}{T_n} - \frac{1}{2}\frac{t_d}{T_n}\right)\right] \tag{d}$$

4. *Plot displacement response.*

Equations (c) and (d) are plotted for $t_d / T_n = 1/8$, 1/4, 1/2, and 1 in Figs. P6.5a, b, c, and d, respectively.

5. *Determine the peak response during* $0 \le t \le t_d$.

The number of peaks in *u(t)* depend on t_d / T_n; the longer the time t_d between the pulses, more such peaks occur. The first peak occurs at $t_o = T_n / 4$ with the deformation u_o given by

$$\frac{u_o}{\dot{u}_{go} / \omega_n} = 1 \tag{e}$$

Thus t_d must be longer than $T_n / 4$ for at least one peak to develop during $0 \le t \le t_d$.

If t_d is shorter than $T_n / 4$ no peak will develop during $0 \le t \le t_d$ and the response simply builds up from zero to $u(t_d)$, where

$$\frac{|u(t_d)|}{\dot{u}_{go} / \omega_n} = \sin\frac{2\pi t_d}{T_n} \tag{f}$$

The absolute maximum deformation during $0 \le t \le t_d$ is

$$\frac{u_o}{\dot{u}_{go} / \omega_n} = \begin{cases} \sin 2\pi t_d / T_n & t_d / T_n \le 1/4 \\ 1 & t_d / T_n \ge 1/4 \end{cases} \tag{g}$$

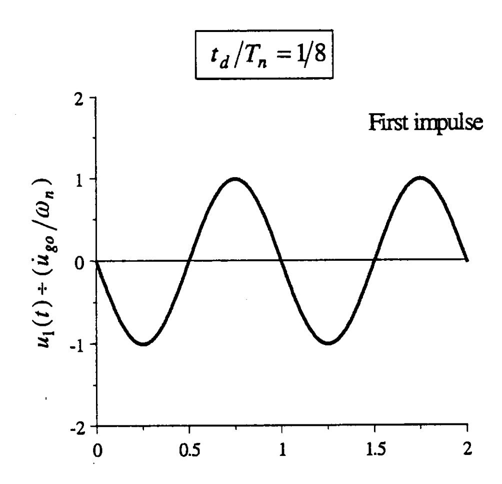
Equation (g) is plotted in Fig. P6.5e.

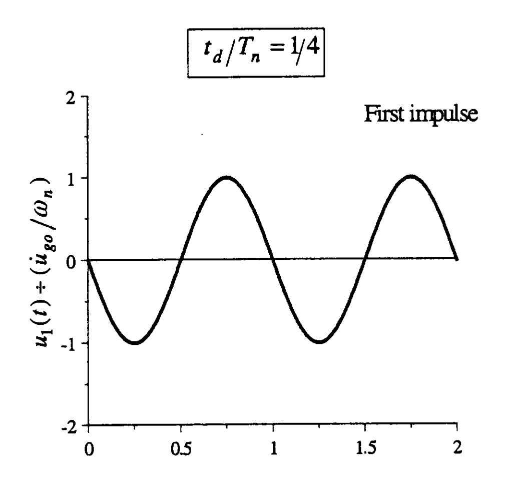
6. *Determine the peak response during* $t \ge t_d$.

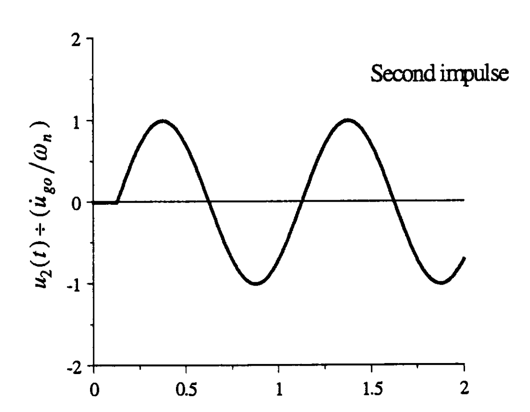
From Eq. (d), the peak deformation u_o during $t \ge t_d$ is given by

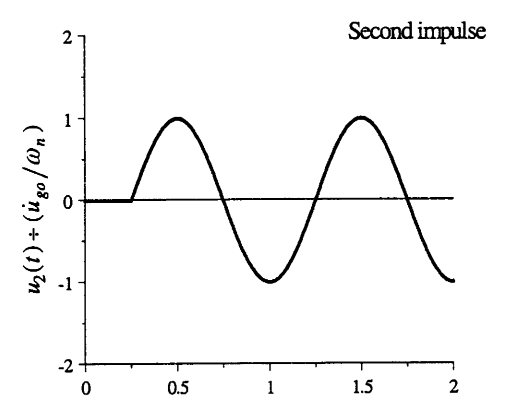
$$\frac{u_o}{\dot{u}_{go} / \omega_n} = 2\left|\sin(\pi t_d / T_n)\right| \tag{h}$$

Equation (h) is plotted in Fig. P6.5e









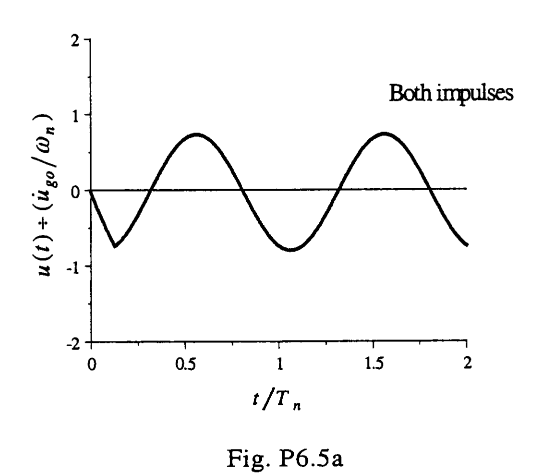


Fig. P6.5a

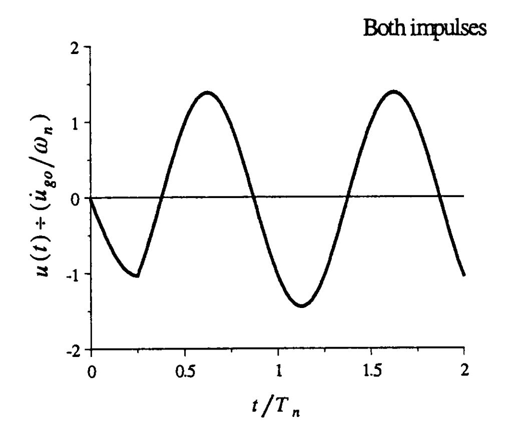
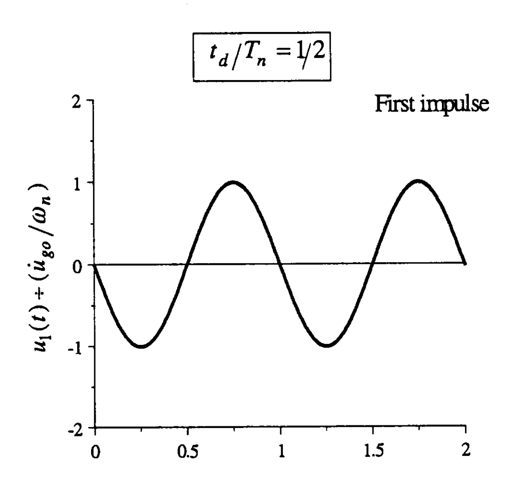
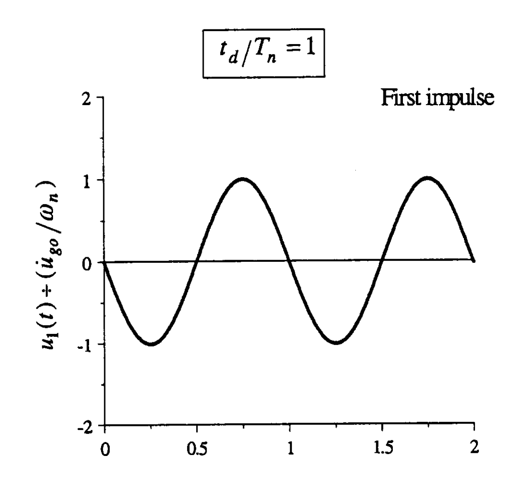
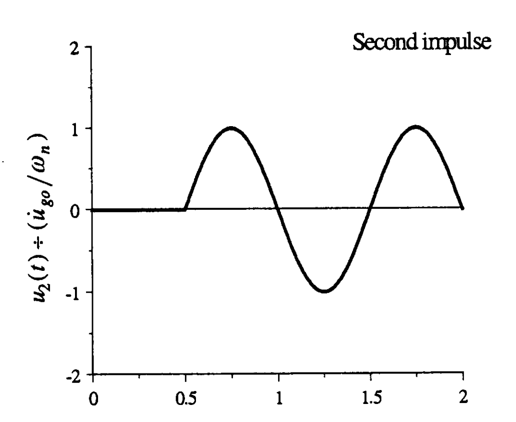
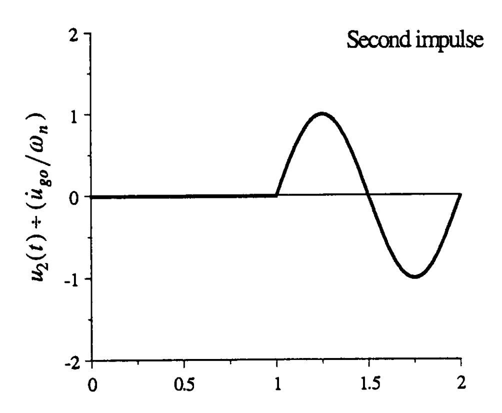


Fig. P6.5b









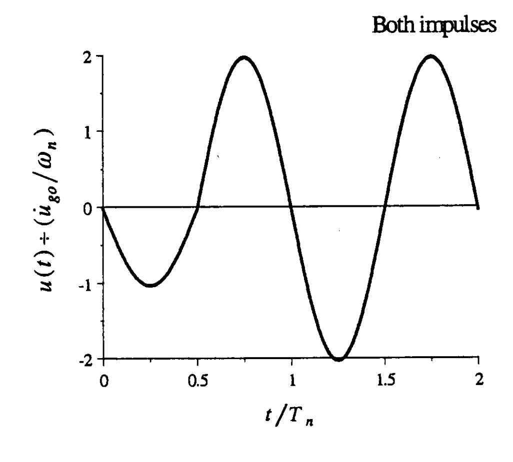


Fig. P6.5c

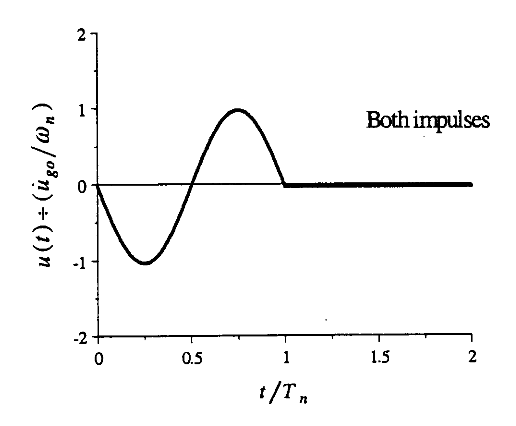


Fig. P6.5d

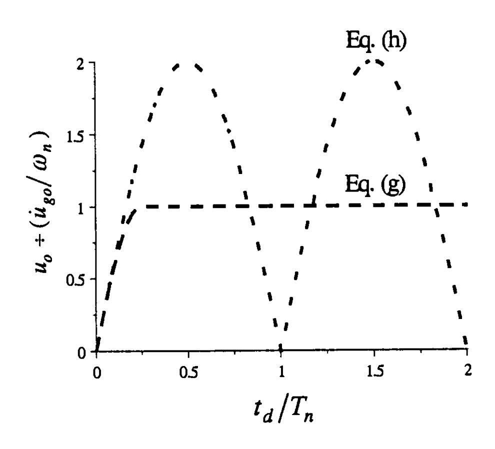


Fig. P6.5e

7. *Determine the overall maximum response.*

From Eqs. (g) and (h), the overall maximum response is given by

$$\frac{u_o}{\dot{u}_{go}/\omega_n} = \begin{cases} 2\left|\sin(\pi t_d/T_n)\right| & t_d/T_n \le 1/4 \\ \max \text{ of } \begin{cases} 1 \\ 2\left|\sin(\pi t_d/T_n)\right| \end{cases} & t_d/T_n \ge 1/4 \end{cases} \tag{i}$$

Equation (i) is plotted t_d/T_n in Fig. P6.5f to obtain the response spectrum.

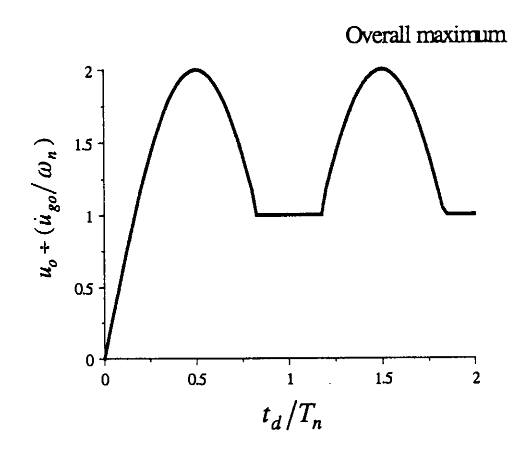


Fig. P6.5f

8. *Determine the pseudo-velocity response spectrum.*

$$\frac{V}{\dot{u}_{go}} = \frac{\omega_n D}{\dot{u}_{go}} = \frac{\omega_n u_o}{\dot{u}_{go}} = \frac{u_o}{\dot{u}_{go}/\omega_n} \tag{j}$$

Thus, the pseudo-velocity response spectrum for $t_d = 0.5$ sec. is given by Eq. (i) with $t_d = 0.5$ sec. This is plotted in Fig. P6.5g

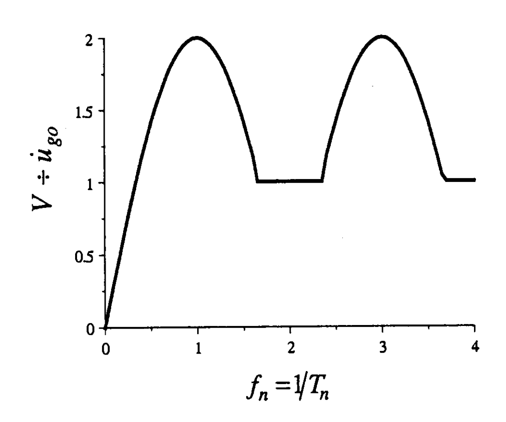


Fig. P6.5.g

Problem 6.6

1. *Determine response to the first impulse.*

The ground motion impulse can be represented by the effective earthquake force,

$$p_{\text{eff}}(t) = -m\ddot{u}_g(t) = -m\dot{u}_{go}\delta(t)$$

The response of the system to the first impulse is the unit impulse response of Eq. (4.1.6) times $-m\dot{u}_{go}$:

$$u_1(t) = -\frac{\dot{u}_{go}}{\omega_n}\sin\omega_n t \qquad \text{(a)}$$

2. *Determine response to second impulse.*

$$u_2(t) = -\frac{\dot{u}_{go}}{\omega_n}\sin\omega_n(t-t_d) \qquad t \ge t_d \qquad \text{(b)}$$

3. *Determine response to both impulses.*

For $0 \le t \le t_d$:

$$u(t) = -\frac{\dot{u}_{go}}{\omega_n}\sin\omega_n t = -\frac{\dot{u}_{go}}{\omega_n}\sin\frac{2\pi t}{T_n} \qquad \text{(c)}$$

For $t \ge t_d$:

$$\begin{aligned} u(t) &= -\frac{\dot{u}_{go}}{\omega_n}[\sin\omega_n t + \sin\omega_n(t-t_d)] \\ &= -\frac{\dot{u}_{go}}{\omega_n}2\cos\frac{\omega_n t_d}{2}\sin\frac{\omega_n(2t-t_d)}{2} \\ &= -\frac{2\dot{u}_{go}}{\omega_n}\left(\cos\frac{\pi t_d}{T_n}\right)\sin\left[2\pi\left(\frac{t}{T_n}-\frac{1}{2}\frac{t_d}{T_n}\right)\right] \end{aligned} \qquad \text{(d)}$$

4. *Plot displacement response.*

Equations (c) and (d) are plotted for $t_d/T_n = 1/8$, 1/4, 1/2, and 1 in Figs. P6.6a, b, c, and d, respectively.

5. *Determine the peak response during* $0 \le t \le t_d$.

The number of peaks in *u(t)* depend on t_d/T_n; the longer the time t_d between the pulses, more such peaks occur. The first peak occurs at $t_o = T_n/4$ with the deformation u_o given by

$$\frac{u_o}{\dot{u}_{go}/\omega_n} = 1 \qquad \text{(e)}$$

Thus t_d must be longer than $T_n/4$ for at least one peak to develop during $0 \le t \le t_d$.

If t_d is shorter than $T_n/4$ no peak will develop during $0 \le t \le t_d$ and the response simply builds up from zero to $u(t_d)$, where

$$\frac{|u(t_d)|}{\dot{u}_{go}/\omega_n} = \sin\frac{2\pi t_d}{T_n} \qquad \text{(f)}$$

The absolute deformation during $0 \le t \le t_d$ is

$$\frac{u_o}{\dot{u}_{go}/\omega_n} = \begin{cases} \sin 2\pi t_d/T_n & t_d/T_n \le 1/4 \\ 1 & t_d/T_n \ge 1/4 \end{cases} \qquad \text{(g)}$$

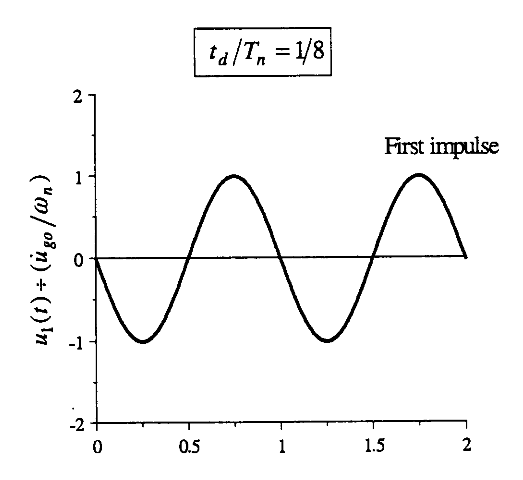
Equation (g) is plotted in Fig. P6.6e.

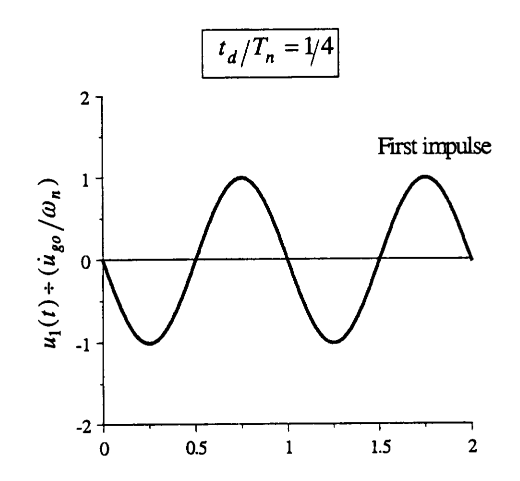
6. *Determine the peak response during* $t \ge t_d$.

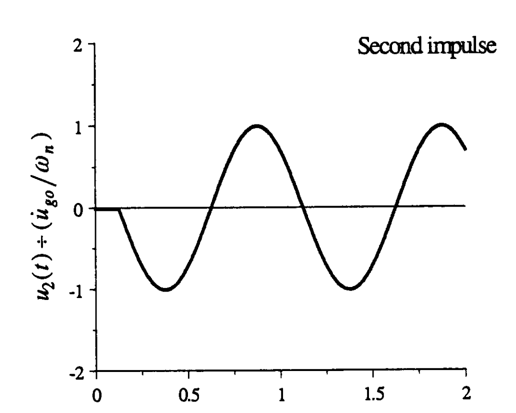
From Eq. (d), the peak deformation u_o during $t \ge t_d$ is given by

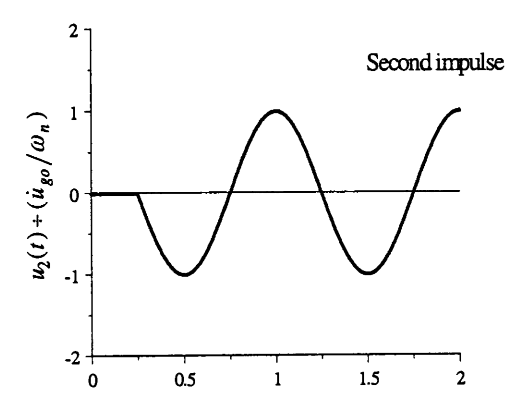
$$\frac{u_o}{\dot{u}_{go}/\omega_n} = 2\left|\cos(\pi t_d/T_n)\right| \qquad \text{(h)}$$

Equation (h) is plotted in Fig. P6.6e









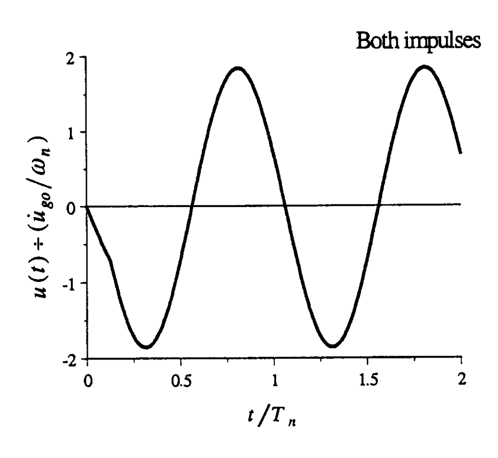


Fig. P6.6a

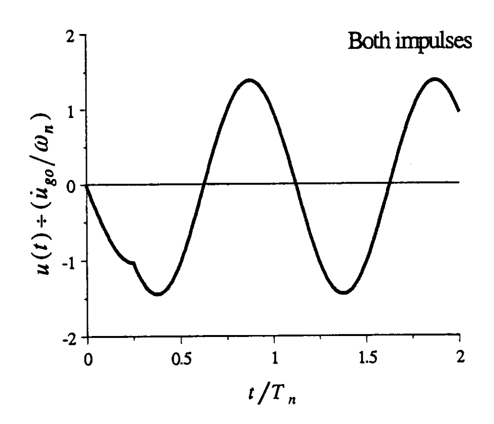
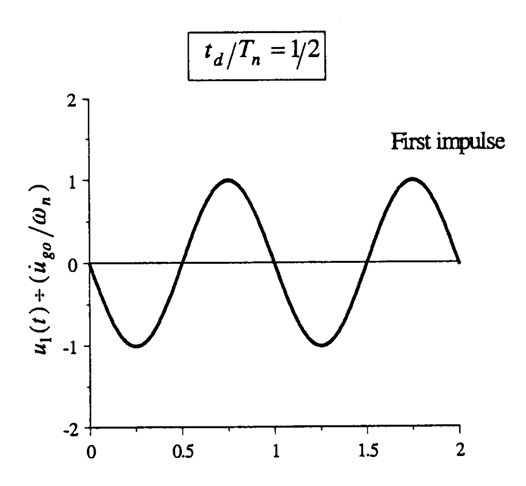
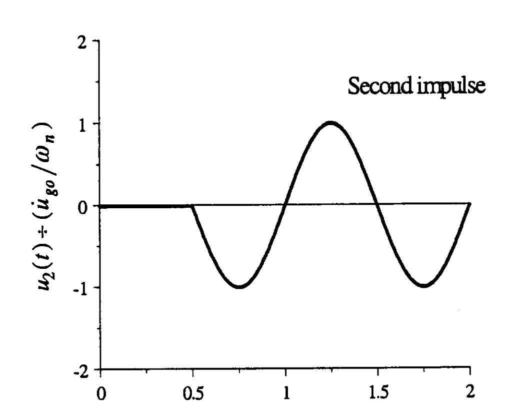


Fig. P6.6b





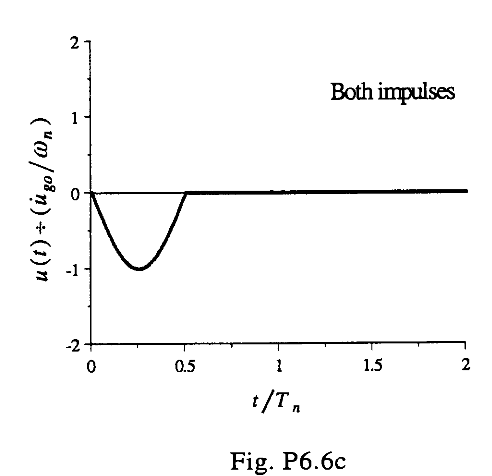
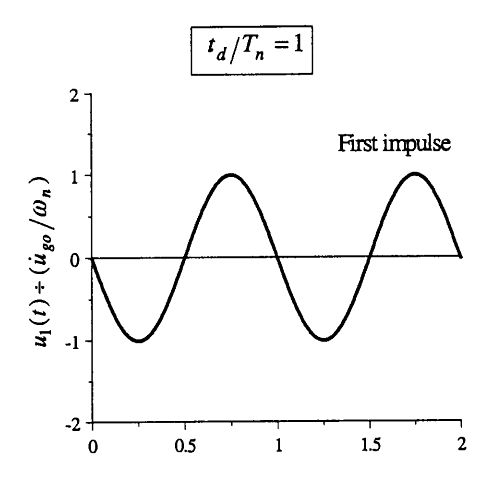
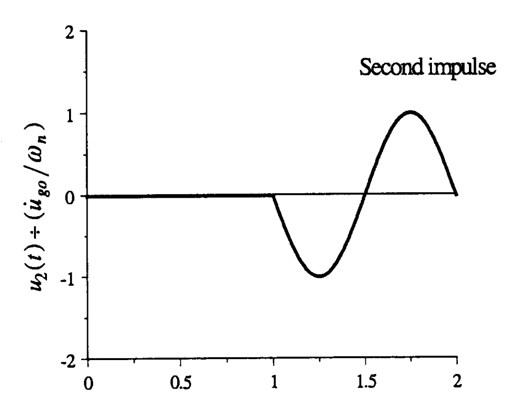


Fig. P6.6c





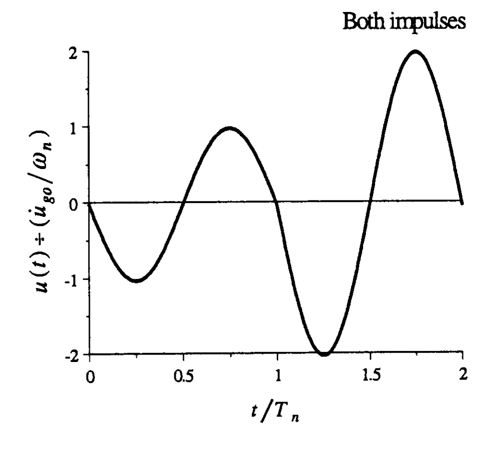


Fig. P6.6d

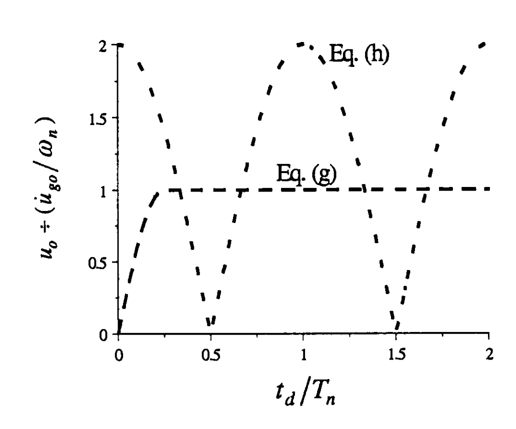


Fig. P6.6e

7. *Determine the overall maximum response.*

From Eqs. (g) and (h), the overall maximum response is given by

$$\frac{u_o}{\dot{u}_{go}/\omega_n} = \begin{cases} 2\left|\cos(\pi t_d/T_n)\right| & t_d/T_n \le 1/4 \\ \text{max. of} \begin{cases} 1 \\ 2\left|\cos(\pi t_d/T_n)\right| \end{cases} & t_d/T_n \ge 1/4 \end{cases} \qquad \text{(i)}$$

Equation (i) is plotted t_d/T_n in Fig. P6.6f to obtain the response spectrum.

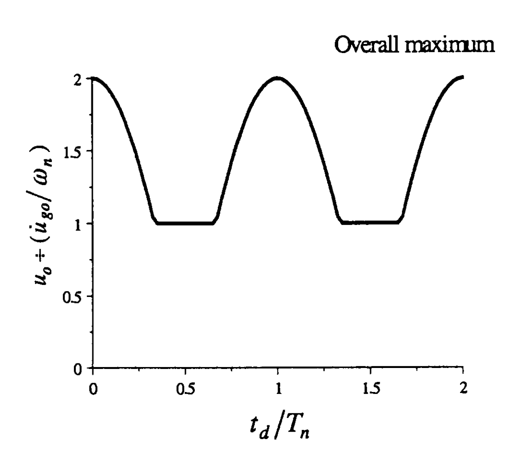


Fig. P6.6f

8. *Determine the pseudo-velocity response spectrum.*

$$\frac{V}{\dot{u}_{go}} = \frac{\omega_n D}{\dot{u}_{go}} = \frac{\omega_n u_o}{\dot{u}_{go}} = \frac{u_o}{\dot{u}_{go}/\omega_n} \qquad \text{(j)}$$

Thus, the pseudo-velocity response spectrum for $t_d = 0.5$ sec. is given by Eq. (i) with $t_d = 0.5$ sec. This is plotted in Fig. P6.6g.

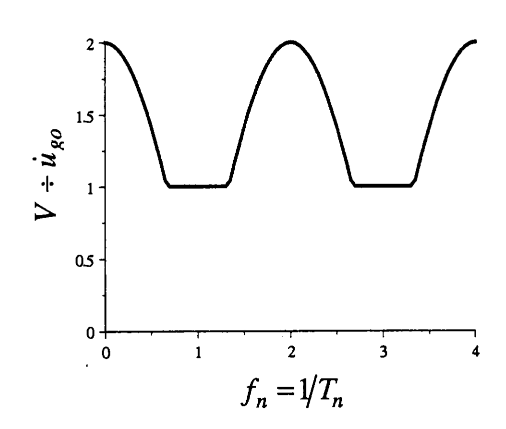


Fig. P6.6g

Problem 6.7

(a) The equation of motion is

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_{go} \sin\left(\frac{2\pi t}{T}\right) = -(p_{\text{eff}})_o \sin\left(\frac{2\pi t}{T}\right) \tag{a}$$

Substituting $(u_{st})_o = (p_{eff})_o/k = \ddot{u}_{go}/\omega_n^2$ and $\omega/\omega_n = T_n/T$ in Eq. (3.2.11) gives the peak deformation

$$D = \frac{\ddot{u}_{go}}{\omega_n^2} \frac{1}{\sqrt{\left[1 - (T_n/T)^2\right]^2 + \left[2\zeta T_n/T\right]^2}} \tag{b}$$

The peak pseudo-acceleration $A = \omega_n^2 D$ is

$$A = \ddot{u}_{go} \frac{1}{\sqrt{\left[1 - (T_n/T)^2\right]^2 + \left[2\zeta T_n/T\right]^2}} \tag{c}$$

Substituting $\omega/\omega_n = T_n/T$ in Eq. (3.6.4) gives the true acceleration

$$\ddot{u}_o^t = \ddot{u}_{go} \left\{ \frac{1 + \left(2\zeta T_n/T\right)^2}{\left[1 - (T_n/T)^2\right]^2 + \left[2\zeta T_n/T\right]^2} \right\}^{1/2} \tag{d}$$

(b) For $\zeta = 0$, Eqs. (c) and (d) become identical:

$$A = \ddot{u}_{go} \frac{1}{\left|1 - (T_n/T)^2\right|}; \quad \ddot{u}_o^t = \ddot{u}_{go} \frac{1}{\left|1 - (T_n/T)^2\right|} \tag{e}$$

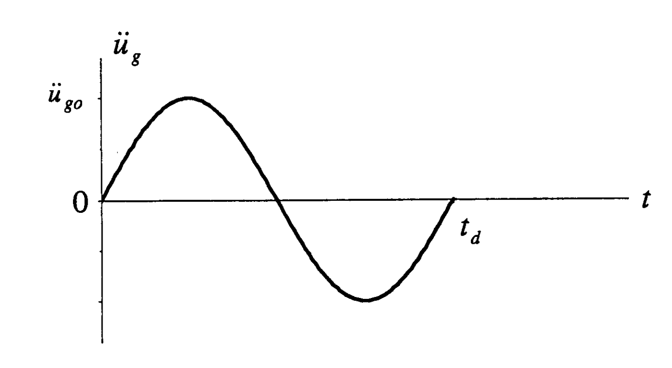
(c) Figure 3.2.6a with the abscissa ω/ω_n replaced by T_n/T and the ordinate by $A/\ddot{u}_{go}$ gives the pseudo-acceleration response spectrum. Figure 3.5.1 with the abscissa ω/ω_n replaced by T_n/T gives the true-acceleration response spectrum.

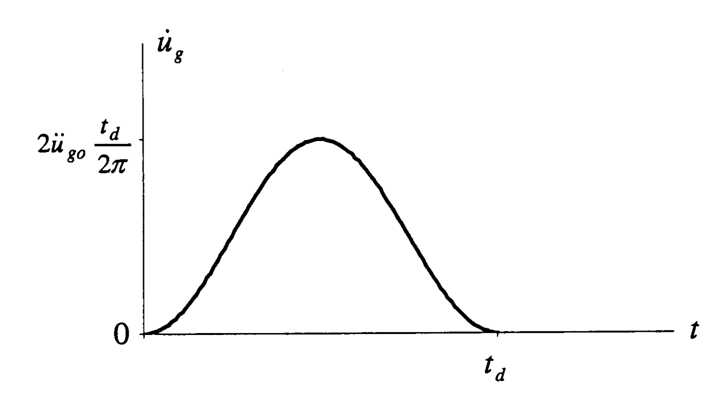
Problem 6.8

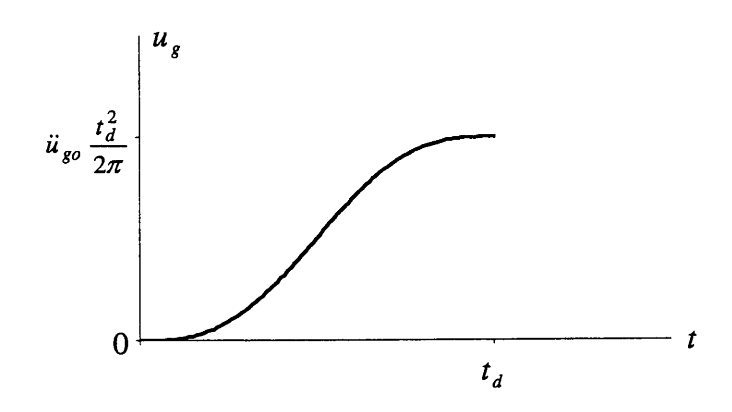
The equation of motion to be solved is

$$m\ddot{u} + ku = p_{\text{eff}}(t) = \begin{cases} -m\ddot{u}_{go}\sin(2\pi t/t_d) & t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(a)}$$

with at rest initial conditions.







1. *Determine response u(t).*

Case 1: $t_d / T_n \neq 1$

Forced Vibration Phase

The response solution is adapted from Eq. (3.1.6b). Substituting $\omega = 2\pi / t_d$, $\omega_n = 2\pi / T_n$, $p_o = m\ddot{u}_{go}$ and including a minus sign in Eq. (3.1.6b) gives

$$\frac{u(t)}{\ddot{u}_{go}/\omega_n^2} = -\frac{1}{1-(T_n/t_d)^2}\left[\sin\left(2\pi\frac{t}{t_d}\right) - \frac{T_n}{t_d}\sin\left(2\pi\frac{t}{T_n}\right)\right] \qquad t \le t_d \qquad \text{(b)}$$

Free Vibration Phase

The motion is described by Eq. (4.7.3) with $u(t_d)$ and $\dot{u}(t_d)$, determined from Eq. (b) :

$$u(t_d) = \frac{\ddot{u}_{go}}{\omega_n^2}\frac{1}{1-(T_n/t_d)^2}\frac{T_n}{t_d}\sin\left(2\pi\frac{t_d}{T_n}\right) \qquad \text{(c.1)}$$

$$\dot{u}(t_d) = \frac{\ddot{u}_{go}}{\omega_n^2}\frac{-1}{1-(T_n/t_d)^2}\frac{2\pi}{t_d}\left[1-\cos\left(2\pi\frac{t_d}{T_n}\right)\right] \qquad \text{(c.2)}$$

Substituting Eq. (c) in Eq. (4.7.3) gives

$$\frac{u(t)}{\ddot{u}_{go}/\omega_n^2} = \frac{1}{1-(T_n/t_d)^2}\frac{T_n}{t_d}\left[\sin\left(\frac{2\pi}{T_n}t_d\right)\cos\frac{2\pi}{T_n}(t-t_d) - \sin\frac{2\pi}{T_n}(t-t_d) + \cos\left(\frac{2\pi}{T_n}t_d\right)\sin\frac{2\pi}{T_n}(t-t_d)\right]$$

$$= \frac{-(T_n/t_d)}{1-(T_n/t_d)^2}\left[\sin\frac{2\pi}{T_n}(t-t_d) - \sin\left(\frac{2\pi}{T_n}t\right)\right]$$

$$= \frac{2(T_n/t_d)\sin(\pi t_d/T_n)}{1-(T_n/t_d)^2}\cos\left[2\pi\left(\frac{t}{T_n} - \frac{1}{2}\frac{t_d}{T_n}\right)\right] \qquad t \ge t_d \qquad \text{(d)}$$

Case 2: $t_d / T_n = 1$

Forced Vibration Phase

The forced response is now given by Eq. (3.1.13b)

$$\frac{u(t)}{\ddot{u}_{go}/\omega_n^2} = -\frac{1}{2}\left[\sin\frac{2\pi t}{T_n} - \frac{2\pi t}{T_n}\cos\frac{2\pi t}{T_n}\right] \quad t \le t_d \qquad \text{(e)}$$

Free Vibration Phase

From Eq. (e) determine

$$u(t_d) = \pi\frac{\ddot{u}_{go}}{\omega_n^2} \quad \text{and} \quad \dot{u}(t_d) = 0 \qquad \text{(f)}$$

The second equation implies that the displacement in the forced vibration phase reaches its maximum at the end of this phase. Substituting Eq. (f) in Eq. (4.7.3) gives

$$\frac{u(t)}{\ddot{u}_{go}/\omega_n^2} = \pi\cos 2\pi\left(\frac{t}{T_n} - 1\right)$$

$$= \pi\cos 2\pi\frac{t}{T_n} \qquad t \ge t_d \qquad \text{(g)}$$

2. *Plot response history.*

The time variation of the normalized deformation, $u(t)/(\ddot{u}_{go}/\omega_n^2)$, given by Eqs. (b) and (d), is plotted in Fig. P6.8a for several values of t_d / T_n. For the special case of $t_d / T_n = 1$, Eqs. (e) and (g) describe the response of the system and these are also plotted in Fig. P6.8a. The static solution is included in these figures.

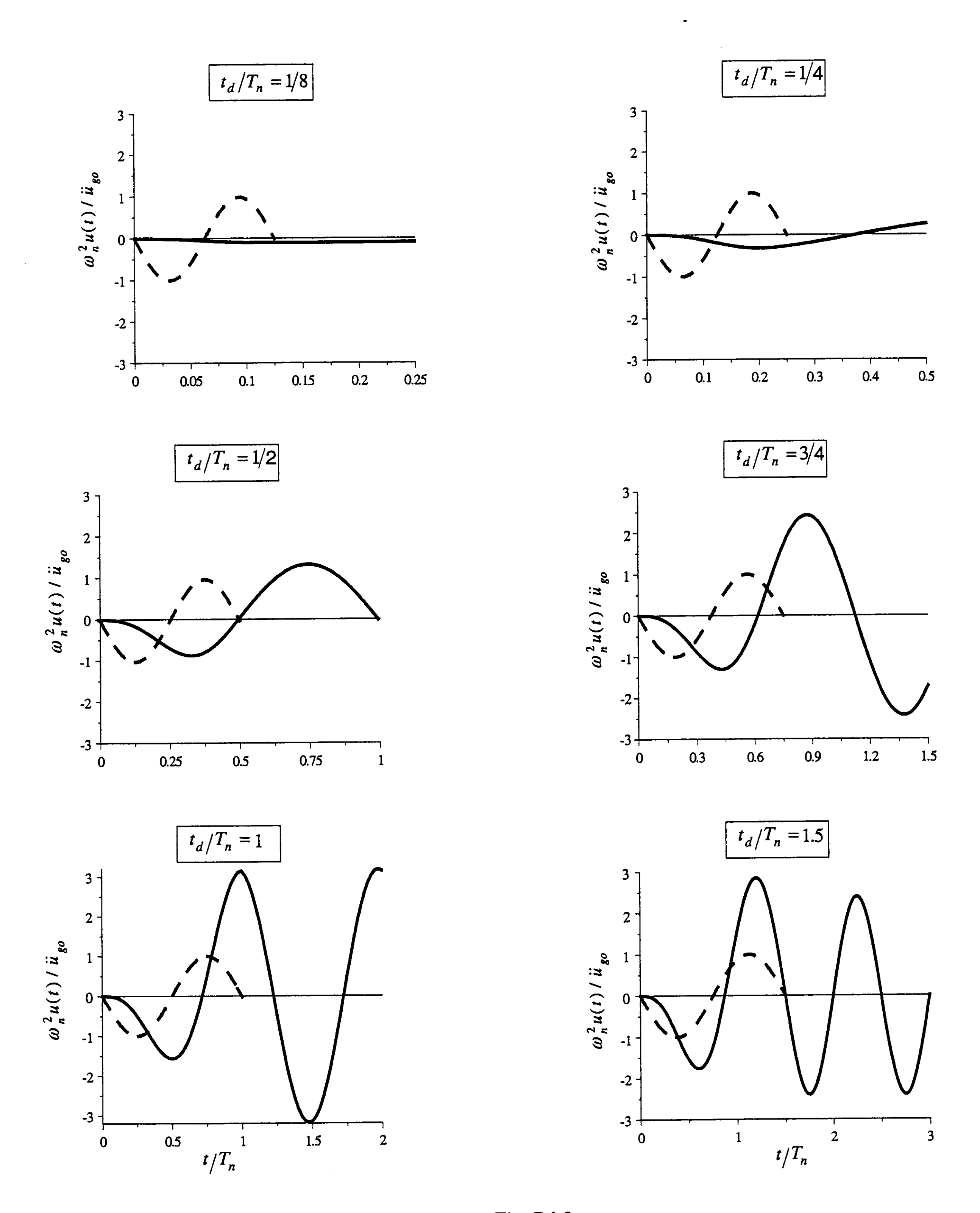

$t_d/T_n = 1/8$
$t_d/T_n = 1/4$
$t_d/T_n = 1/2$
$t_d/T_n = 3/4$
$t_d/T_n = 1$
$t_d/T_n = 1.5$
$\omega_n^2 u(t) / \ddot{u}_{go}$
t/T_n

Fig. P6.8a

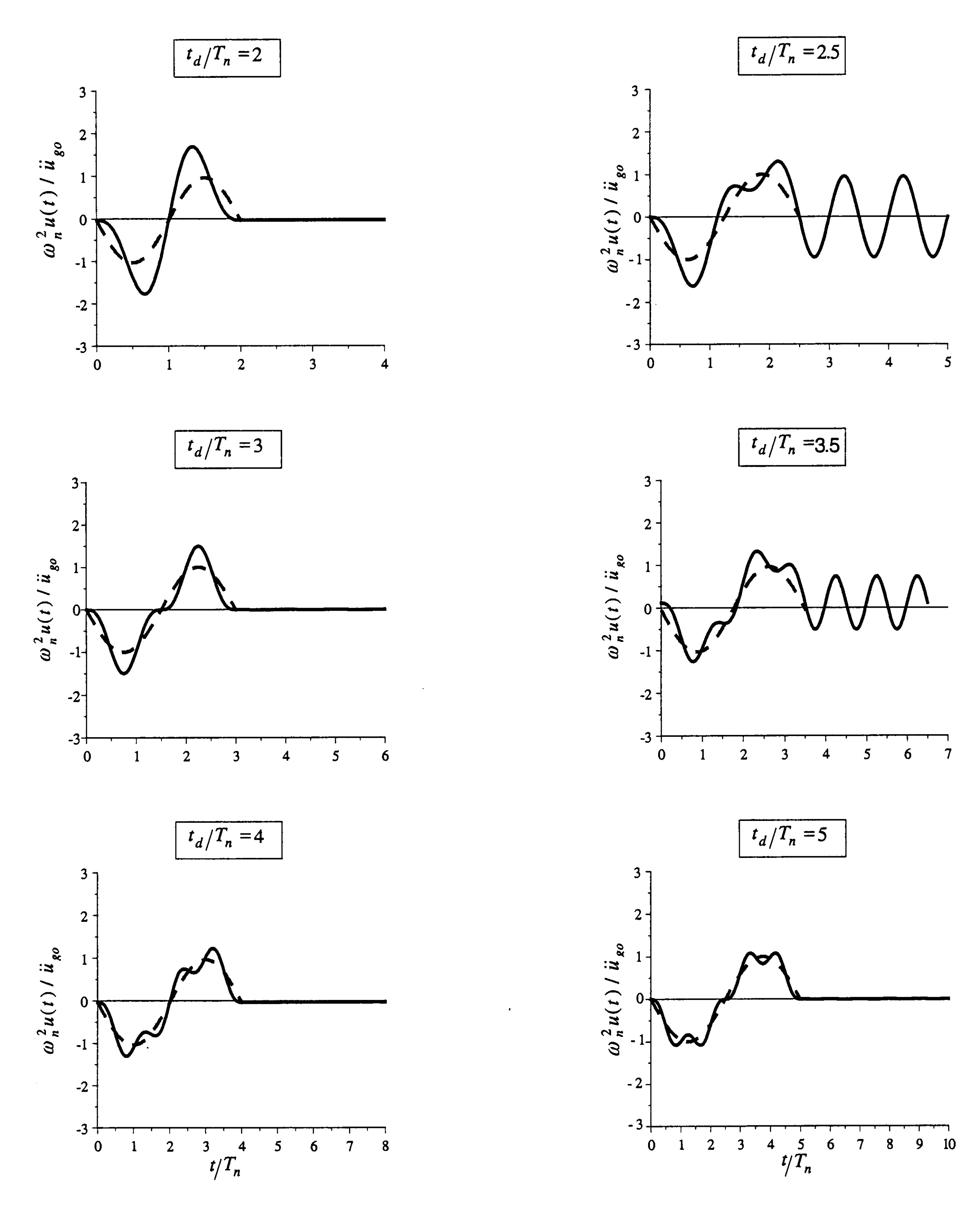


Fig. P6.8a (continued)

3. *Determine maximum response.*

The pseudo acceleration

$$A = \omega_n^2 u_o \tag{h}$$

will be determined from $u_o \equiv \max_t |u(t)|$.

During the forced vibration phase, the number of local maxima and minima depends on t_d / T_n ; the longer the pulse duration, more such peaks occur. These peaks occur at time instants t_0 when $\dot{u}(t) = 0$. This condition applied to Eq. (b) gives

$$\cos\frac{2\pi t_0}{t_d} = \cos\frac{2\pi t_0}{T_n}$$

$$(t_0)_l = \frac{l}{1+(t_d/T_n)} t_d \qquad l=1, 2, 3...... \tag{i}$$

Only those l for which $(t_0)_l < t_d$ are relevant. Substituting Eq. (i) into Eq. (b) and using Eq. (h) gives

$$\frac{A}{\ddot{u}_{go}} = \left| \frac{1}{1-(T_n/t_d)^2} * \left[\sin\frac{2\pi l}{1+(t_d/T_n)} - \frac{T_n}{t_d}\sin\frac{2\pi l}{1+(T_n/t_d)} \right] \right| \tag{j}$$

At least one local maximum occurs during the ground acceleration pulse, irrespective of the t_d / T_n value. If $t_d / T_n > 1/2$ the displacement reverses in sign during the excitation and has a negative value at the end of the excitation. If $t_d / T_n > 1$ a local minimum develops during the ground acceleration pulse. If $t_d / T_n > 2$ more than one local maximum and/or more than one local minimum may develop. We define

$$u_{max} = \max_t u(t) \qquad u_{min} = \min_t u(t)$$

Figure P6.8b shows $u_{max}/(u_{st})_o$ and $-u_{min}/(u_{st})_o$ plotted as a function of t_d / T_n. The response spectrum for the larger of the two values during the acceleration pulse is shown as 'Forced Response' in Figure P6.8c.

During the free vibration phase, the response is given by Eq. (d) and its amplitude is u_o; using Eq. (h) gives

$$\frac{A}{\ddot{u}_{go}} = \left| \frac{2(T_n/t_d)\sin(\pi t_d/T_n)}{(T_n/t_d)^2 - 1} \right| \tag{k}$$

This equation is plotted in Fig. P6.8c.

For the special case of $t_d / T_n = 1$, the maximum response during the forced vibration can be determined from Eq. (e):

$$\dot{u}(t) = 0 \Rightarrow t_0 = T_n \quad \text{and}$$

$$\frac{\omega_n^2 u(t_0)}{\ddot{u}_{go}} = \pi \Rightarrow \frac{A}{\ddot{u}_{go}} = \pi \tag{l}$$

Similarly, the maximum response during free vibration can be determined from Eq. (g):

$$\frac{A}{\ddot{u}_{go}} = \pi \tag{m}$$

The overall maximum response is the larger of the two maxima determined separately for the forced and free vibration phases. Figure P6.8c shows that if $t_d > T_n$, the overall maximum is the largest peak that develops during the force pulse. On the other hand, if $t_d < T_n$, the overall maximum is given by the peak response during the free vibration phase. For the special case of $t_d = T_n$, as mentioned earlier, the two individual maxima are equal. The overall maximum response is plotted against t_d / T_n in Fig. P6.3d; for each t_d / T_n it is the larger of the two plots in Fig. P6.8c. This is the pseudo acceleration response spectrum for the full-cycle sine pulse ground motion.

4. *True Acceleration Response Spectrum.*

As shown in Section 6.3, for undamped systems:

$$\ddot{u}_o^t = \omega_n^2 u_o$$

Thus the true acceleration response spectrum is also given by Fig. P6.8b-d with the ordinate axis showing $\ddot{u}_o^t / \ddot{u}_{go}$.

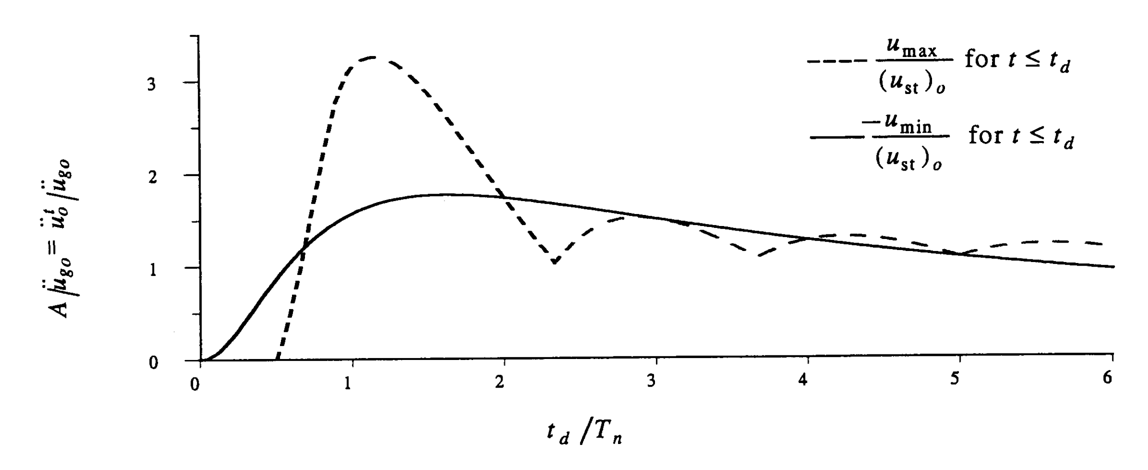


Fig. P6.8b

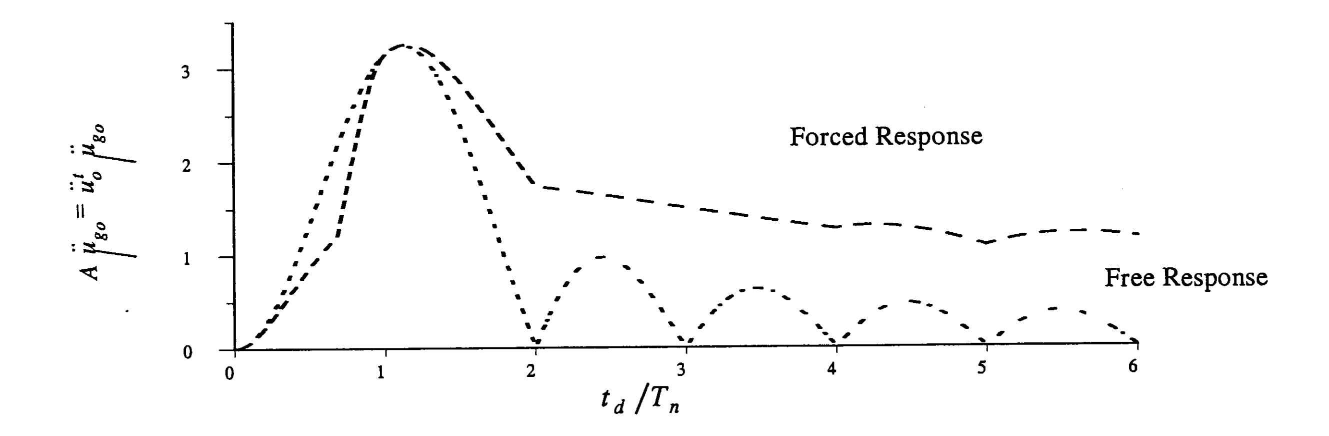


Fig. P6.8c

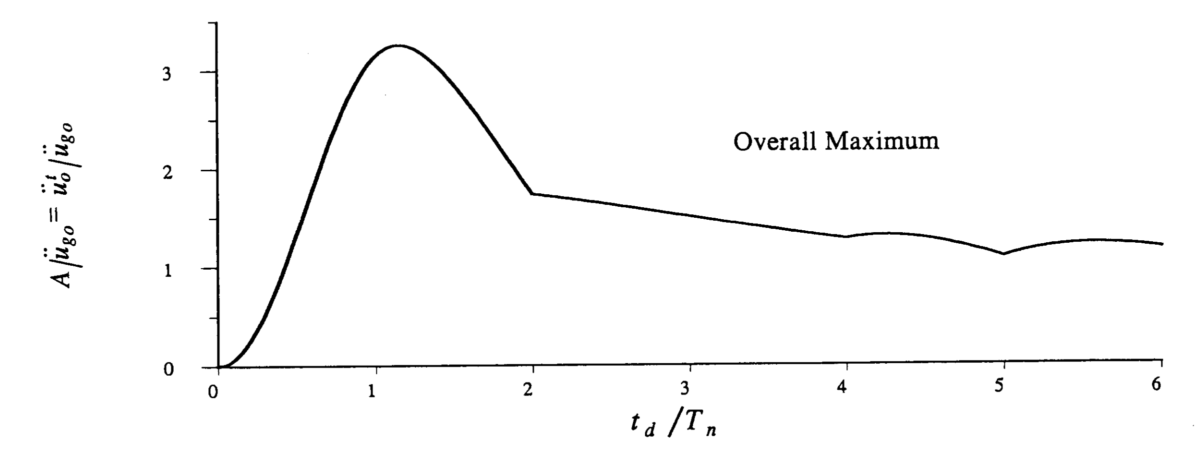


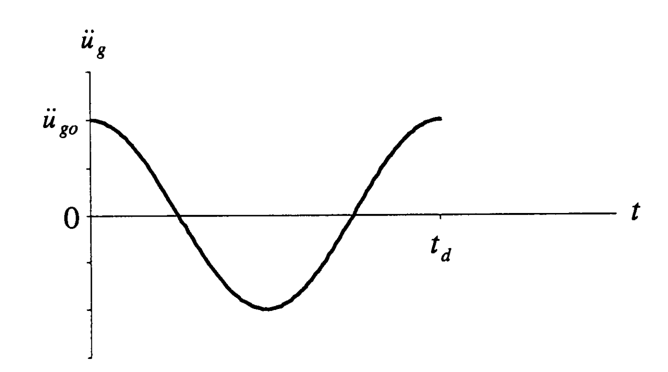
Fig. P6.8d

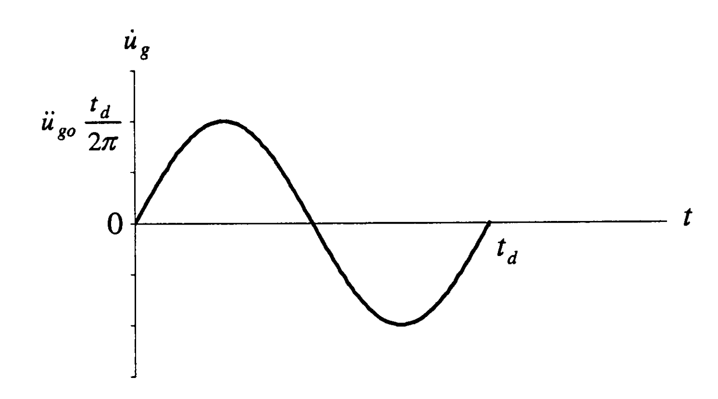
Problem 6.9

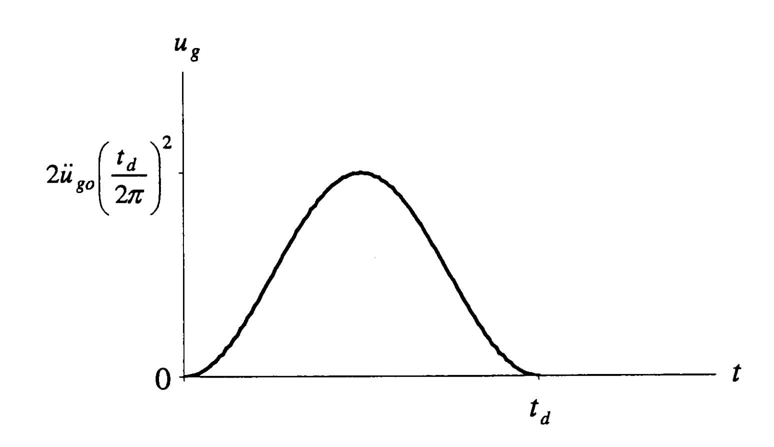
The equation of motion to be solved is

$$m\ddot{u} + ku = p_{\text{eff}}(t) = \begin{cases} -m\ddot{u}_{go}\cos(2\pi t/t_d) & t \le t_d \\ 0 & t \ge t_d \end{cases} \tag{a}$$

with at rest initial conditions.







1. *Determine response u(t).*

Case 1: $t_d/T_n \ne 1$

Forced Vibration Phase

The response solution of Eq. (a) is

$$\frac{u(t)}{\ddot{u}_{go}/\omega_n^2} = \frac{1}{1-(T_n/t_d)^2}\left[\cos\left(2\pi\frac{t}{T_n}\right) - \cos\left(2\pi\frac{t}{t_d}\right)\right] \quad t \le t_d \tag{b}$$

Free Vibration Phase

The motion is described by Eq. (4.7.3) with $u(t_d)$ and $\dot{u}(t_d)$ determined from Eq. (b) :

$$u(t_d) = \frac{\ddot{u}_{go}}{\omega_n^2}\frac{1}{1-(T_n/t_d)^2}\left[\cos\left(2\pi\frac{t_d}{T_n}\right) - 1\right] \tag{c.1}$$

$$\dot{u}(t_d) = \frac{\ddot{u}_{go}}{\omega_n}\frac{-1}{1-(T_n/t_d)^2}\sin\left(2\pi\frac{t_d}{T_n}\right) \tag{c.2}$$

Substituting Eq. (c) in Eq. (4.7.3) gives

$$\frac{u(t)}{\ddot{u}_{go}/\omega_n^2} = \frac{1}{1-(T_n/t_d)^2}\left\{\left[\cos\left(\frac{2\pi}{T_n}t_d\right) - 1\right]\cos\frac{2\pi}{T_n}(t-t_d) - \sin\left(\frac{2\pi}{T_n}t_d\right)\sin\frac{2\pi}{T_n}(t-t_d)\right\}$$

$$= \frac{-1}{1-(T_n/t_d)^2}\left[-\cos\frac{2\pi}{T_n}t_d + \cos\frac{2\pi}{T_n}(t-t_d)\right]$$

$$= -\frac{2\sin(\pi t_d/T_n)}{1-(T_n/t_d)^2}\sin\left[2\pi\left(\frac{t}{T_n} - \frac{1}{2}\frac{t_d}{T_n}\right)\right] \quad t \ge t_d \tag{d}$$

Case 2: $t_d/T_n = 1$

Forced Vibration Phase

The forced response is

$$\frac{u(t)}{\ddot{u}_{go}/\omega_n^2} = -\frac{\pi t}{T_n}\sin\frac{2\pi t}{T_n} \quad t \le t_d \tag{e}$$

Free Vibration Phase

From Eq. (e) determine

$$u(t_d) = 0 \quad \text{and} \quad \dot{u}(t_d) = -\frac{\ddot{u}_{go} t_d}{2} \qquad \text{(f)}$$

The second equation implies that the displacement in the forced vibration phase reaches its maximum at the end of this phase. Substituting Eq. (f) in Eq. (4.7.3) gives

$$\frac{u(t)}{\ddot{u}_{go}/\omega_n^2} = -\pi \sin 2\pi\left(\frac{t}{T_n} - 1\right)$$

$$= -\pi \sin 2\pi \frac{t}{T_n} \qquad t \geq t_d \qquad \text{(g)}$$

2. *Plot response history.*

The time variation of the normalized deformation, $u(t)/(\ddot{u}_{go}/\omega_n^2)$, given by Eqs. (b) and (d), is plotted in Fig. P6.9a for several values of t_d / T_n. For the special case of $t_d / T_n = 1$, Eqs. (e) and (g) describe the response of the system and these are also plotted in Fig. P6.9a. The static solution is included in these figures.

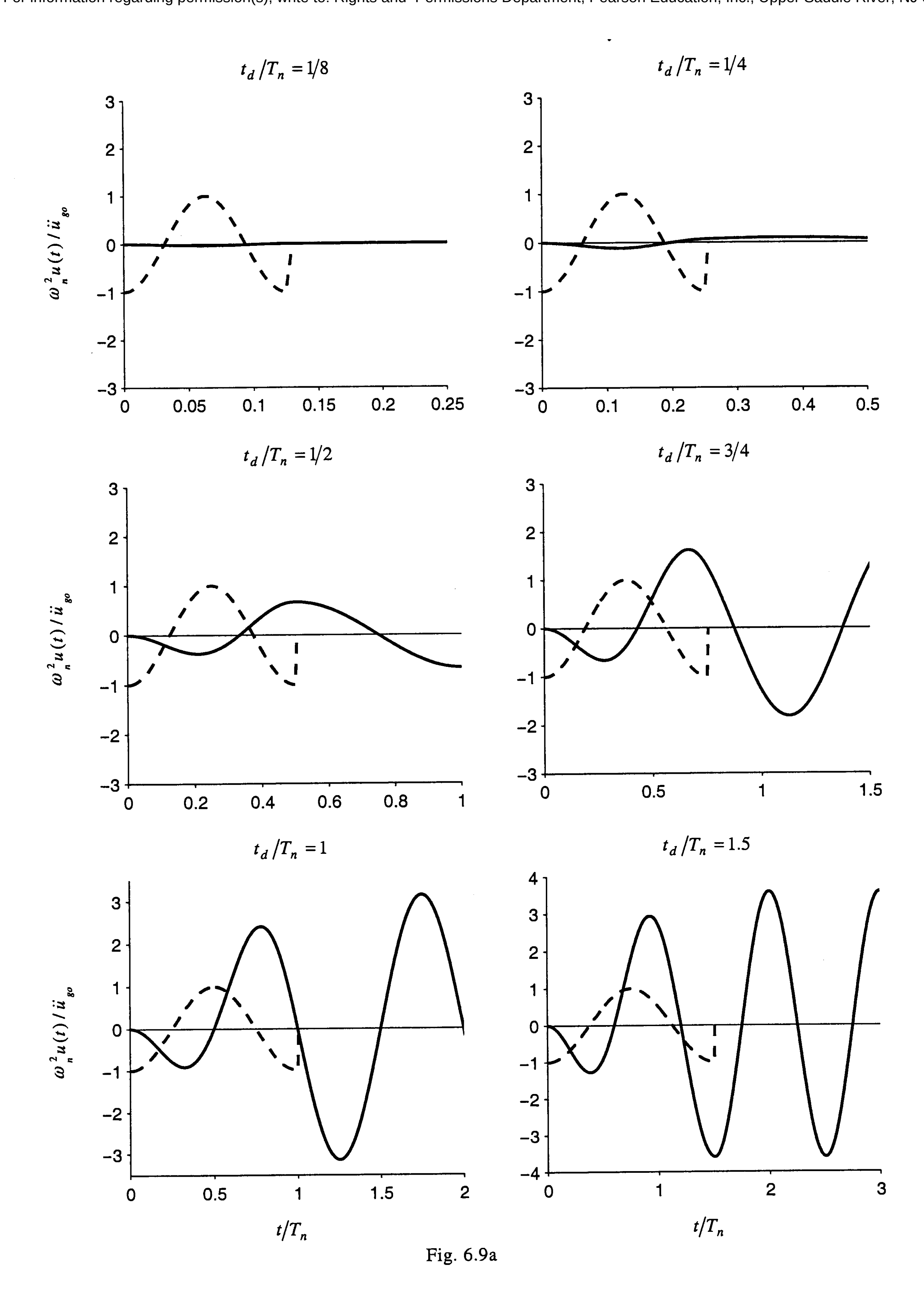

$t_d/T_n = 1/8$
$t_d/T_n = 1/4$
$t_d/T_n = 1/2$
$t_d/T_n = 3/4$
$t_d/T_n = 1$
$t_d/T_n = 1.5$
$\omega_n^2 u(t) / \ddot{u}_{go}$
t/T_n

Fig. 6.9a

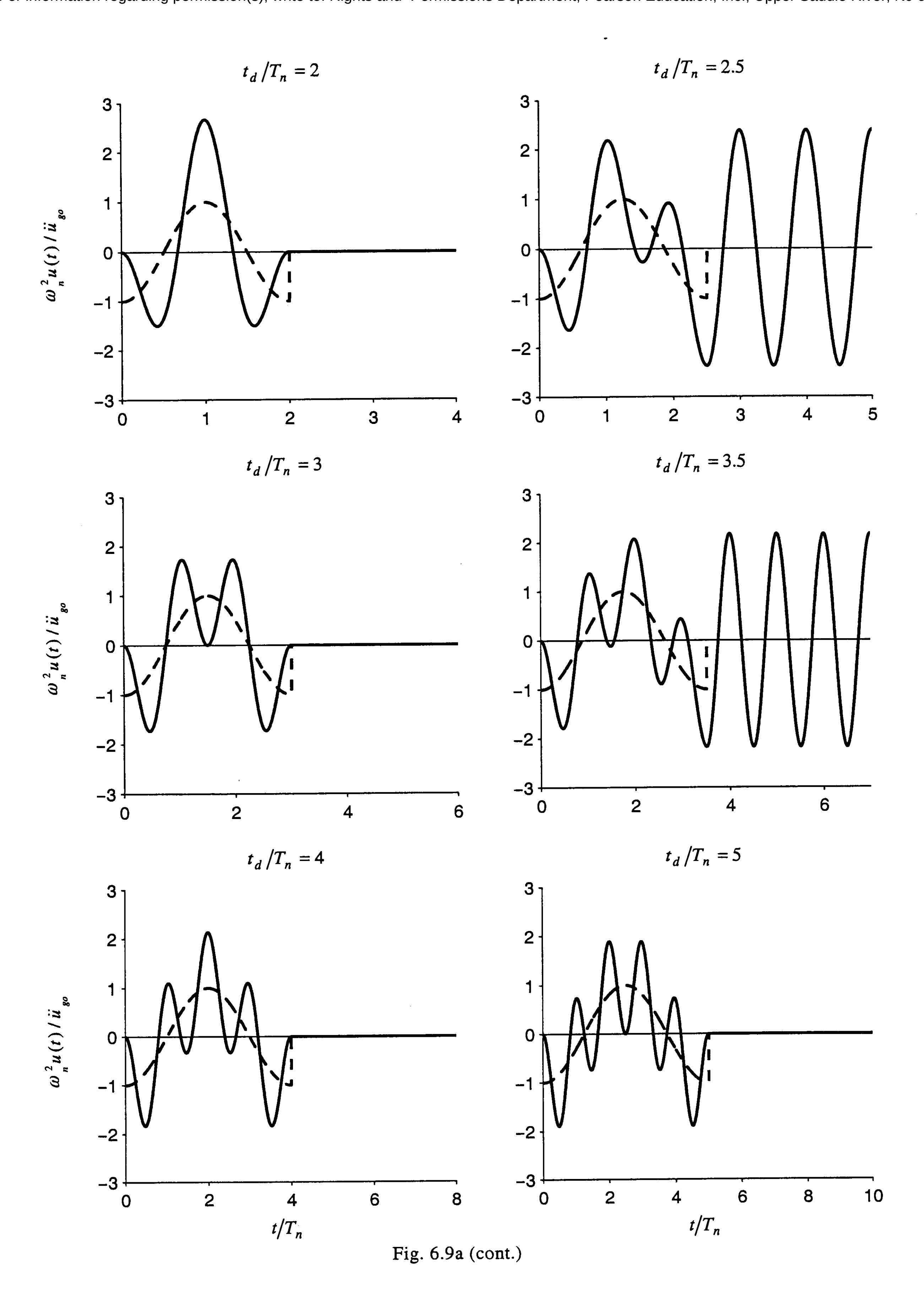

$t_d/T_n = 2$
$t_d/T_n = 2.5$
$t_d/T_n = 3$
$t_d/T_n = 3.5$
$t_d/T_n = 4$
$t_d/T_n = 5$
$\omega_n^2 u(t) / \ddot{u}_{go}$
t/T_n

Fig. 6.9a (cont.)

3. *Determine maximum response.*

The pseudo acceleration

$$A = \omega_n^2 u_o \qquad \text{(h)}$$

will be determined from $u_o \equiv \max_t |u(t)|$. We define

$$u_{\max} = \max_t \, u(t) \qquad u_{\min} = \min_t u(t)$$

Figure P6.9b shows $u_{\max}/(u_{st})_o$ and $-u_{\min}/(u_{st})_o$ plotted as a function of t_d / T_n. The response spectrum for the larger of the two values during the acceleration pulse is shown as "Forced Response" in Fig. P6.9c.

During the free vibration phase, the response is given by Eq. (d) and its amplitude is u_o; using Eq. (h) gives

$$\frac{A}{\ddot{u}_{go}} = \left| \frac{2\sin(\pi t_d / T_n)}{(T_n / t_d)^2 - 1} \right| \qquad \text{(i)}$$

This equation is plotted in Fig. P6.9c.

For the special case of $t_d / T_n = 1$, the maximum response during the forced vibration can be determined from Eq. (e):

$$\dot{u}(t) = 0 \Rightarrow t_0 = 3T_n / 4 \quad \text{and}$$

$$\frac{\omega_n^2 u(t_0)}{\ddot{u}_{go}} = 3\pi / 4 \Rightarrow \frac{A}{\ddot{u}_{go}} = 3\pi / 4 \qquad \text{(j)}$$

Similarly, the maximum response during free vibration can be determined from Eq. (g):

$$\frac{A}{\ddot{u}_{go}} = \pi \qquad \text{(k)}$$

The overall maximum response is the larger of the two maxima determined separately for the forced and free vibration phases. The overall maximum response is plotted against t_d / T_n in Fig. P6.9d; for each t_d / T_n it is the larger of the two plots in Fig. P6.9c. This is the pseudo-acceleration response spectrum for the full-cycle cosine pulse ground motion.

4. *True acceleration response spectrum.*

As shown in Section 6.3, for undamped systems:

$$\ddot{u}_o^t = \omega_n^2 u_o$$

Thus the true acceleration response spectrum is also given by Figs. P6.9b-d with the ordinate axis showing $\ddot{u}_o^t / \ddot{u}_{go}$.

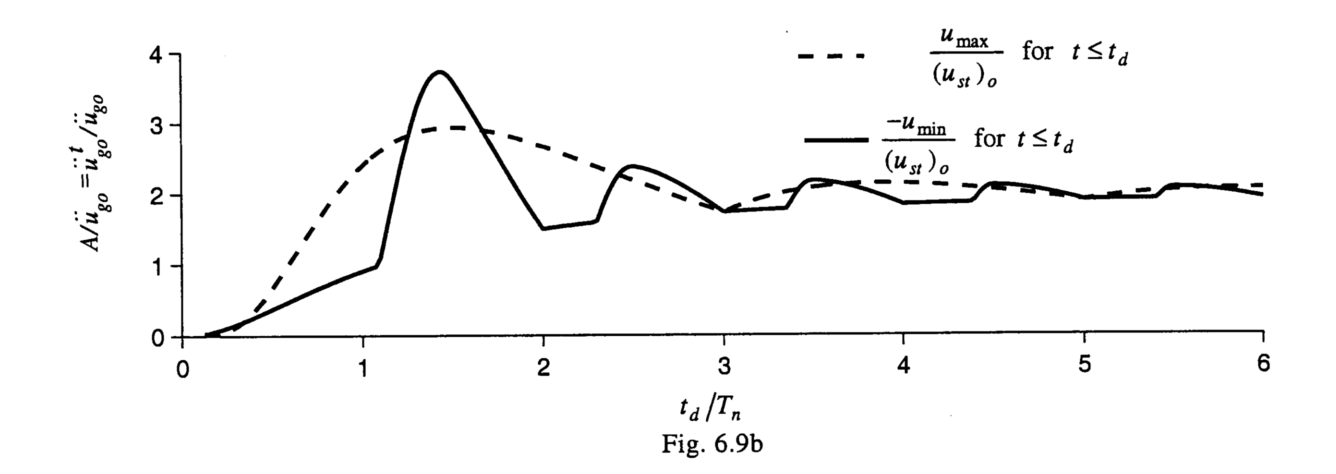


Fig. 6.9b

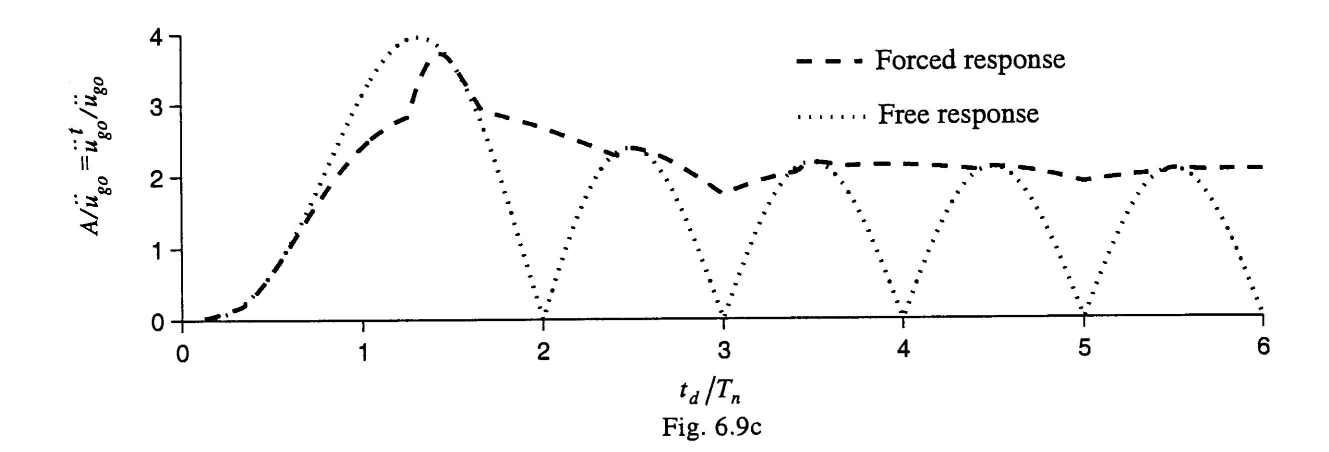


Fig. 6.9c

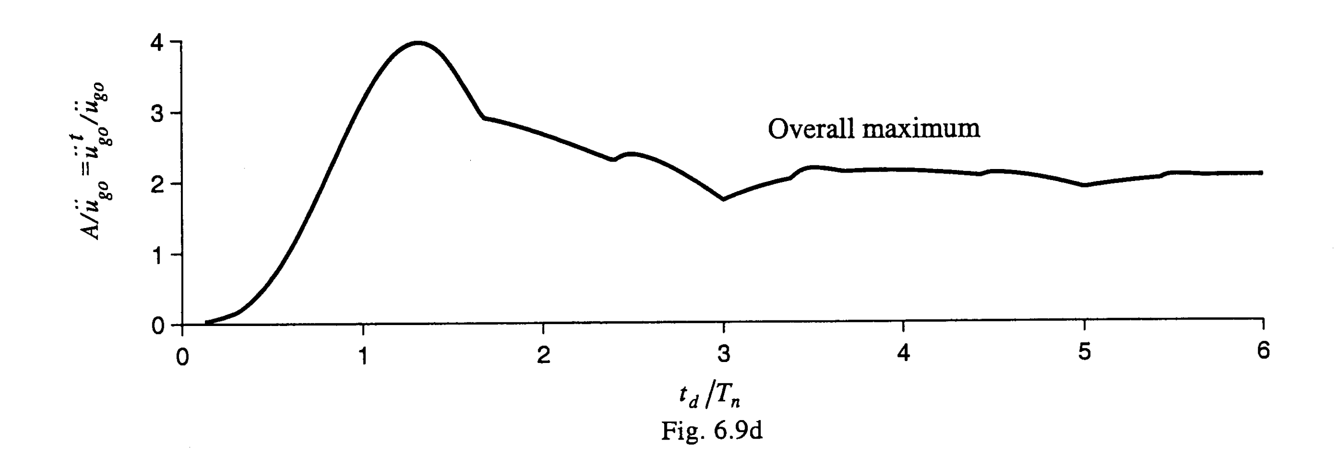


Fig. 6.9d

Problem 6.10

The lateral stiffness of the SDF system is

$$k = \frac{3EI}{L^3} = \frac{3\,(29 \cdot 10^3)\,28.1}{(10.12)^3} = 1.415 \text{ kips/in.}$$

The total weight of the pipe is $18.97 \times 10 = 189.7$ lbs, which may be neglected relative to the lumped weight. Thus

$$m = \frac{w}{g} = \frac{3}{386.4} = 0.0078 \text{ kip}-\text{sec}^2/\text{in.}$$

The natural frequency and period are

$$\omega_n = \sqrt{k/m} = 13.47 \text{ rads/sec}$$

$$T_n = 2\pi/\omega_n = 0.47 \text{ sec}$$

At $T_n = 0.47$ sec the response spectrum curve for $\zeta = 5\%$ gives $A = 0.9g$.

The peak value of deformation is

$$u_o = \frac{A}{\omega_n^2} = \frac{0.9\,(386)}{(13.47)^2} = 1.91 \text{ in.}$$

The peak value of the equivalent static force is

$$f_{So} = mA = w\left(\frac{A}{g}\right) = 3 \times 0.9 = 2.7 \text{ kips}$$

The bending moment diagram increases from zero at the top to the maximum moment at the base:

$$M = 2.7 \times 10 = 27 \text{ kip}-\text{ft}$$

The maximum bending stress is

$$\sigma = \frac{Mc}{I} = \frac{(27 \times 12)\,(6.625/2)}{28.1} = 38.2 \text{ ksi}$$

Problem 6.11

For each system we compute $\omega_n = \sqrt{k/m}$ and $T_n = 2\pi/\omega_n$. For the computed T_n and $\zeta = 5\%$ we read one of D, V or A from the design spectrum of Fig. 6.9.4 multiplied by 0.50; and compute the other two among D, V and A. The peak deformation $u_o = D$ and the base shear is $V_{bo} = (A/g)\,w$. These results are summarized in the accompanying table.

System	ω_n sec^{-1}	T_n sec	V in./sec	D in.	A/g	V_{bo} kips
(a)	3.93	1.60	55.2	14.1	0.562	56.2
(b)	5.56	1.13	55.2	9.93	0.795	79.5
(c)	3.93	1.60	55.2	14.1	0.562	112.4

Comparing the response of systems (a) and (b), we observe that stiffening the tower shortens the natural period and reduces the design deformation, but increases the base shear; the latter is a disadvantage.

Comparing systems (a) and (c), both have the same natural period and spectral ordinates. However, system (c) has twice the mass of system (a), which doubles the base shear.

The preceding comments are restricted to systems in the constant-V region of the spectrum.

Problem 6.12

For each system we compute $\omega_n = \sqrt{k/m}$ and $T_n = 2\pi/\omega_n$. For the computed T_n and $\zeta = 5\%$ we read A from the design spectrum of Fig. 6.9.5 multiplied by 0.50 and compute D and V. The peak deformation $u_o = D$ and the base shear is $V_{bo} = (A/g)w$. The results are summarized in the accompanying table.

System	ω_n sec^{-1}	T_n sec	V in./sec	D in.	A/g	V_{bo} kips
(a)	9.82	0.64	53.24	5.42	1.355	21.7
(b)	13.89	0.45	37.79	2.71	1.355	21.7
(c)	9.82	0.64	53.24	5.42	1.355	43.4

Comparing the response of systems (a) and (b), we observe that stiffening the tower shortens the natural period and reduces the design deformation; however the base shear remains unchanged.

Comparing systems (a) and (c), both have the same natural period and spectral ordinates. The deformations of the two systems are the same. However, system (c) has twice the mass of system (a), which doubles the base shear.

The preceding comments are restricted to systems in the constant-A region of the spectrum.

Problem 6.13

For each system we compute $\omega_n = \sqrt{k/m}$ and $T_n = 2\pi/\omega_n$. For the computed T_n and $\zeta = 5\%$ we read A from the design spectrum of Fig. 6.9.5 multiplied by 0.50 and compute D and V. The peak deformation $u_o = D$ and the base shear is $V_{bo} = (A/g)w$. The results are summarized in the accompanying table.

System	ω_n sec^{-1}	T_n sec	V in./sec	D in.	A/g	V_{bo} kips
(a)	0.98	6.40	35.5	36.2	0.09	144.7
(b)	1.39	4.52	50.3	36.2	0.18	289.4
(c)	0.98	6.40	35.5	36.2	0.09	289.4

Comparing the response of systems (a) and (b), we observe that stiffening the tower shortens the natural period and increases the base shear; however the deformations remain unchanged.

Comparing systems (a) and (c), both have the same natural period and spectral ordinates. The deformations of the two systems are the same. However, system (c) has twice the mass of system (a), which doubles the base shear.

The preceding comments are restricted to systems in the constant-D region of the spectrum.

Problem 6.14

(a) *Frame with rigid beam* ($EI_b = \infty$)

$$k = 2\left[\frac{12EI_c}{h^3}\right] = \frac{24\,(3\times10^3)\,(10^4/12)}{(12\times12)^3} = 20.09 \text{ kips/in.}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{20.09}{10/386}} = 27.85 \text{ rads/sec}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.226 \text{ sec}$$

For $T_n = 0.226$ sec and $\zeta = 5\%$, Fig. 6.9.5 scaled by 0.50 gives

$$\frac{A}{g} = 1.355 \Rightarrow D = \frac{A}{\omega_n^2} = \frac{1.355\times386}{(27.85)^2} = 0.674 \text{ in.}$$

Design quantities

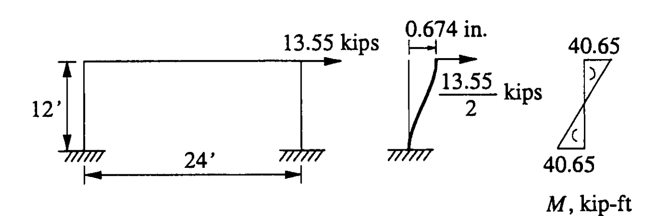
Deformation, $u_o = 0.674$ in.

Lateral force, $f_{So} = (A/g)\,w = 13.55$ kips

Bending moments at top and bottom of columns:

$$M = \left(\frac{13.55}{2}\right)\frac{h}{2} = \frac{13.55\times12}{4} = 40.65 \text{ kip-ft}$$

Bending moments in the columns are shown in the accompanying diagram.



(b) *Frame with flexible beam* ($EI = 0$)

$$k = 2\left[\frac{3EI_c}{h^3}\right] = 5.02 \text{ kips/in.}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 13.92 \text{ rads/sec}; \quad T_n = 0.452 \text{ sec}$$

For $T_n = 0.452$ sec and $\zeta = 5\%$, Fig. 6.9.5 gives

$$\frac{A}{g} = 1.355 \Rightarrow D = \frac{A}{\omega_n^2} = \frac{1.355\times386}{(13.92)^2} = 2.7 \text{ in.}$$

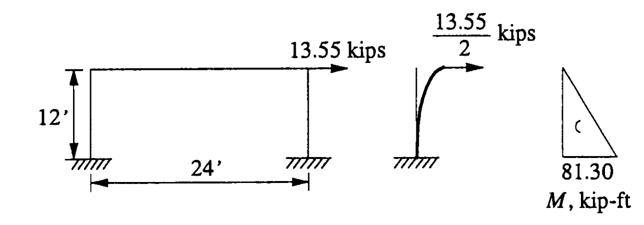
Design quantities

$u_o = 2.7$ in.

$$f_{So} = \left(\frac{A}{g}\right)w = 13.55 \text{ kips}$$

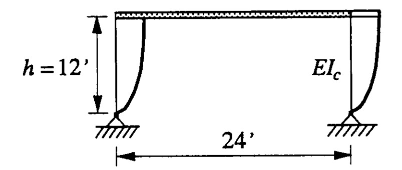
The bending moment at the base of columns is

$$M = \left(\frac{13.55}{2}\right)h = \frac{13.55\times12}{2} = 81.30 \text{ kip-ft}$$



A rigid beam increases the lateral stiffness by a factor of 4 and shortens the natural period by a factor of 2. In the constant-A region of the spectrum, this change in T_n does not affect the lateral force; however, the maximum bending moment is halved. The design deformation is reduced by a factor of four.

Problem 6.15



$$k = 2\left[\frac{3EI_c}{h^3}\right] = \frac{6\,(3 \times 10^3)\,(10^4/12)}{(12 \times 12)^3} = 5.02 \text{ kips/in.}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.02}{10/386}} = 13.92 \text{ rads/sec}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.452 \text{ sec}$$

For $T_n = 0.452$ sec and $\zeta = 5\%$, Fig. 6.9.5 scaled by 0.5 gives

$$\frac{A}{g} = 1.355 \Rightarrow D = \frac{A}{\omega_n^2} = \frac{1.355 \times 386}{(13.92)^2} = 2.7 \text{ in.}$$

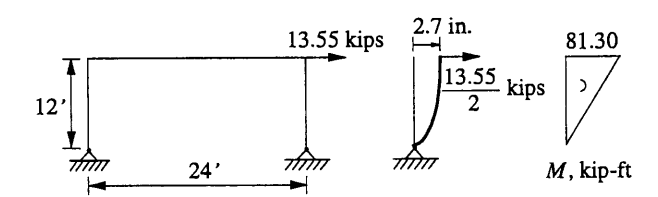
Design quantities

$$u_o = 2.7 \text{ in.}$$

$$f_{So} = \left(\frac{A}{g}\right) w = 1355 \text{ kips}$$

The bending moment diagram for each column is as shown; at the top

$$M = \left(\frac{1355}{2}\right) h = \frac{1355 \times 12}{2} = 81.30 \text{ kip} - \text{ft}$$



Influence of base fixity

If the columns are hinged at the base instead of clamped, the lateral stiffness is reduced by a factor of 4 and the vibration period is lengthened by a factor of 2. In the constant-A region of the spectrum, this change in T_n does not affect the lateral force; however, the maximum bending moment is doubled. The design deformation is quadrupled.

Problem 6.16

From Example 1.2:

$$w = 30 \times 30 \times 20 = 18{,}000 \text{ lbs} = 18.0 \text{ kips}$$

$$m = w/g = 0.04663 \text{ kip-sec}^2/\text{in.}$$

$$k_{\text{N-S}} = 38.58 \text{ kips/in.}$$

$$k_{\text{E-W}} = 119.6 \text{ kips/in.}$$

(a) North-South excitation

1. *Determine the natural vibration period.*

$$T_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k_{\text{N-S}}}} = 2\pi\sqrt{\frac{0.04663}{38.58}}$$

$$T_n = 0.219 \text{ sec}$$

2. *Determine the pseudo-acceleration.*

From Fig. 6.9.5 scaled by 0.25, the pseudo-acceleration for $T_n = 0.219$ sec is

$$A = 0.25(2.71g) = 0.6775g$$

3. *Compute peak responses.*

The peak lateral displacement u_o is

$$u_o \equiv D = \left(\frac{T_n}{2\pi}\right)^2 A = \left(\frac{0.219}{2\pi}\right)^2 (0.6775 \times 386)$$

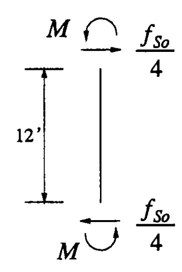
$$u_o = 0.316 \text{ in.}$$

To determine bending moments in the columns, we first determine the equivalent static force:

$$f_{So} = \frac{A}{g} w = 0.6775 \times 18.0 = 12.19 \text{ kips}$$

The bending moments in the columns are determined from the static analysis of the frame subjected to the equivalent static force, $f_{So} = 12.19$ kips.

Each column carries 1/4th of the force:



$$2M = \frac{f_{So}}{4} \times 12 = \frac{12.19}{4} \times 12$$

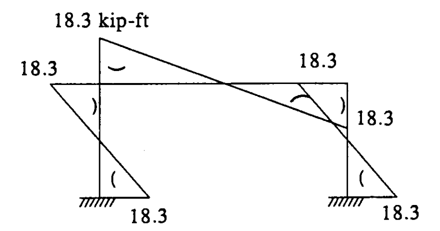
$$M = 18.3 \text{ kip-ft}$$

Alternatively, from Eq. (A1.1) in the book with $u_a = u_o$, $u_b = \theta_a = \theta_b = 0$,

$$M = \frac{6EI}{h^2} u_o = \frac{6 \times 29000 \times 82.8}{(12 \times 12)^2} \times \frac{0.316}{12}$$

$$M = 18.3 \text{ kip-ft}$$

The bending moment diagram drawn on the compression side is shown in the accompanying figure.



(b) East-West excitation

1. *Determine the natural vibration period.*

$$T_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k_{\text{E-W}}}} = 2\pi\sqrt{\frac{0.04663}{119.6}} = 0.124 \text{ sec}$$

2. *Determine the pseudo-acceleration.*

From Fig. 6.9.5 scaled by 0.25 the pseudo-acceleration for $T_n = 0.124$ sec is

$$A = 0.25(11.70 \times 0.124^{0.704} g) = 0.6728g$$

3. *Compute peak responses.*

The peak lateral displacement u_o is

$$u_o \equiv D = \left(\frac{T_n}{2\pi}\right)^2 A = \left(\frac{0.124}{2\pi}\right)^2 (0.6728 \times 386)$$

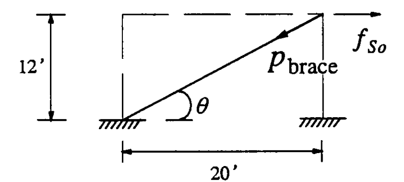
$$u_o = 0.1013 \text{ in.}$$

The equivalent static force is

$$f_{So} = \frac{A}{g} w = 0.6728 \times 18.0 = 12.11 \text{ kips}$$

Neglecting the lateral resistance of the columns, the axial force in each brace is

$$p_{\text{brace}} = \frac{f_{So}/4}{\cos\theta} = \frac{12.11/4}{0.8575} = 3.53 \text{ kips}$$


12'
p_{brace}
f_{So}
θ
20'

Problem 6.17

1. *Compute T_n.*

$$k = \sum \frac{3EI}{L^3} = \frac{3\left(3\times10^3\right)10^4}{12}\left[\frac{1}{(10\times12)^3}+\frac{1}{(20\times12)^3}\right]$$

$$= 4.340 + 0.543 = 4.883 \text{ kip/in.}$$

$$T_n = 2\pi\sqrt{\frac{10/386}{4.883}} = 0.458 \text{ sec} \quad ; \; \omega_n = 13.73 \text{ rads/sec}$$

2. *Compute peak deformation u_o.*

From spectrum:

$$D = \frac{A}{\omega_n^2} = \frac{0.25\times2.71\times386}{(13.73)^2} = 1.39 \text{ in.}$$

$$u_o = D = 1.39 \text{ in.}$$

3. *Compute bending moments.*

Method 1: $M = \frac{3EI}{L^2}u_o$

$$M_{\text{short}} = \frac{3\left(3\times10^3\right)}{(10\times12)^2}\frac{10^4}{12}1.39 = 724 \text{ k-in} = 60.3 \text{ k-ft}$$

$$M_{\text{long}} = \frac{3\left(3\times10^3\right)}{(20\times12)^2}\frac{10^4}{12}1.39 = 181 \text{ k-in} = 15.1 \text{ k-ft}$$

Method 2: From u_o calculate lateral force for each column and then the bending moment.

$$(f_s)_{\text{short}} = k_{\text{short}}u_o = 4.340\times1.39 = 6.03 \text{ kip}$$
$$M_{\text{short}} = 6.03\times10 = 60.3 \text{ k-ft}$$
$$(f_s)_{\text{long}} = k_{\text{long}}u_o = 0.543\times1.39 = 0.755 \text{ kip}$$
$$M_{\text{long}} = 0.755\times20 = 15.1 \text{ k-ft}$$

The bending moment diagrams for both columns are shown.

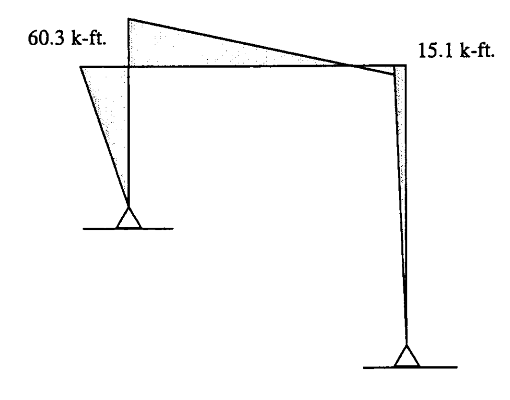
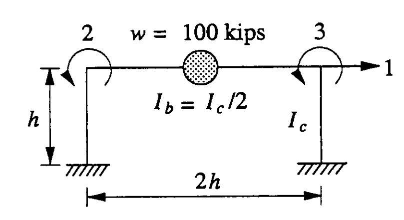


Fig. P6.17

Problem 6.18



Data: $E = 30 \times 10^3$ ksi; $I_c = 320$ in.^{4}; $h = 12$ ft; $\zeta = 5\%$

1. *Determine lateral stiffness of the frame.*

Following the procedure of Example 1.1 of the text, the 3×3 stiffness matrix is

$$\mathbf{k} = \frac{EI_c}{h^3}\begin{bmatrix} 24 & 6h & 6h \\ 6h & 5h^2 & \frac{1}{2}h^2 \\ 6h & \frac{1}{2}h^2 & 5h^2 \end{bmatrix}$$

The equilibrium equations are

$$\frac{EI_c}{h^3}\begin{bmatrix} 24 & 6h & 6h \\ 6h & 5h^2 & \frac{1}{2}h^2 \\ 6h & \frac{1}{2}h^2 & 5h^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_S \\ 0 \\ 0 \end{Bmatrix} \qquad \text{(a)}$$

From the second and third equations, the joint rotations can be expressed as

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = -\begin{bmatrix} 5h^2 & \frac{1}{2}h^2 \\ \frac{1}{2}h^2 & 5h^2 \end{bmatrix}^{-1}\begin{Bmatrix} 6h \\ 6h \end{Bmatrix} u_1 = -\frac{12}{11h}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} u_1 \qquad \text{(b)}$$

Substituting Eq. (b) into the first of three equations in Eq. (a) gives

$$f_S = \left(\frac{24EI_c}{h^3} - \frac{EI_c}{h^3}\frac{12}{11h}\langle 6h \quad 6h \rangle \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) u_1 = \frac{120}{11}\frac{EI_c}{h^3} u_1 \qquad \text{(c)}$$

Thus the lateral stiffness of the frame is

$$k_{lat} = \frac{120}{11}\frac{EI_c}{h^3} = \frac{120}{11}\frac{(30 \times 10^3)(320)}{(12 \times 12)^3} = 35.07 \text{ kips/in.}$$

2. *Calculate natural period.*

$$m = \frac{w}{g} = \frac{100}{386} = 0.2591 \text{ kip-sec}^2/\text{in.}$$

$$\omega_n = \sqrt{\frac{k_{lat}}{m}} = \sqrt{\frac{35.08}{0.2591}} = 11.64 \text{ rads/sec}$$

$$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{11.64} = 0.540 \text{ sec}$$

3. *Determine spectral ordinate.*

From the design spectrum of Fig. 6.9.5, scaled by 0.5

$$A = 0.5(2.71g) = 1.355g$$

$$D = \frac{A}{\omega_n^2} = \frac{1.355(386)}{(11.64)^2} = 3.86 \text{ in.}$$

4. *Determine peak responses.*

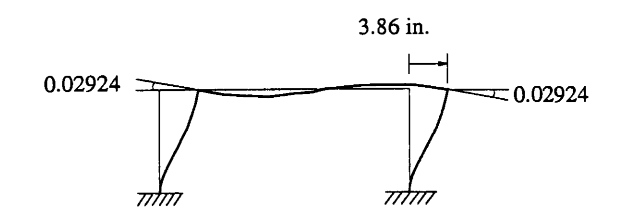
The peak lateral displacement is

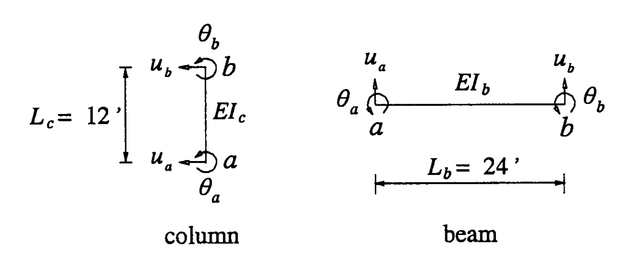
$$u_{1o} = 3.86 \text{ in.}$$

To determine bending moments we recover joint rotations from Eq. (b):

$$\begin{Bmatrix} u_{2o} \\ u_{3o} \end{Bmatrix} = -\frac{12}{11h}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} u_{1o} = -\frac{12}{11(12 \times 12)}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}(3.86) = -0.02924\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

The deflected shape at peak response is





Bending moments in columns:

$$M_a = \frac{4EI_c}{L_c}\theta_a + \frac{2EI_c}{L_c}\theta_b + \frac{6EI_c}{L_c^2}u_a - \frac{6EI_c}{L_c^2}u_b$$

$$M_b = \frac{2EI_c}{L_c}\theta_a + \frac{4EI_c}{L_c}\theta_b + \frac{6EI_c}{L_c^2}u_a - \frac{6EI_c}{L_c^2}u_b$$

Substituting $\theta_a = 0$, $\theta_b = -0.02924$, $u_a = 0$, and $u_b = -3.86$ and values for E, I_c and L_c gives

$$M_a = 6824 \text{ kip-in.}$$

$$M_b = 2925 \text{ kip-in.}$$

Bending moments in beam:

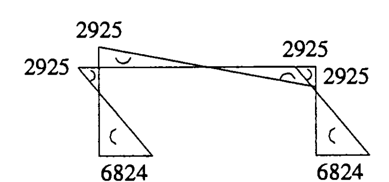
$$M_a = \frac{4EI_b}{L_b}\theta_a + \frac{2EI_b}{L_b}\theta_b + \frac{6EI_b}{L_b^2}u_a - \frac{6EI_b}{L_b^2}u_b$$

$$M_b = \frac{2EI_b}{L_b}\theta_a + \frac{4EI_b}{L_b}\theta_b + \frac{6EI_b}{L_b^2}u_a - \frac{6EI_b}{L_b^2}u_b$$

Substituting $\theta_a = \theta_b = -0.02924$, $u_a = u_b = 0$, and values for E, I_b and L_b gives

$$M_a = M_b = -2925 \text{ kip-in.}$$

The bending moment diagram drawn on the compression side is shown in the accompanying figure; the units are kip-in.



Problem 6.19

Data: $E = 30 \times 10^3$ ksi; $I_c = 320$ in^4; $h = 12$ft; $\zeta = 5\%$

1. *Determine lateral stiffness of the frame.*

The lateral stiffness of this frame was computed in Problem 1.16:

$$k_{\text{lat}} = \frac{2EI_c}{h^3} = \frac{2(30\times10^3)(320)}{(12\times12)^3} = 6.43 \quad \text{kips/in.}$$

2. *Calculate natural period.*

$$m = \frac{w}{g} = \frac{100}{386} = 0.2591 \quad \text{kip-sec}^2/\text{in.}$$

$$\omega_n = \sqrt{\frac{k_{\text{lat}}}{m}} = \sqrt{\frac{6.43}{0.2591}} = 4.982 \quad \text{rad/sec}$$

$$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{4.982} = 1.261 \quad \text{sec}$$

3. *Determine spectral ordinate.*

From the design spectrum of Fig. 6.9.5, scaled by 0.5,

$$A = 0.5\left(1.80 \times 1.261^{-1}\right)g = 0.7136g$$

$$D = \frac{A}{\omega_n^2} = \frac{0.7136 \times 386}{4.982^2} = 11.10 \quad \text{in.}$$

4. *Determine peak responses.*

The peak lateral displacement is

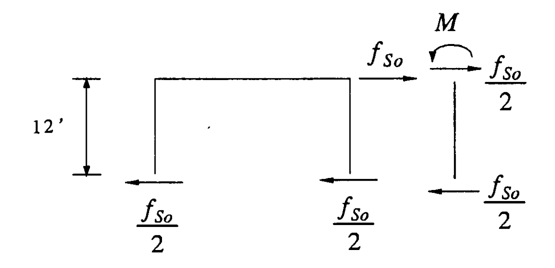
$$u_o \equiv D = 11.10 \quad \text{in.}$$

To determine bending moments, we first determine the equivalent static force

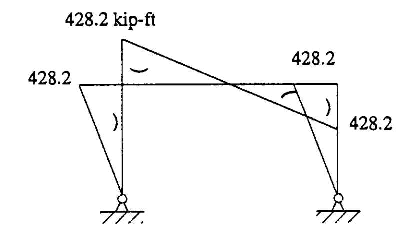
$$f_{So} = k_{\text{lat}} u_o = 6.43 \times 11.10 = 71.4 \quad \text{kips}$$

Using symmetry, the moment at the beam-column joint is

$$M = \frac{f_{So}}{2} h = \frac{71.4}{2} \times 12 = 428.2 \quad \text{kip-ft}$$



Moment diagram:



5. *Influence of base fixity.*

The above results together with those from Problem 6.18 are summarized:

Base	u_o (in)	M_{base} (kip-ft)	M_{top} (kip-ft)
Hinged	11.1	0	428.2
Clamped	3.86	568.7	243.8

Base fixity shortens the period, reduces the design deformation, creates a larger bending moment at the base, but reduces the bending moment at the beam-column joints.

Problem 6.20

1. *Determine structural properties.*

Lateral Stiffness: $k = \dfrac{48EI}{h^3} = 118.6 \text{ kips/in.}$

Mass: $m = \dfrac{w}{g} = 0.2591 \text{ kip-sec}^2/\text{in.}$

Natural Frequency: $\omega_n = \sqrt{k/m} = 21.4 \text{ rad/sec}$

Natural Period: $T_n = 2\pi/\omega_n = 0.294 \text{ sec}$

2. *Determine peak lateral displacement.*

From the design spectrum of Fig. 6.9.5, scaled to (1/3)g,

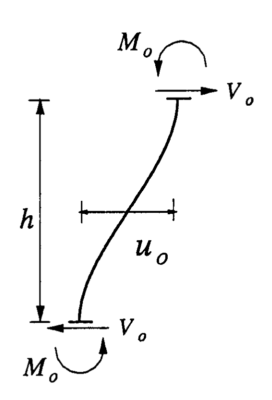
$$A = \frac{1}{3} \times 2.71g = 0.90g$$

$$D = \frac{A}{\omega_n^2} = \frac{0.90 \times 386}{21.4^2} = 0.76 \text{ in.}$$

$$u_o = D = 0.76 \text{ in.}$$

3. *Compute bending stress.*

For one



Column:

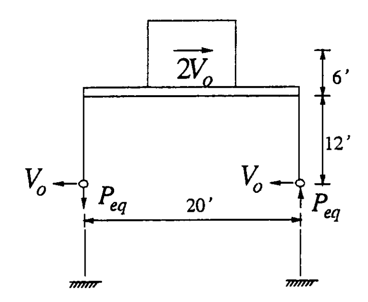
$$V_o = \frac{12EI}{h^3} u_o = \frac{k}{4} u_o = \frac{118.6}{4} \times 0.76 = 22.5 \text{ kips}$$

$$M_o = V_o \frac{h}{2} = 22.5 \times \frac{24 \times 12}{2} = 3240 \text{ kip-in.}$$

$$f_b = \frac{M_o}{S} = \frac{3240}{170} = 19.06 \text{ ksi}$$

4. *Compute stress due to axial forces.*

The sketch represents half of the structure, i.e., one pair of columns with rigid platform and rigid column bases; there is a point of inflection at mid-height of each column. Hence, taking moments about one of the inflection points, we get the column axial forces due to earthquake to be



$$P_{eq} = \frac{2V_o(12+6)}{20} = 40.5 \text{ kips}$$

The axial force in each column due to gravity load is

$$P_{grav} = \frac{w}{4} = 25 \text{ kips}$$

The total axial stress is

$$f_a = \frac{P_{eq} + P_{grav}}{A} = \frac{40.5 + 25}{20} = 3.28 \text{ ksi}$$

5. *Determine total stress.*

$$f_o = f_b + f_a = 19.06 + 3.28 = 22.34 \text{ ksi}$$

Problem 6.21

The equation of motion to be solved is

$$I_O\ddot{u}_\theta + (k_x d^2 + k_y b^2)u_\theta = -I_O\ddot{u}_{g\theta}; \quad \ddot{u}_{g\theta} = \delta(t) \qquad \text{(a)}$$

We have solved a related equation:

$$m\ddot{u} + ku = p(t); \quad p(t) = \delta(t) \qquad \text{(b)}$$

and its solution is given by Eq. (4.1.7) specialized for $\tau =$ 0:

$$u(t) = h(t) = \frac{1}{m\omega_n}\sin\omega_n t \qquad \text{(c)}$$

Therefore, solution to Eq. (a) is Eq. (c) multiplied by $-I_O$ with m replaced by I_O:

$$u_\theta(t) = -\frac{I_O}{I_O\omega_n}\sin\omega_n t$$

or (d)

$$u_\theta(t) = -\frac{1}{\omega_n}\sin\omega_n t$$

where

$$\omega_n = \left(\frac{k_x d^2 + k_y b^2}{I_O}\right)^{\frac{1}{2}} \qquad \text{(e)}$$

Problem 6.22

1. *Define structural properties.*

From Example 2.4:

$$b = 30 \text{ ft}; \; d = 20 \text{ ft}; \; h = 12 \text{ ft}; \; w = 0.1 \text{ kip/ft}^2$$

$$k_x = 1.5 \text{ kips/in.}; \; k_y = 1.0 \text{ kip/in.}$$

$$I_O = 201.86 \text{ kip-sec}^2\text{-ft}$$

$$k_\theta = 18000 \text{ kip-ft/rad}$$

$$\omega_n = \sqrt{\frac{k_\theta}{I_O}} = 9.44 \text{ rad/sec}$$

$$T_n = 0.67 \text{ sec}$$

2. *Write equation of motion.*

$$I_O \ddot{u}_\theta + k_\theta u_\theta = -I_O \ddot{u}_{g\theta}(t)$$

or

$$\ddot{u}_\theta + \omega_n^2 u_\theta = -\ddot{u}_{g\theta}(t)$$

Including damping gives

$$\ddot{u}_\theta + 2\zeta\omega_n \dot{u}_\theta + \omega_n^2 u_\theta = -\ddot{u}_{g\theta}(t)$$

3. *Determine spectral ordinate.*

From Fig. 6.9.5 scaled by 0.05, for $T_n = 0.67$ sec. and $\zeta = 5\%$,

$$A = 0.05(1.80 \times 0.67^{-1} g) = 0.1343g$$

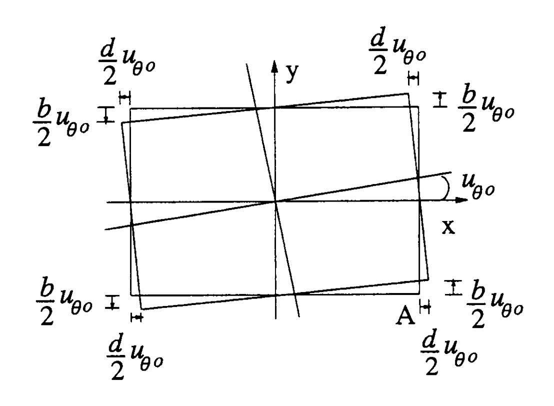
4. *Determine peak rotation.*

The peak value of rotation is

$$u_{\theta o} = \frac{2}{b}\frac{A}{\omega_n^2} = \frac{2}{30 \times 12} \times \frac{0.1343 \times 386}{(9.44)^2} = 0.00323 \text{ rad}$$

5. *Determine displacement at each corner of the roof slab.*

The corner displacements are shown in the accompanying figure, where



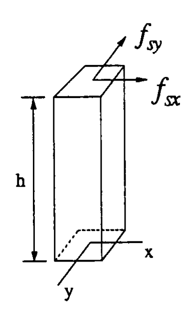
$$\frac{b}{2}u_{\theta o} = \frac{30 \times 12}{2} \times 0.00323 = 0.582 \text{ in}$$

$$\frac{d}{2}u_{\theta o} = \frac{20 \times 12}{2} \times 0.00323 = 0.388 \text{ in}$$

6. *Determine base torque T_o.*

$$T_o = k_\theta u_{\theta o} = 18000 \times 0.00323 = 58.2 \text{ kip-ft}$$

7. *Determine bending moment at top and base of column A.*



$$f_{sx} = k_x\left(\frac{d}{2}u_{\theta o}\right) = 0.388 \text{ kips}$$

$$f_{sy} = k_y\left(\frac{b}{2}u_{\theta o}\right) = 0.873 \text{ kips}$$

For a column clamped at both ends

$$M_y = \frac{1}{2} f_{sx} h = \frac{1}{2} \times 0.873 \times 12 = 5.24 \text{ kip-ft}$$

$$M_x = \frac{1}{2} f_{sy} h = \frac{1}{2} \times 0.388 \times 12 = 2.33 \text{ kip-ft}$$

Bending moments in other columns are the same; the relative direction can be determined from the direction of displacements.

Problem 6.23

(a) With reference to Fig. P6.23a, the design spectrum is determined by the following steps:

1. The peak parameters for the ground motion: $\ddot{u}_{go} = 0.50\text{g}$, $\dot{u}_{go} = 24$ in./sec, and $u_{go} = 18$ in. are plotted.

2. From Table 6.9.1 the amplification factors for the 84.1 percentile and 2% damping are obtained:

$$\alpha_A = 3.66,\ \alpha_V = 2.92, \text{ and } \alpha_D = 2.42.$$

3-5. The ordinates for three branches of the spectrum are:

$$b\text{-}c\text{: } A = 0.50\text{g} \times 3.66 = 1.83\text{g}$$

$$c\text{-}d\text{: } V = 24 \times 2.92 = 70.08 \text{ in./sec}$$

$$d\text{-}e\text{: } D = 18 \times 2.42 = 43.56 \text{ in.}$$

6. The line $A = 0.50\text{g}$ is plotted for $T_n < 1/33$ sec and $D = 18$ in. for $T_n > 33$ sec.

7. The transition line b-a is drawn to connect the point $A = 1.83\text{g}$ at $T_n = 1/8$ sec to $A = 0.50\text{g}$ at $T_n = 1/33$ sec. Similarly, the transition line e-f is drawn to connect the point $D = 43.56$ in. at $T_n = 10$ sec to $u_{go} = 18$ in. at $T_n = 33$ sec.

The same procedure is used to determine the median spectrum, using the amplification factors $\alpha_A = 2.74$, $\alpha_V = 2.03$, and $\alpha_D = 1.63$.

(b) Determine T_c and T_d.

At T_c, $A = 1.83\text{g}$ and $V = 70.08$ in./sec

$$A = \frac{2\pi}{T_n} V \Rightarrow T_n = 2\pi \frac{V}{A} = 2\pi \left(\frac{70.08}{1.83 \times 386} \right) = 0.623 \text{ sec}$$

At T_d, $V = 70.08$ in./sec and $D = 43.56$ in.

$$V = \frac{2\pi}{T_n} D \Rightarrow T_n = 2\pi \frac{D}{V} = 2\pi \left(\frac{43.56}{70.08} \right) = 3.91 \text{ sec}$$

Determine equations for $A(T_n)/\text{g}$.

- $T_n \le 1/33$ sec $\qquad \dfrac{A(T_n)}{\text{g}} = 0.5$

- $1/8 \text{ sec} \le T_n \le 0.623$ sec $\qquad \dfrac{A(T_n)}{\text{g}} = 1.83$

- $0.623 \text{ sec} < T_n \le 3.91$ sec

$$\frac{A(T_n)}{\text{g}} = \frac{2\pi V}{\text{g} T_n} = \frac{2\pi(70.08)}{\text{g} T_n} = 1.14\, T_n^{-1}$$

- $3.91 \text{ sec} < T_n \le 10$ sec

$$\frac{A(T_n)}{\text{g}} = \left(\frac{2\pi}{T_n} \right)^2 \frac{D}{\text{g}} = \left(\frac{2\pi}{T_n} \right)^2 \frac{43.56}{\text{g}} = 4.46\, T_n^{-2}$$

- $T_n \ge 33$ sec

$$\frac{A(T_n)}{\text{g}} = \left(\frac{2\pi}{T_n} \right)^2 \frac{u_{go}}{\text{g}} = 1.84\, T_n^{-2}$$

- $1/33 \text{ sec} < T_n \le 1/8$ sec: Find equation to straight line on log-log paper connecting point b with coordinates $A/\text{g} = 1.83$ and $T_n = 1/8$ sec to point a with coordinates $A/\text{g} = 0.50$ and $T_n = 1/33$ sec. The equation is

$$\frac{A(T_n)}{\text{g}} = 12.28\, T_n^{0.916}$$

- $10 \text{ sec} < T_n \le 33$ sec: Find equation to straight line on log-log paper connecting point e with coordinates $D = 43.56$ in. and $T_n = 10$ sec to point f with coordinates $D = 18$ in. and $T_n = 33$ sec. The equation is

$$\frac{A(T_n)}{\text{g}} = 24.49\, T_n^{-2.74}$$

Pseudo-acceleration spectrum

The preceding equations are plotted in Fig. P6.23b using log-log scale.

(c) The same spectrum is plotted using linear scales in Fig. P6.23c.

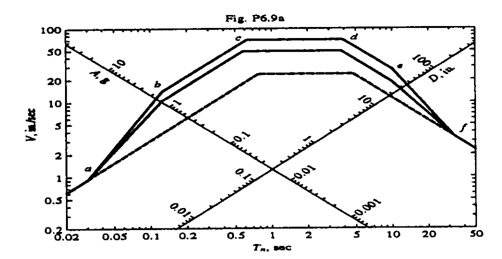


Fig. 6.23a

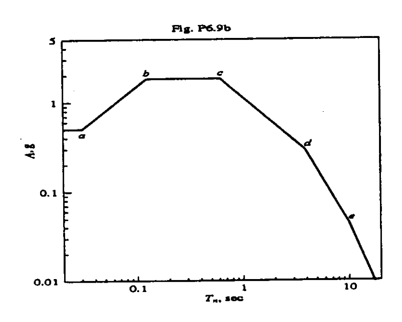


Fig. 6.23b

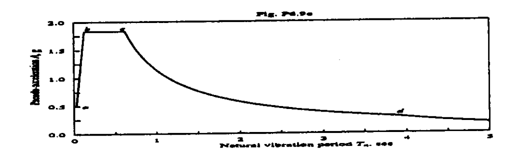


Fig. 6.23c

Problem 7.1

(a) For vibration amplitudes smaller than u_y, the system is vibrating in the linear elastic range. The period of these oscillations is

$$T_n = \frac{2\pi}{\omega_n} = 2\pi \div \sqrt{\frac{kg}{w}} = 2\pi \div \sqrt{\frac{2.112 \times 386}{5.2}} = 0.502 \text{ sec.}$$

and the damping ratio $\zeta = 2$ %, as given.

(b) A system vibrating at amplitudes larger than u_y yields; because it is no longer linear, a natural period or damping ratio cannot be defined.

(c) For the corresponding linear system,

$T_n = 0.502$ sec. and $\zeta = 2$ %

(d) The peak value f_o of the equivalent static force for the associated linear system due to the El Centro ground motion was determined in Example 6.3: $f_o = 5.72$ kips. For ground motion scaled up by a factor of 3,

$$f_o = 17.16 \text{ kips}$$

The yield strength of the elastoplastic system is given:

$$f_y = 5.55 \text{ kips}$$

From Eq. (7.2.1)

$$\bar{f}_y = \frac{f_y}{f_o} = \frac{5.55}{17.16} = 0.323$$

From Eq. (7.2.2)

$$R_y = \frac{f_o}{f_y} = \frac{17.16}{5.55} = 3.09$$

Problem 7.2

The central difference method is modified to include a nonlinear restoring force; the remainder of the algorithm is unchanged from the implementation for a linear system. An outline of the algorithm is listed as follows (for zero damping):

Steps

1.0 *Initial calculations*

$T_n = 0.5$ sec. $\quad f_o = 1.37w$

$\bar{f}_y = 0.125 \Rightarrow f_y = \bar{f}_y f_o = 0.125(1.37w) = 0.171w$

$\Delta t = 0.02$ sec. $\quad u_0 = 0 \quad \dot{u}_0 = 0$

Note: $p(t) = -m\ddot{u}_g(t)$; $p_i = -m(\ddot{u}_g)_i$

1.1 $\ddot{u}_0 = \dfrac{p_0 - ku_0}{m} = \dfrac{-m\ddot{u}_{g0} - ku_0}{m}$

$= -(\ddot{u}_g)_0 - \omega_n^2 u_0 = 0$

1.2 $u_{-1} = u_0 - (\Delta t)\dot{u}_0 + \dfrac{(\Delta t)^2}{2}\ddot{u}_0 = 0$

1.3 $\hat{k} = \dfrac{m}{(\Delta t)^2} = 2500m$

1.4 $a = \dfrac{m}{(\Delta t)^2} = 2500m$

2.0 *Calculations for each time step*

2.1 $\hat{p}_i = -m(\ddot{u}_g)_i - au_{i-1} - (f_s)_i + \dfrac{2m}{(\Delta t)^2}u_i$

2.2 $u_{i+1} = \dfrac{\hat{p}_i}{\hat{k}}$

$= \left[-(\ddot{u}_g)_i - \dfrac{(f_s)_i}{m}\right](\Delta t)^2 - u_{i-1} + 2$

2.3 *Determine restoring force at time step i+1*

2.3.1 $(\Delta f_s)_i = k(u_{i+1} - u_i)$

2.3.2 $(f_s)_{i+1} = (f_s)_i + (\Delta f_s)_i$

2.3.3 If $\left|(f_s)_{i+1}\right| > f_y$ then

$\dot{u}_i = \dfrac{u_{i+1} - u_{i-1}}{2\Delta t}$

$(f_s)_{i+1} = sign(\dot{u}_i) f_y$

Computational steps 2.1-2.3.3 are repeated for $i = 0, 1, 2, 3, \ldots$ leading to Table P7.2 which summarizes the results for the first 0.2 second duration. The results are also shown in Figs. P7.2a-b.

Table P7.2 Numerical solution by Central Difference Method for the first 0.2 seconds

t_i	p_i/m	$(f_s)_i/w$	u_{i-1}	u_i	$\hat{p}_i/m$	u_{i+1}
0.00	0	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	-2.4343	-0.1538	0.0000	0.0000	-2.4343	-0.0010
0.04	-1.4065	-0.3867	0.0000	-0.0010	-6.1214	-0.0024
0.06	-0.3825	-0.6193	-0.0010	-0.0024	-9.8043	-0.0039
0.08	-1.6538	-0.9173	-0.0024	-0.0039	-14.5217	-0.0058
0.10	-2.9289	-1.3423	-0.0039	-0.0058	-21.2508	-0.0085
0.12	-4.2002	-1.9479	-0.0058	-0.0085	-30.8377	-0.0123
0.14	-2.6352	-2.5969	-0.0085	-0.0123	-41.1120	-0.0164
0.16	-1.0703	-3.1494	-0.0123	-0.0164	-49.8598	-0.0199
0.18	0.4946	-3.4718	-0.0164	-0.0199	-54.9635	-0.0220
0.20	-1.4220	-3.6647	-0.0199	-0.0220	-58.0174	-0.0232

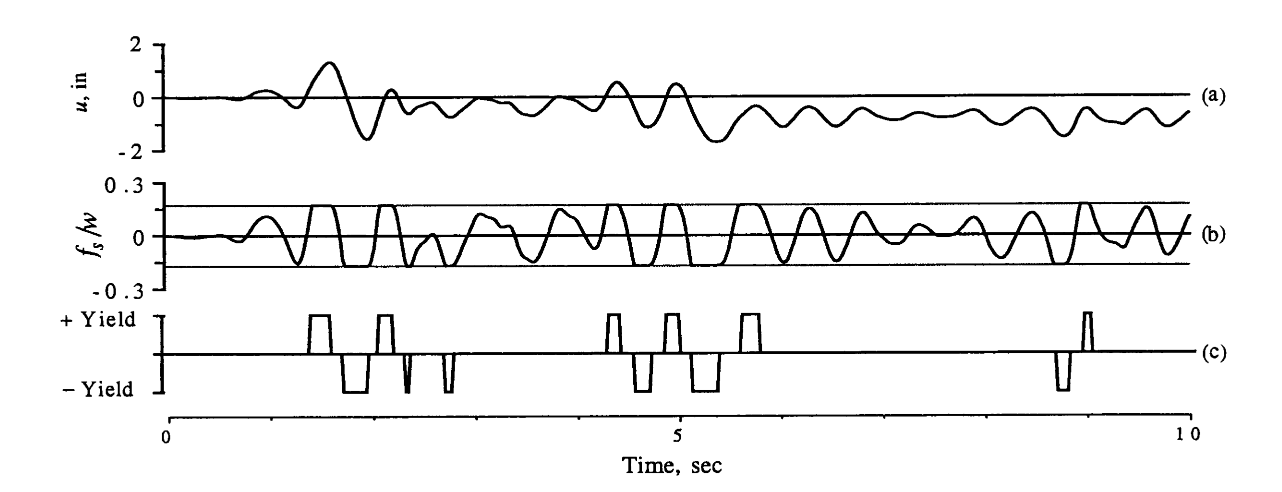


Fig. P7.2a

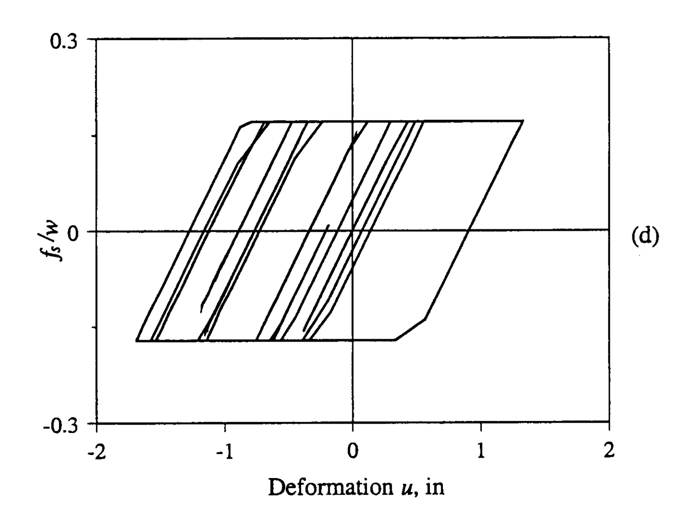


Fig. P7.2b

Problem 7.3

1. *Determine response of the corresponding linear system.*

$u(t)$ is given in Fig. 7.4.3a; $u_o = 2.25$ in.

$$D \equiv u_o = 2.25 \text{ in.}$$

$$\frac{A}{g} = \frac{\omega_n^2 D}{g} = \left(\frac{2\pi}{0.5}\right)^2 \frac{2.25}{386} = 0.9205$$

$$f_o = \frac{A}{g} w = 0.9205w$$

2. *Determine response of the elastoplastic system with* $\bar{f}_y = 0.5 \Rightarrow f_y = 0.4602\text{w}$.

$u(t)$ is given in Fig. 7.4.3b.

3. *Determine response of the system with half the strength of Case 2.*

$$\bar{f}_y = 0.25 \Rightarrow f_y = 0.2301w.$$

$u(t)$ is given in Fig. 7.4.3c; $u_m = 1.75$ in.

$$u_y = \frac{f_y}{k} = \frac{0.2301mg}{k} = 0.2301g\left(\frac{T_n}{2\pi}\right)^2 = 0.5625 \text{ in.}$$

$$\mu(t) = \frac{u(t)}{u_y} = \frac{u(t) \text{ in Fig. 7.4.3c}}{0.5625}$$

$$\mu = \frac{u_m}{u_y} = \frac{1.75}{0.5625} = 3.11$$

4. *Determine response of the system of case 2 to double the El Centro ground motion.*

$$T_n = 0.5 \text{ sec.}, \zeta = 5\%, f_y = 0.4602w$$

Excitation: $2 \times \ddot{u}_g(t)$ of Fig. 6.1.4; $\ddot{u}_{go} = 0.638g$.

The deformation response $u(t)$ is computed by the average acceleration method using a time step $\Delta t = 0.02$ sec, which was further subdivided to detect the transition from elastic to plastic branches and vice versa, in the force deformation relation. The results are shown in Fig. P7.3, which gives $u_m = 3.50$ in.

$$u_y = \frac{f_y}{k} = \frac{0.4602mg}{k} = 0.4602g\left(\frac{T_n}{2\pi}\right)^2 = 1.125 \text{ in.}$$

$$\mu(t) = \frac{u(t)}{u_y} = \frac{u(t) \text{ in Fig. P7.3}}{1.125}$$

$$\mu = \frac{u_m}{u_y} = \frac{3.50}{1.125} = 3.11$$

5. *Verification.*

The ductility factor μ is the same in Cases 3 and 4. So are the $\mu(t)$; this is obvious by comparing $u(t)$ in Fig. 7.4.3c ÷ u_y (=0.5625 in.) and $u(t)$ in Fig. P7.3 ÷ u_y (=1.125 in.).

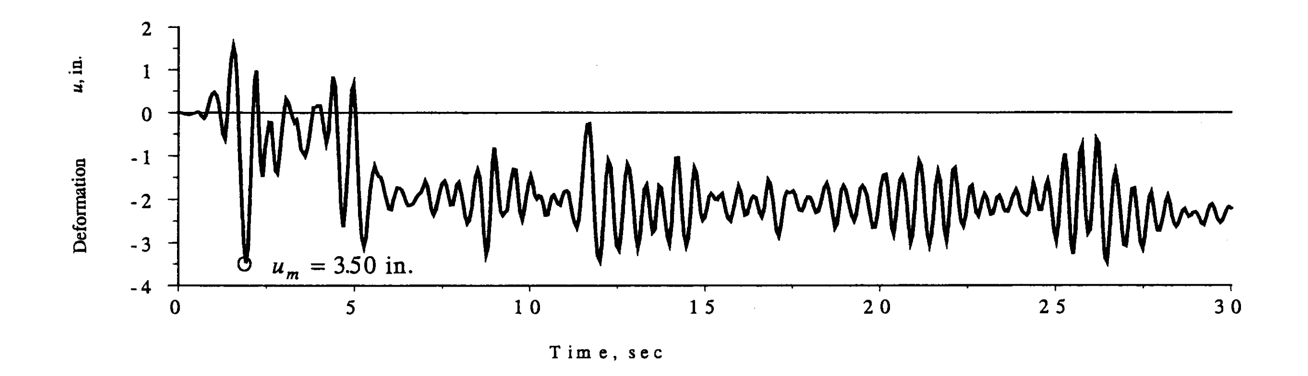


Fig. P7.3

Problem 7.4

(a) Excitation: El Centro ground motion.

1. *Determine response of the corresponding linear system.*

u(t) is given in Fig. 7.4.3a; $u_o = 2.25$ in.

$$D \equiv u_o = 2.25 \text{ in.}$$

$$\frac{A}{g} = \frac{\omega_n^2 D}{g} = \left(\frac{2\pi}{0.5}\right)^2 \frac{2.25}{386} = 0.9205$$

$$f_o = \frac{A}{g} w = 0.9205w$$

2. *Determine response of the elastoplastic system with* $\bar{f}_y = 0.25 \Rightarrow f_y = 0.2301$w.

u(t) is given in Fig. 7.4.3c, $u_m = 1.75$ in.

$$\mu = \frac{1}{\bar{f}_y}\frac{u_m}{u_o} = \frac{1}{0.25}\frac{1.75}{2.25} = 3.11$$

(b) Excitation: $2 \times$ El Centro ground motion.

3. *Determine response of the corresponding linear system.*

u(t) is twice of that for case 1; $u_o = 4.50$ in.

$$D \equiv u_o = 4.50 \text{ in.}$$

$$f_o = 1.851w$$

4. *Determine response of the elastoplastic system with* $\bar{f}_y = 0.25 \Rightarrow f_y = 0.4602$w.

The deformation response *u(t)* is computed by the average acceleration method using a time step Δt = 0.02 sec, which was further subdivided to detect the transition from elastic to plastic branches and vice versa, in the force-deformation relation. The results are shown in Fig. P7.4; $u_m = 3.50$ in.

$$u_y = \frac{f_y}{k} = \frac{0.4602mg}{k} = 0.4602g\left(\frac{T_n}{2\pi}\right)^2 = 1.125 \text{ in.}$$

$$\mu(t) = \frac{u(t)}{u_y} = \frac{u(t) \text{ in Fig. P7.15}}{1.125}$$

$$\mu = \frac{u_m}{u_y} = \frac{3.50}{1.125} = 3.11$$

The ductility factors calculated for the two systems with normalized strength $\bar{f}_y = 0.25$ are independent of the factor by which the ground motion is scaled.

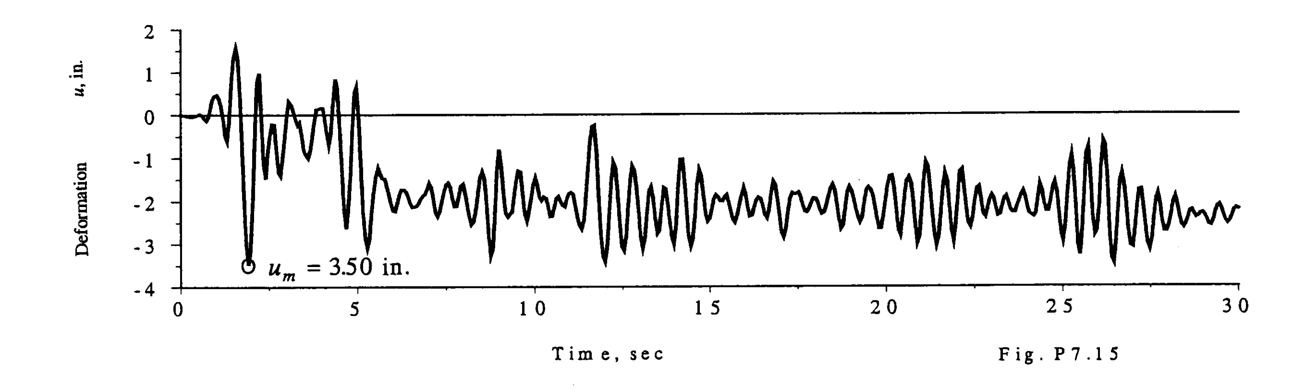


Fig. P7.4

Problem 7.5

(a) For $\bar{f}_y = 0.5$, from Fig. 7.4.3, $u_m = 1.62$ in., and $u_o = 2.25$ in. Substituting these data in Eq. (7.2.4) gives

$$\mu = \frac{1}{\bar{f}_y}\frac{u_m}{u_o} = \frac{1}{0.5}\frac{1.62}{2.25} = 1.44$$

(b) For $\bar{f}_y = 0.25$, $u_m = 1.75$ in. and $u_o = 2.25$ in.

$$\mu = \frac{1}{0.25}\frac{1.75}{2.25} = 3.11$$

(c) For $\bar{f}_y = 0.125$, $u_m = 2.07$ in. and $u_o = 2.25$ in.

$$\mu = \frac{1}{0.125}\frac{2.07}{2.25} = 7.36$$

Problem 7.6

(a) The design spectrum for elastic systems with $\zeta = 2\%$ and the selected ground motion was constructed in Problem 6.23 and is reproduced in Figs. P7.6a, b and c. The inelastic spectrum for $\mu = 3$ is determined by the following steps:

1-3. The ordinates for the various branches are

$$b'\text{-}c': A_y = (1.83\text{g}) \div \sqrt{2\mu - 1} = 0.818\text{g}$$

$$c'\text{-}d': V_y = 70.08 \div \mu = 23.36 \text{ in./sec}$$

$$d'\text{-}e': D_y = 43.56 \div \mu = 14.52 \text{ in.}$$

4. The ordinate for point f' is $D_y = 18/\mu = 6$ in.. Join points f' and e' by a straight line. For $T_n > 33$ sec, $D_y = 6$ in.

5. Join a', which is the same as a, to point b'.

6. Draw the line $A_y = 0.50\text{g}$ for $T_n < 1/33$ sec.

The resulting design spectrum is shown in Fig. P7.3a.

(b) and (c). The two spectra of Fig P7.6a are replotted in Figs. P7.6b and P7.6c.

(d) Determine $T_{a'}$, $T_{b'}$, $T_{c'}$, $T_{d'}$, $T_{e'}$ and $T_{f'}$.

$$T_{a'} = 1/33,\ T_{b'} = T_b = 1/8,\ T_{d'} = T_d = 3.91,$$

$$T_{e'} = T_e = 10,\ T_{f'} = T_f = 33;\ \text{all in sec.}$$

At $T_{c'}$, $A_y = 0.818\text{g}$ and $V_y = 23.36$ in./sec

$$A_y = \frac{2\pi}{T_n} V_y \Rightarrow$$

$$T_n = 2\pi \frac{V_y}{A_y} = 2\pi\left(\frac{23.36}{0.818\text{g}}\right) = 0.465 \text{ sec}$$

(e) *Determine equations for* $A_y(T_n)/\text{g}$.

With reference to the data in Problem 6.23,

- $T_n \le 1/33$ sec: $\dfrac{A_y(T_n)}{\text{g}} = 0.5$
- $1/33 \text{ sec} < T_n \le 1/8$ sec: Find equation to the straight line on log-log paper connecting points a' and b':
  $$\frac{A_y(T_n)}{\text{g}} = 1.687\, T_n^{0.348}$$
- $1/8 \text{ sec} < T_n \le 0.465$ sec: $\dfrac{A_y(T_n)}{\text{g}} = 0.818$
- $0.465 \text{ sec} < T_n \le 3.91$ sec:
  $$\frac{A_y(T_n)}{\text{g}} = \frac{1.14\, T_n^{-1}}{\mu} = 0.380\, T_n^{-1}$$
- $3.91 \text{ sec} < T_n \le 10$ sec:
  $$\frac{A_y(T_n)}{\text{g}} = \frac{4.46\, T_n^{-2}}{\mu} = 1.487\, T_n^{-2}$$
- $10 \text{ sec} < T_n \le 33$ sec
  $$\frac{A_y(T_n)}{\text{g}} = \frac{24.49\, T_n^{-2.74}}{\mu} = 8.16\, T_n^{-2.74}$$
- $T_n > 33$ sec:
  $$\frac{A_y(T_n)}{\text{g}} = \frac{1.84\, T_n^{-2}}{\mu} = 0.614\, T_n^{-2}$$

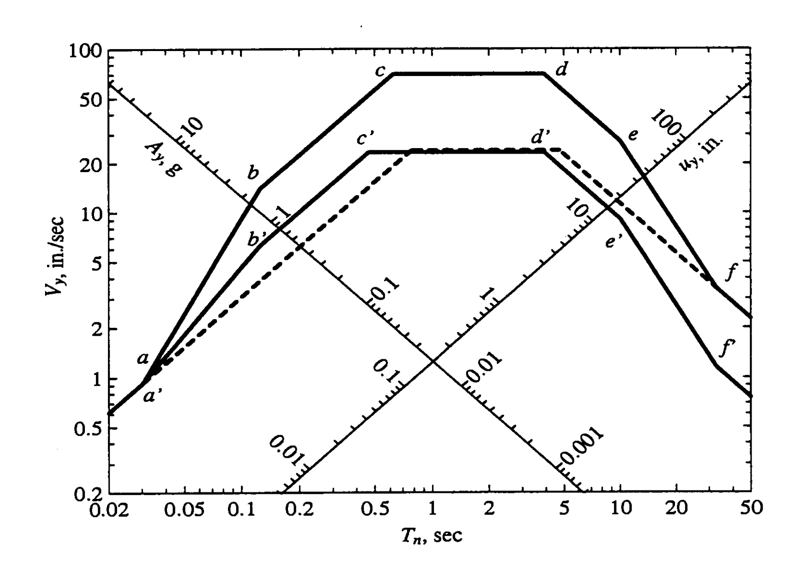


Fig. P7.6a

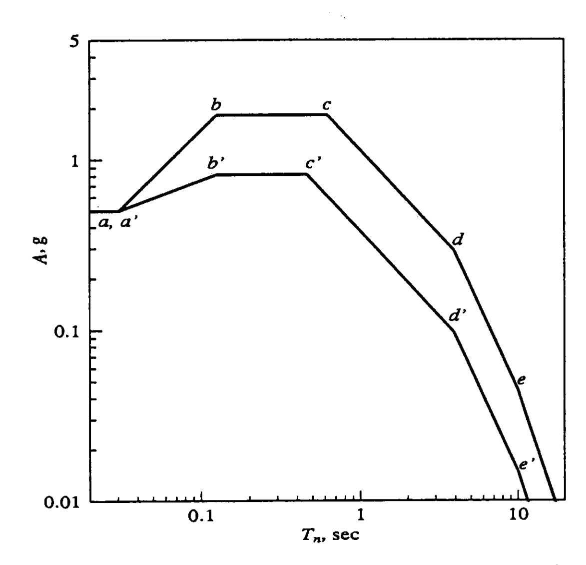


Fig. P7.6b

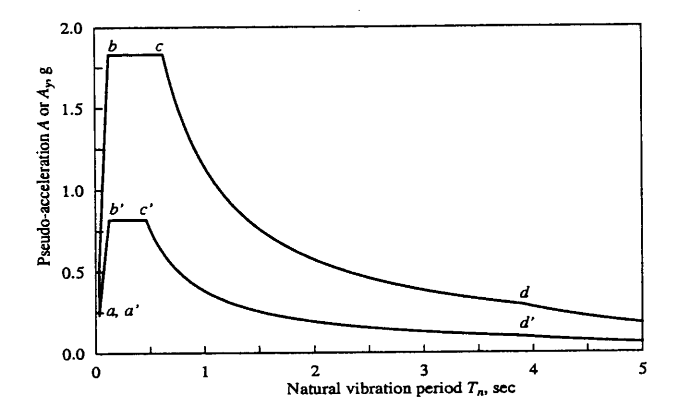

Pseudo-acceleration A or A_y, g
2.0
1.5
1.0
0.5
0.0
b
c
b'
c'
a, a'
d
d'
0
1
2
3
4
5
Natural vibration period T_n, sec

Fig. P7.6c

Problem 7.7

(a) $T_n = 0.02$ *sec*

This system is on the $T_n < T_a$ branch of the design spectrum (Fig. 6.9.4) and Eqs. (7.11.3) apply. For a linearly elastic system, $A = (1g)\,0.50 = 0.5g$. Then

$$f_o = \left(\frac{A}{g}\right) w = 0.5w$$

$$u_o = \frac{A}{\omega_n^2} = \left(\frac{T_n}{2\pi}\right)^2 A = 1.955 \times 10^{-3} \text{ in.}$$

Substituting for f_o and u_o in Eq. (7.11.3) gives the following results for $\mu = 1$, 2, 4 and 8:

μ	f_y/w	u_m, in.
1	0.50	1.955×10^{-3}
2	0.50	3.910×10^{-3}
4	0.50	7.820×10^{-3}
8	0.50	15.640×10^{-3}

Yielding has no influence on the design force but the design deformation is increased proportional to the allowable ductility.

(b) $T_n = 0.2$ *sec*

For this ystem $T_b < T_n < T_c'$ and T_n is on the constant-A_y branch of the design spectrum for all values of $\mu = 1$ to 8 [(Figs. 7.11.1 and 7.11.5) and Eq. (7.11.4) apply]. For a linearly elastic system, $A = (2.71g)\,0.5 = 1.355g$. Then

$$f_o = \left(\frac{A}{g}\right) w = 1.355w$$

$$u_o = \frac{A}{\omega_n^2} = \left(\frac{T_n}{2\pi}\right)^2 A = 0.53 \text{ in.}$$

Substituting for f_o and u_o in Eq. (7.11.4) gives the following results for $\mu = 1, 2, 4$ and 8:

μ	f_y/w	u_m, in.
1	1.355	0.530
2	0.782	0.612
4	0.512	0.801
8	0.350	1.095

Yielding reduces the design force by the factor of $\sqrt{2\mu - 1}$, but increases the design deformation by the factor $\mu/\sqrt{2\mu - 1}$.

(c) $T_n = 2$ *sec*

This system is on the $T_n > T_c$ part of the design spectrum (Fig. 6.9.4) and Eq. (7.11.5) apply. For a linearly elastic system,

$$V = (110.4)(0.50) = 55.2$$

$$A = \left(\frac{2\pi}{T_n}\right) V = 0.448g$$

$$f_o = \left(\frac{A}{g}\right) w = 0.448w$$

Problem 7.8

For a system with $T_n = 2$ sec, Fig. 6.9.5 gives $A = (1.80g/2)0.5 = 0.45g$. Equation (7.12.1) gives

$$\frac{A_y}{g} = \frac{f_y}{w} = \frac{0.112w}{w} = 0.112$$

and Eq. (7.12.2) leads to

$$R_y = \frac{A}{A_y} = \frac{0.45g}{0.112g} = 4$$

Knowing R_y, μ can be computed from Eq. (7.11.2) for $T_n = 2$ sec:

$$\mu = R_y = 4$$

Then Eq. (7.12.3) gives

$$u_m = 4\frac{1}{4}\left(\frac{2}{2\pi}\right)^2 0.45g = 17.6 \text{ in.}$$

Problem 7.9

For transverse ground motion the viaduct can be idealized as an SDF system with its lateral stiffness computed from

$$k = \frac{3EI}{h^3} \qquad \text{(a)}$$

where E is the elastic modulus of concrete, I is the effective moment of inertia of the reinforced concrete cross-section, and h is the column length. Based on the American Concrete Institute design provisions ACI 318-95, the effective EI for circular columns subjected to lateral load is given by

$$EI = E_c I_g \left(0.2 + 2\rho_t \gamma^2 \frac{E_s}{E_c} \right) \qquad \text{(b)}$$

where I_g is the second moment of area of the gross cross-section, E_c and E_s are the elastic moduli of concrete and reinforcing steel, ρ_t is the longitudinal reinforcement ratio, and γ is the ratio of the distances from the center of the column to the center of the outermost reinforcing bars and to the column edge.

We selected the following system properties: concrete strength = 4 ksi, steel strength = 60 ksi, and $\gamma = 0.9$.

The mass of the idealized SDF system is the tributary mass for one bent, i.e., the mass of 130 ft length of the superstructure:

$$m = \frac{w}{g} = \frac{1690}{386} = 4.378 \text{ kip} - \text{sec}^2/\text{in} \qquad \text{(c)}$$

The step-by-step procedure described in Section 7.12 is now implemented as follows:

1. An initial estimate of $u_y = 0.78$ in.

2. The plastic rotation acceptable at the base of the column is $\theta_p = 0.02$ radians.

3. The design displacement given by Eq. (7.12.5) is

$$u_m = u_y + h\theta_p = 0.78 + 156 \times 0.02 = 3.90 \text{ in.}$$

and the design ductility factor is

$$\mu = u_m / u_y = 3.90/0.78 = 5$$

4. The deformation design spectrum for inelastic systems is shown in Fig. P7.9 for $\mu = 5$. Corresponding to u_m = 3.9 in., this spectrum gives $T_n = 0.444$ sec and k is computed by Eq. (7.12.6):

$$k = \left(\frac{2\pi}{0.444} \right)^2 4.378 = 878.4 \text{ kips/in}$$

5. The yield strength is given by Eq. (7.12.7):

$$f_y = ku_y = 878.4 \times 0.78 = 685.1 \text{ kips}$$

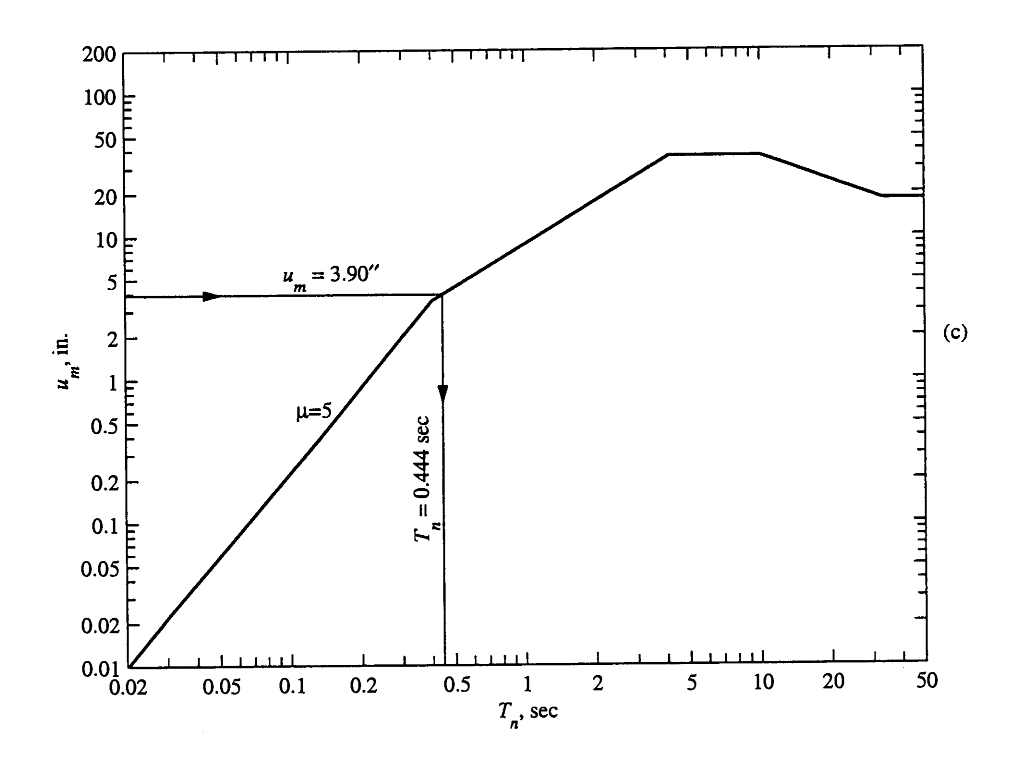
6. The circular column is then designed using ACI 318-95 for axial force due to dead load of 1690 kips due to the superstructure plus 55 kips due to self weight of the column and the bending moment due to lateral force = f_y: $M = hf_y = 106{,}876$ kip-in. For the resulting column design, $\rho_t = 3.5\%$, flexural strength = 118,800 kip-in, and lateral strength = 761.5 kips. For $\rho_t = 3.5\%$, Eq. (b) gives $EI = 1.54 \times 10^9 \text{ kip} - \text{in}^2$; using this EI value Eq. (a) gives k = 1219.9 kips/in. The yield deformation is $u_y = f_y / k$ = 761.5/1219.9 = 0.62 in.

7. Since the yield deformation computed in Step 6 differs significantly from the initial estimate of $u_y = 0.78$ in, iteration is necessary. The results of such iterations are summarized in Table P7.3.

The procedure converged after four iterations giving a column design with $\rho_t = 2.9\%$. This column has an initial stiffness, k = 1058.7 kips/in and lateral yield strength, f_y = 671.5 kips.

Table P7.9

No.	u_y (in.)	u_m (in.)	μ	T_n (secs)	k (kips/in.)	f_y (kips)	ρ_t (%)	Design f_y (kips)	Design k (kips/in.)	u_y (in.)
1	0.78	3.90	5.00	0.444	878.4	685.2	3.50	761.5	1219.9	0.62
2	0.62	3.74	6.00	0.426	953.0	594.9	2.80	664.6	1046.5	0.64
3	0.64	3.76	5.91	0.427	947.5	601.7	2.90	671.5	1058.7	0.63
4	0.63	3.75	5.92	0.427	947.9	601.2	2.90	671.5	1058.7	0.63

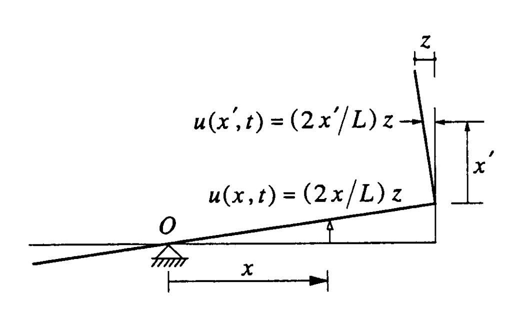


200
100
50
20
10
5
2
1
0.5
0.2
0.1
0.05
0.02
0.01
0.02
0.05
0.1
0.2
0.5
1
2
5
10
20
50
u_m, in.
T_n, sec
$u_m = 3.90''$
μ=5
$T_n = 0.444$ sec
(c)

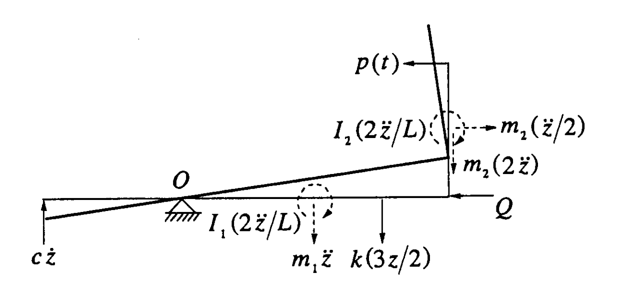
Fig. P7.9

Problem 8.1

1. *Determine the shape function.*



2. *Draw the free body diagram and write the equilibrium equation.*



$$\sum M_O = 0 \Rightarrow$$

$$I_1 \frac{2\ddot{z}}{L} + m_1 \ddot{z} \frac{L}{2} + I_2 \frac{2\ddot{z}}{L} + m_2 (2\ddot{z}) L + m_2 \frac{\ddot{z}}{2} \frac{L}{4}$$

$$+ c\dot{z} \frac{L}{2} + k \frac{3z}{2} \frac{3L}{4} = p(t) \frac{L}{2}$$

Substituting $I_1 = m_1 L^2/12$ and $I_2 = m_2 L^2/128$ gives

$$\left(\frac{2m_1 L}{3} + \frac{137 m_2 L}{64} \right) \ddot{z} + \left(\frac{cL}{2} \right) \dot{z} + \left(\frac{9kL}{8} \right) z = p(t) \frac{L}{2} \qquad \text{(a)}$$

The equation of motion (after dividing by L) is

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p}(t) \qquad \text{(b)}$$

where

$$\tilde{m} = \frac{2m_1}{3} + \frac{137 m_2}{64}, \tilde{c} = \frac{c}{2}, \tilde{k} = \frac{9k}{8}$$

and

$$\tilde{p}(t) = \frac{p(t)}{2} \qquad \text{(c)}$$

The relation between z and the rotation θ about fulcrum O is

$$z = \left(\frac{L}{2} \right) \theta \qquad \text{(d)}$$

Substituting Eq. (d) into Eq. (a) leads to the same equation of motion as in Example 8.1.

3. *Determine natural frequency and damping ratio.*

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} \qquad \zeta = \frac{\tilde{c}}{2\sqrt{\tilde{k}\tilde{m}}} \qquad \text{(e)}$$

which are the same as in Example 8.1.

4. *Solve the equation of motion.*

$$\tilde{p}(t) = \frac{p(t)}{2} = \frac{p_o}{2} \equiv \tilde{p}_o$$

$$z(t) = \frac{\tilde{p}_o}{\tilde{k}} (1 - \cos \omega_n t) = \frac{4p_o}{9k} (1 - \cos \omega_n t) \qquad \text{(f)}$$

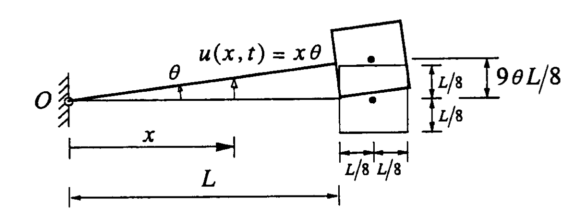
5. *Determine displacements.*

$$u(x,t) = \frac{2x}{L} z(t) \qquad u(x',t) = \frac{2x'}{L} z(t) \qquad \text{(g)}$$

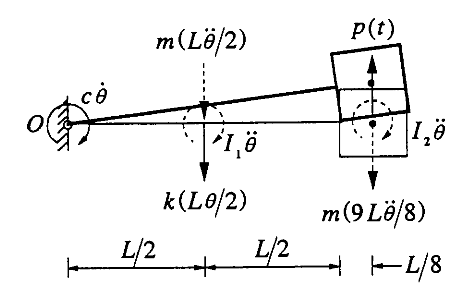
which are identical to the results in Example 8.1. Thus the results are independent of the choice of generalized displacement.

Problem 8.2

1. *Determine the shape function.*



2. *Draw the free body diagram and write the equilibrium equation.*



$$\sum M_O = 0 \Rightarrow$$

$$I_1\ddot{\theta} + \left(\frac{mL\ddot{\theta}}{2}\right)\frac{L}{2} + I_2\ddot{\theta} + \left(\frac{9mL\ddot{\theta}}{8}\right)\frac{9L}{8} + c\dot{\theta}$$

$$+ \left(\frac{kL\theta}{2}\right)\frac{L}{2} = p(t)\,\frac{9L}{8}$$

Substituting $I_1 = mL^2/12$ and $I_2 = (m/12)\left[(L/4)^2 + (L/4)^2\right] = mL^2/96$ gives

$$\tilde{m}\ddot{\theta} + \tilde{c}\dot{\theta} + \tilde{k}\theta = \tilde{p}(t) \qquad \text{(a)}$$

where

$$\tilde{m} = \frac{103mL^2}{64},\ \tilde{c} = c,\ \tilde{k} = \frac{kL^2}{4},\ \text{and}\ \tilde{p}(t) = \frac{9L}{8}\,p(t) \qquad \text{(b)}$$

3. *Determine natural frequency and damping ratio.*

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{\frac{16k}{103m}}; \quad \zeta = \frac{\tilde{c}}{2\sqrt{\tilde{k}\tilde{m}}} = \frac{8c}{\sqrt{103kmL^4}} \qquad \text{(c)}$$

4. *Solve the equation of motion.*

For $p(t) = \delta(t)$, the solution of Eq. (a) is

$$\theta(t) = \frac{9L/8}{\tilde{m}\omega_D} e^{-\zeta\omega_n t} \sin\omega_D t = \frac{72}{103mL\omega_D} e^{-\zeta\omega_n t} \sin\omega_D t \qquad \text{(d)}$$

where $\omega_D = \omega_n\sqrt{1-\zeta^2}$.

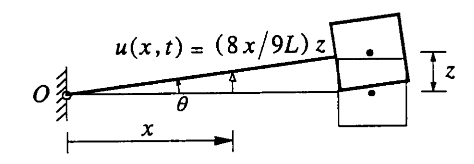
5. *Determine displacements.*

$$u(x,t) = x\,\theta(t) \qquad \text{(e)}$$

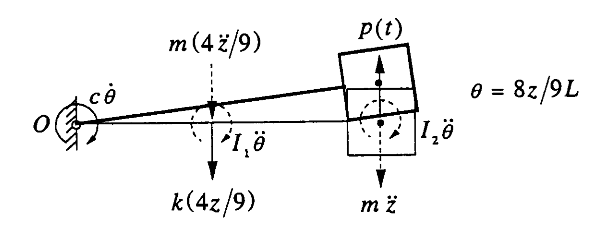
where $\theta(t)$ is given by Eq. (d).

Problem 8.3

1. *Determine the shape function.*



2. *Draw the free body diagram and write the equilibrium equation.*



$$\sum M_O = 0 \Rightarrow$$

$$I_1 \frac{8\ddot{z}}{9L} + \frac{4m\ddot{z}}{9}\frac{L}{2} + I_2 \frac{8\ddot{z}}{9L} + m\ddot{z}\frac{9L}{8} + c\frac{8\dot{z}}{9L}$$

$$+ \frac{4kz}{9}\frac{L}{2} = p(t)\frac{9L}{8}$$

Substituting $I_1 = mL^2/12$ and $I_2 = mL^2/96$ and dividing by L gives

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p}(t) \tag{a}$$

where

$$\tilde{m} = \frac{103m}{72}, \tilde{c} = \frac{8c}{9L^2}, \tilde{k} = \frac{2k}{9}, \text{and } \tilde{p}(t) = \frac{9}{8}p(t) \tag{b}$$

3. *Determine natural frequency and damping ratio.*

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{\frac{16k}{103m}}; \quad \zeta = \frac{\tilde{c}}{2\sqrt{\tilde{k}\tilde{m}}} = \frac{8c}{\sqrt{103kmL^4}} \tag{c}$$

4. *Solve the equation of motion.*

For $p(t) = \delta(t)$, the solution of Eq. (a) is

$$z(t) = \frac{9/8}{\tilde{m}\omega_D} e^{-\zeta\omega_n t} \sin \omega_D t = \frac{81}{103m\omega_D} e^{-\zeta\omega_n t} \sin \omega_D t \tag{d}$$

where $\omega_D = \omega_n\sqrt{1 - \zeta^2}$.

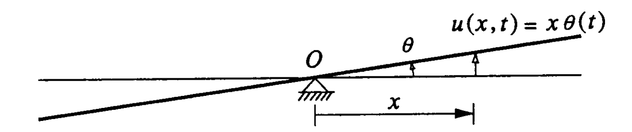
5. *Determine displacements.*

$$u(x,t) = \frac{8x}{9L} z = \frac{72x}{103mL\omega_D} e^{-\zeta\omega_n t} \sin \omega_D t \tag{e}$$

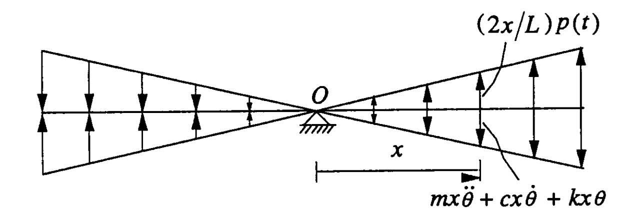
This result is identical to Eqs. (d) and (e) in Problem 8.2, i.e., the response is independent of the choice of generalized displacement.

Problem 8.4

1. *Determine the shape function.*



2. *Draw the free body diagram and write the equilibrium equation.*



$$\sum M_O = 0 \Rightarrow$$

$$\int_{-L/2}^{L/2} (mx\ddot{\theta} + cx\dot{\theta} + kx\theta)\, x\, dx = \int_{-L/2}^{L/2} \left[\frac{2x}{L} p(t)\right] x\, dx$$

or

$$\frac{mL^3}{12}\ddot{\theta} + \frac{cL^3}{12}\dot{\theta} + \frac{kL^3}{12}\theta = \frac{L^2}{6} p(t)$$

The equation of motion is

$$\tilde{m}\ddot{\theta} + \tilde{c}\dot{\theta} + \tilde{k}\theta = \tilde{p}(t)$$

where

$$\tilde{m} = \frac{mL^3}{12},\ \tilde{c} = \frac{cL^3}{12},\ \tilde{k} = \frac{kL^3}{12},$$

and

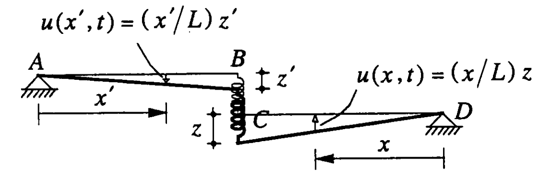
$$\tilde{p}(t) = \frac{L^2}{6} p(t)$$

3. *Determine natural frequency and damping ratio.*

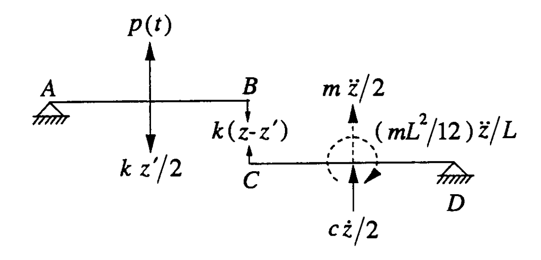
$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{\frac{k}{m}};\ \zeta = \frac{\tilde{c}}{2\sqrt{\tilde{k}\tilde{m}}} = \frac{c}{2\sqrt{km}}$$

Problem 8.5

1. *Determine the shape function.*



2. *Draw the free body diagram and write the equilibrium equation.*



$\sum M_A = 0$ for bar $AB \Rightarrow$

$$\frac{kz'}{2}\frac{L}{2} + p(t)\frac{L}{2} - k(z - z')L = 0 \Rightarrow$$

$$z' = \frac{4z}{5} - \frac{2}{5}\frac{p(t)}{k} \qquad \text{(a)}$$

The force in spring BC is

$$f_S = k(z - z') = \frac{kz}{5} + \frac{2}{5}p(t) \qquad \text{(b)}$$

$\sum M_D = 0$ for bar $CD \Rightarrow$

$$\frac{mL^2}{12}\frac{\ddot{z}}{L} + \frac{m\ddot{z}}{2}\frac{L}{2} + \frac{c\dot{z}}{2}\frac{L}{2} + f_S L = 0 \qquad \text{(c)}$$

Substituting Eq. (b) in Eq. (c) gives

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p}(t) \qquad \text{(d)}$$

where

$$\tilde{m} = \frac{m}{3}, \tilde{c} = \frac{c}{4}, \tilde{k} = \frac{k}{5}, \text{and } \tilde{p}(t) = -\frac{2}{5}p(t) \qquad \text{(e)}$$

3. *Determine natural frequency and damping ratio.*

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \sqrt{0.6\frac{k}{m}}; \zeta = \frac{\tilde{c}}{2\sqrt{\tilde{k}\tilde{m}}} = 0.484\frac{c}{\sqrt{km}}$$

Problem 8.6

1. *Determine the generalized properties.*

$$\tilde{m} = \int_0^L m(x)\left[\psi(x)\right]^2 dx$$

$$= m\int_0^L \left(\frac{3}{2}\frac{x^2}{L^2} - \frac{1}{2}\frac{x^3}{L^3}\right)^2 dx = \frac{33}{140}mL$$

$$\tilde{k} = \int_0^L EI(x)\left[\psi''(x)\right]^2 dx$$

$$= EI\int_0^L \left(\frac{3}{L^2} - \frac{3x}{L^3}\right)^2 dx = \frac{3EI}{L^3}$$

$$\tilde{L} = \int_0^L m(x)\,\psi(x)\,dx$$

$$= m\int_0^L \left(\frac{3}{2}\frac{x^2}{L^2} - \frac{1}{2}\frac{x^3}{L^3}\right) dx = \frac{3mL}{8}$$

2. *Determine the natural period.*

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \frac{3.57}{L^2}\sqrt{\frac{EI}{m}} = 1.76\,;\; T_n = 3.57 \text{ sec}$$

3. *Formulate the equation of motion.*

$$\ddot{z} + \omega_n^2 z = -\tilde{\Gamma}\,\ddot{u}_g(t);\; \tilde{\Gamma} = 1.59$$

4. *Determine the peak value of $z(t)$.*

For $T_n = 3.57$ sec and $\zeta = 0.05$, the design spectrum gives

$$\frac{A}{g} = 0.25\left(\frac{1.80}{3.57}\right) = 0.126\,;\; D = \frac{A}{\omega_n^2} = 15.7 \text{ in.}$$

$$\therefore\; z_o = \tilde{\Gamma}D = 25.0 \text{ in.}$$

5. *Determine peak displacements of the tower.*

$$u_o(x) = z_o\,\psi(x) = 25.0\left(\frac{3}{2}\frac{x^2}{L^2} - \frac{1}{2}\frac{x^3}{L^3}\right) \text{in.}$$

6. *Determine equivalent static forces.*

$$f_o(x) = \tilde{\Gamma}m(x)\,\psi(x)\,A = 11.19\left(\frac{3}{2}\frac{x^2}{L^2} - \frac{1}{2}\frac{x^3}{L^3}\right) \text{kip/ft}$$

7. *Compute shear and bending moment at mid-height and at the base.*

$$V_o(L/2) = \int_{L/2}^{L} f_o(\xi)\,d\xi = 2{,}151 \text{ kips}$$

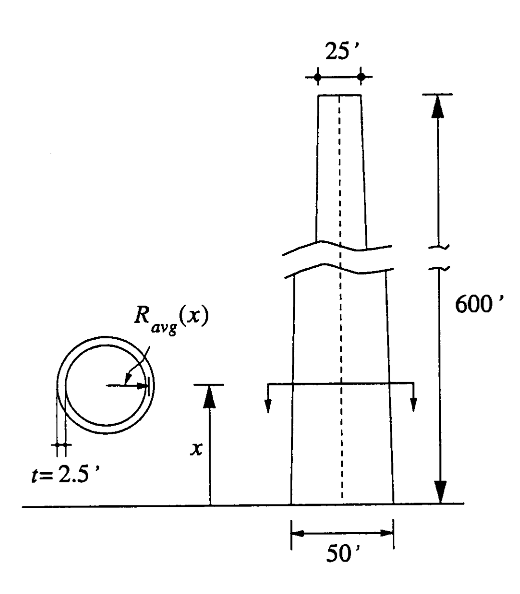
$$\mathcal{M}_o(L/2) = \int_{L/2}^{L} \left(\xi - \frac{L}{2}\right) f_o(\xi)\,d\xi$$

$$= 0.3806 \times 10^6 \text{ kip-ft}$$

$$V_{bo} = \int_0^L f_o(\xi)\,d\xi = 2{,}518 \text{ kips}$$

$$\mathcal{M}_o = \int_0^L \xi f_o(\xi)\,d\xi = 1.108 \times 10^6 \text{ kip-ft}$$

Problem 8.7



1. *Determine properties of the chimney.*

$L = 600$ ft

$$R_{avg}(x) = 23.75 - \frac{12.5}{600}x$$

$$A(x) = 2\pi R_{avg}(x)\, t = 373.06 - 0.327x \ \text{ft}^2$$

$$m(x) = \frac{150}{32.2}A(x) = 1.738 - \left(1.523 \times 10^{-3}\right)x \ \text{kip}-\text{sec}^2/\text{ft}^2$$

$$I(x) = \pi R_{avg}^3(x)\, t = 105{,}216 - 276.9x + 0.243x^2 - \left(7.102 \times 10^{-5}\right)x^3 \ \text{ft}^4$$

$$EI(x) = 5.454 \times 10^{10} - \left(1.435 \times 10^8\right)x + \left(1.259 \times 10^5\right)x^2 - 36.82x^3 \ \text{kip}-\text{ft}^2$$

2. *Determine $\tilde{m}$, $\tilde{k}$, $\tilde{\Gamma}$ and T_n.*

$$\tilde{m} = \int_0^L m(x)\left[\psi(x)\right]^2 dx = 134.5 \ \text{kip-sec}^2/\text{ft}$$

$$\tilde{k} = \int_0^L EI(x)\left[\psi''(x)\right]^2 dx = 483.5 \ \text{kips/ft}$$

$$\tilde{L} = \int_0^L m(x)\,\psi(x)\, dx = 231.6 \ \text{kip}-\text{sec}^2/\text{ft}$$

$$\tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}} = 1.722$$

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = 1.896 \ \text{rad/sec}; \ T_n = \frac{2\pi}{\omega_n} = 3.313 \ \text{sec}$$

3. *Determine the peak value of $z(t)$.*

For $T_n = 3.313$ sec and $\zeta = 0.05$, the design spectrum gives

$$\frac{A}{g} = 0.25\left(\frac{1.80}{3.313}\right) = 0.1358\,; \ D = \frac{A}{\omega_n^2} = 14.58 \ \text{in.}$$

$$\therefore \ z_o = \tilde{\Gamma} D = 25.1 \ \text{in.}$$

4. *Determine peak displacements at the top.*

$$u_o(L) = \psi(L)\, z_o = 25.1 \ \text{in.}$$

5. *Determine equivalent static forces.*

$$f_o(x) = \tilde{\Gamma} m(x)\,\psi(x)\, A = 1.722\left[1.738 - \left(1.523 \times 10^{-3}\right)x\right] \times \left[1 - \cos\frac{\pi x}{2L}\right](0.1358\text{g}) \ \text{kip/ft}$$

6. *Determine shear and bending moment at mid-height and at the base.*

$$V_o(L/2) = \int_{L/2}^{L} f_o(\xi)\, d\xi = 1{,}426 \ \text{kips}$$

$$\mathcal{M}_o(L/2) = \int_{L/2}^{L}\left(\xi - \frac{L}{2}\right) f_o(\xi)\, d\xi = 0.2399 \times 10^6 \ \text{kip-ft}$$

$$V_{bo} = \int_0^L f_o(\xi) d\xi = 1{,}739 \ \text{kips}$$

$$\mathcal{M}_{bo} = \int_0^L \xi f_o(\xi)\, d\xi = 0.7368 \times 10^6 \ \text{kip-ft}$$

Problem 8.8

1. *Determine properties of the chimney.*

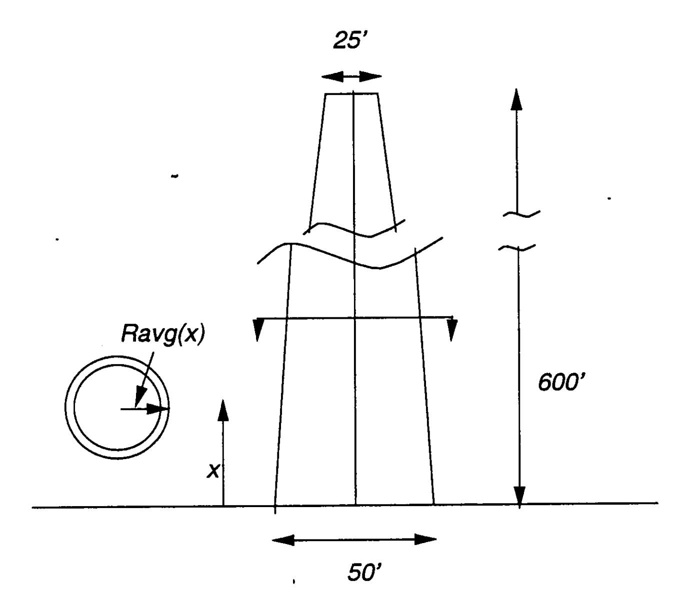
L=600 ft.

$$R_{avg}(x) = 23.75 - \frac{12.5}{600}x$$

$$A(x) = 2\pi R_{avg}(x)t = 373.06 - 0.327x \ \text{ft}^2$$

$$m(x) = \frac{150}{32.2}A(x)$$

$$= 1.738 - (1.523 \times 10^{-3})x \ \text{kip-sec}^2/\text{ft}^2$$



$$I(x) = \pi R_{avg}{}^3(x)t = 105{,}216 - 276.9x^3$$

$$+0.243x^2 - (7.102 \times 10^{-5})x^3 \ ft^4$$

$$EI(x) = 5.454 \times 10^{10} - (1.435 \times 10^8)x$$

$$+(1.259 \times 10^5)x^2 - 36.82x^3 \ \text{kip-ft}^2$$

2. *Determine* $\tilde{m}, \tilde{k}, \tilde{\Gamma}$, and T_n.

$$\tilde{m} = \int_0^L m(x)[\psi(x)]^2 dx = 140.61 \text{kip-sec}^2/\text{ft}$$

$$\tilde{k} = \int_0^L EI(x)[\psi''(x)]^2 dx = 515.93 \text{kips/ft}$$

$$\tilde{L} = \int_0^L m(x)\psi(x)dx = 240.27 \text{kip-sec}^2/\text{ft}$$

$$\tilde{\Gamma} = \tilde{L}/\tilde{m} = 1.709$$

$$\omega_n = \sqrt{\tilde{k}/\tilde{m}} = 1.915 \, rad/\text{sec} \ ; \ T_n = 2\pi/\omega_n = 3.28 \text{sec}$$

3. *Determine the peak value of z(t).*

For $T_n = 3.28$ sec and $\zeta = 0.05$, the design spectrum gives:

$$\frac{A}{g} = 0.25(1.80/3.28) = 0.137$$

$$D = A/\omega_n{}^2 = 14.45 \text{ in.}$$

$$z_o = \tilde{\Gamma}D = 24.7 \text{ in.}$$

4. *Determine peak dispalcement at the top.*

$$u_o(L) = \psi(L)z_o = 24.7 in$$

5. *Determine equivalent static forces.*

$$f_o(x) = \tilde{\Gamma}m(x)\psi(x)A$$

$$= 1.709\left[1.738 - (1.523 \times 10^{-3})x\right] \times$$

$$\left(\frac{3}{2}\frac{x^2}{L^2} - \frac{1}{2}\frac{x^3}{L^3}\right) \times 0.137g$$

6. *Determine shear and bending moment at mid-height and at base.*

$$V_o(L/2) = \int_{L/2}^{L} f_o(\xi)\,d\xi = 1465 \text{ kips}$$

$$V_{bo} = \int_0^L f_o(\xi)d\xi = 1811 \text{ kips}$$

$$\mathcal{M}_o(L/2) = \int_{L/2}^{L} \left(\xi - \frac{L}{2}\right) f_o(\xi)\,d\xi$$

$$= 0.244 \times 10^6 \text{ kip-ft}$$

$$\mathcal{M}_{bo} = \int_0^L \xi f_o(\xi)\,d\xi = 0.759 \times 10^6 \text{ kip-ft}$$

Problem 8.9

The excitation force over the height is

$$p(x,t) = \left(\frac{x}{L}\right) p(t)$$

1. *Formulate equation of motion.*

$$\tilde{m}\ddot{z} + \tilde{k}z = \tilde{p}(t)$$

$$\tilde{m} = 134.5 \text{ kip-sec}^2/\text{ft}; \; \tilde{k} = 483.5 \text{ kips/ft};$$

$$\tilde{p}(t) = \int_0^L p(x,t)\, \psi(x)\, dx$$

$$= \int_0^L \left[\frac{x}{L} p(t)\right]\left(1 - \cos\frac{\pi x}{2L}\right) dx = 161.2\, p(t)$$

2. *Solve the equation of motion.*

Note that $t_d/T_n = 0.25/3.313 = 0.075$. Because $t_d/T_n \ll 0.25$, the excitation can be approximated as a pure impulse. Adapting Eq. (4.10.3) to this problem gives

$$z_o = \frac{1}{\tilde{m}\omega_n} \int_0^{t_d} \tilde{p}(t) dt = \frac{161.2}{(134.5)\,(1.896)} \int_0^{0.25} p(t) dt$$

$$= 0.316 \text{ ft} = 3.79 \text{ in.}$$

3. *Determine peak responses.*

$$u_o(L) = \psi(L)\, z_o = 3.79 \text{ in.}$$

$$f_o(x) = \omega_n^2 m(x)\, \psi(x)\, z_o = 1.136 m(x)\, \psi(x) \;\; \text{kip/ft}$$

$$V_o(L/2) = \int_{L/2}^{L} f_o(\xi)\, d\xi = 216 \text{ kips}$$

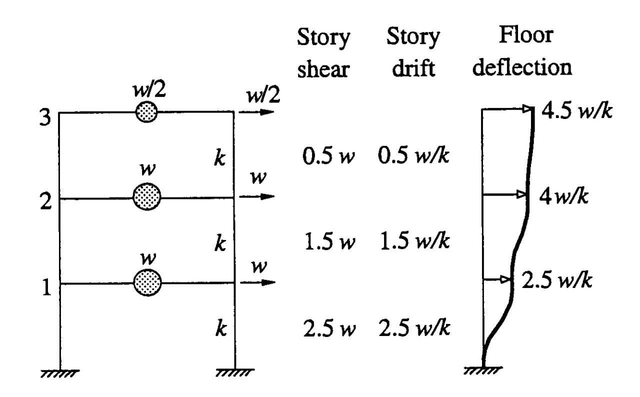
$$\mathcal{M}_o(L/2) = \int_{L/2}^{L} \left(\xi - \frac{L}{2}\right) f_o(\xi)\, d\xi = 36{,}290 \text{ kip-ft}$$

$$V_{bo} = \int_0^L f_o(\xi)\, d\xi = 263 \text{ kips};$$

$$\mathcal{M}_{bo} = \int_0^L \xi f_o(\xi)\, d\xi = 111{,}485 \text{ kip-ft}$$

Problem 8.10

1. *Determine the shape function.*



Shape vector:

$$\psi = \langle 5/9 \quad 8/9 \quad 1 \rangle^T$$

2. *Determine generalized properties.*

$$\tilde{m} = \sum_{j=1}^{3} m_j \psi_j^2 = \frac{259}{162}\frac{w}{g}$$

$$\tilde{k} = \sum_{j=1}^{3} k_j (\psi_j - \psi_{j-1})^2 = \frac{35k}{81}$$

$$\tilde{L} = \sum_{j=1}^{3} m_j \psi_j = \frac{35}{18}\frac{w}{g}$$

3. *Determine the natural period.*

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = 0.520\sqrt{\frac{kg}{w}}$$

where

$$w = 100 \text{ kips}$$

$$k = 2\left[\frac{12\,(29000 \times 1400)}{(12 \times 12)^3}\right] = 326.3 \text{ kips/in.}$$

Thus

$$\omega_n = 0.520\sqrt{\frac{326.3 \times 386}{100}} = 18.45 \text{ rads/sec}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.3405 \text{ sec}$$

4. *Determine the equation of motion.*

$$\ddot{z} + \omega_n^2 z = -\tilde{\Gamma}\ddot{u}_g(t);\ \tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}} = 1.216$$

5. *Determine the peak value of $z(t)$.*

For $T_n = 0.3405$ sec and $\zeta = 0.05$, the design spectrum gives

$$A = 0.25\,(2.71\text{g}) = 0.6775\text{g};$$

$$D = \frac{A}{\omega_n^2} = 0.768 \text{ in.}$$

$$z_o = \tilde{\Gamma} D = 1.216\,(0.768) = 0.934 \text{ in.}$$

6. *Determine floor displacements.*

$$u_{jo} = \psi_j z_o \Rightarrow$$

$$u_{1o} = 0.519 \text{ in.},\ u_{2o} = 0.830 \text{ in.},\ u_{3o} = 0.934 \text{ in.}$$

7. *Determine equivalent static forces.*

$$f_{jo} = \tilde{\Gamma} m_j \psi_j A \Rightarrow$$

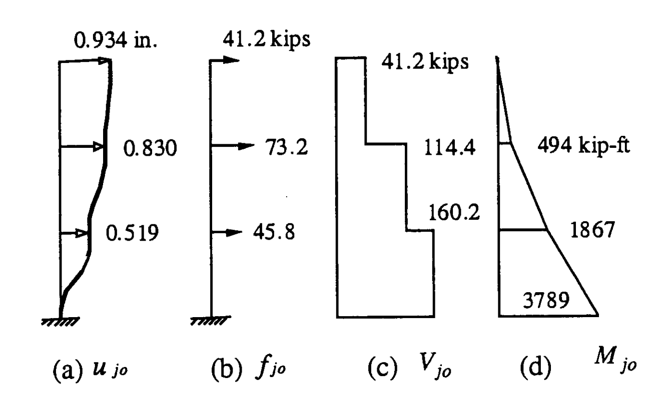
$$f_{1o} = 1.216\left(\frac{100}{\text{g}}\right)\left(\frac{5}{9}\right)(0.6775\text{g}) = 45.8 \text{ kips}$$

$$f_{2o} = 1.216\left(\frac{100}{\text{g}}\right)\left(\frac{8}{9}\right)(0.6775\text{g}) = 73.2 \text{ kips}$$

$$f_{3o} = 1.216\left(\frac{50}{g}\right)(1)\,(0.6775\text{g}) = 41.2 \text{ kips}$$

8. *Determine story shears and overturning moments.*

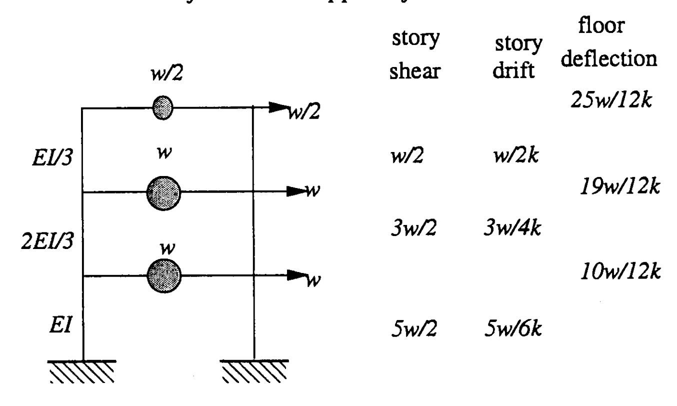
Static analysis of the structure with lateral forces f_{jo} gives the story shears in Fig. (c) and story overturning moments in Fig. (d).



(a) u_{jo} (b) f_{jo} (c) V_{jo} (d) M_{jo}

Problem 8.11

1. *Determine deflections to applied forces.*



In this figure, the stiffness of the top story is

$$k = 2\times\frac{12(EI/3)}{h^3} = \frac{8EI}{h^3}$$

2. *Determine shape vector:*

$$\psi = \langle 2/5 \quad 19/25 \quad 1\rangle^T$$

3. *Determine generalized properties.*

$$\tilde{m} = \sum_{j=1}^{3} m_j \psi_j^2 = 1.237\, w/g$$

$$\tilde{k} = \sum_{j=1}^{3} k_j(\psi_j - \psi_{j-1})^2 = 0.797\, k$$

$$\tilde{L} = \sum_{j=1}^{3} m_j \psi_j = 1.66\, w/g$$

4. *Determine the natural period.*

$$\omega_n = \sqrt{\tilde{k}/\tilde{m}} = 0.803\sqrt{k/m}$$

For $E = 29000$ ksi, $I = 1400\ \text{in}^4$., and $h = 12$ ft,

$k = 108.77$ kip/in

For w = 100 kips, m = w/g = 0.259 kip-sec^2/in.

$$\omega_n = 16.45 \text{ rad/sec}$$

$$T_n = 2\pi/\omega_n = 0.381 \text{sec}$$

5. *Determine the equation of motion.*

$$\tilde{\Gamma} = \tilde{L}/\tilde{m} = 1.342$$

$$\ddot{z} + \omega_n^2 z = -1.342\, \ddot{u}_g(t)$$

6. *Determine the peak value of z(t).*

For $T_n = 0.381$sec and $\zeta = 0.05$,the design spectrum gives:

$$A/g = 2.71\times 0.25 = 0.677\ ;\ D = \frac{A}{\omega_n^2} = 0.966 \text{ in.}$$

$$z_o = \tilde{\Gamma} D = 1.297 \text{ in.}$$

7. *Determine floor displacements.*

$$u_{jo} = \psi_j z_o \Rightarrow$$

$u_{10} = 0.519$; $u_{20} = 0.985$; $u_{30} = 1.297$; all in inches (Fig. a).

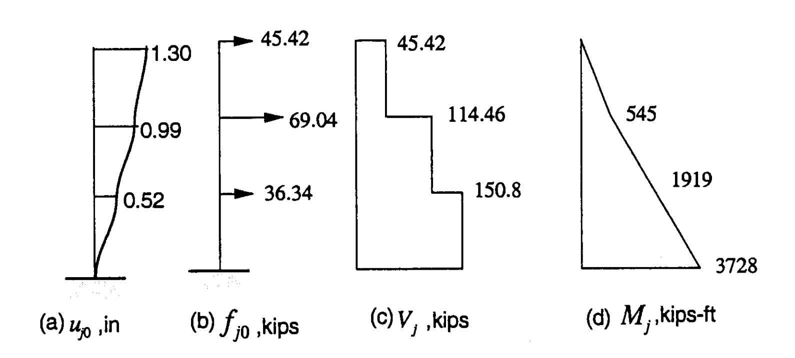
8. *Determine equivalent static forces.*

$$f_{jo} = \tilde{\Gamma} m_j \psi_j A = 350.69\, m_j \psi_j$$

$\Rightarrow f_{10} = 36.34$; $f_{20} = 69.04$; $f_{30} = 45.42$; all in kips (Fig. b).

9. *Determine story forces.*

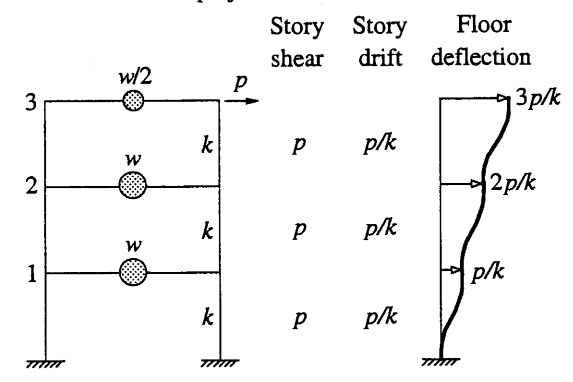
Static analysis of the structure due to external forces f_{j0} gives the story shears and floor overturning moments in Figs. (c) and (d).



(a) u_{j0} ,in (b) f_{j0} ,kips (c) V_j ,kips (d) M_j ,kips-ft

Problem 8.12

1. *Determine the shape function.*



Shape vector:

$$\psi = \langle 1/3 \quad 2/3 \quad 1 \rangle^T$$

2. *Determine generalized properties.*

$$\tilde{m} = \sum_{j=1}^{3} m_j \psi_j^2 = \frac{19}{18}\frac{w}{g}$$

$$\tilde{k} = \sum_{j=1}^{3} k_j (\psi_j - \psi_{j-1})^2 = \frac{k}{3}$$

$$\tilde{L} = \sum_{j=1}^{3} m_j \psi_j = \frac{3}{2}\frac{w}{g}$$

$$\tilde{\Gamma} = \frac{\tilde{L}}{\tilde{m}} = 1.421$$

3. *Determine the natural period.*

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = 0.5620\sqrt{\frac{kg}{w}}$$

where $w = 100$ kips and $k = 326.3$ kips/in. (from Problem 8.9).

$$\omega_n = 19.94 \text{ rads/sec}; \; T_n = 0.315 \text{ sec}$$

4. *Determine the peak value of* $z(t)$.

For $T_n = 0.315$ sec and $\zeta = 0.05$, the design spectrum gives

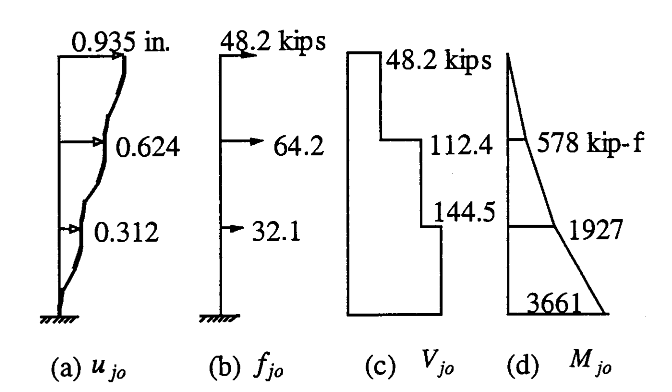
$$A = 0.25\,(2.71\text{g}) = 0.6775\text{g};$$

$$D = \frac{A}{\omega_n^2} = 0.6577 \text{ in.}$$

$$z_o = \tilde{\Gamma} D = 0.935 \text{ in.}$$

5. *Determine peak responses.*

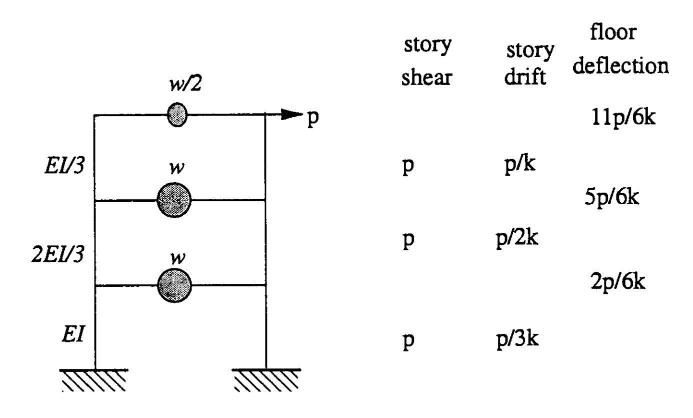
The floor displacements using Eq. (8.4.14) are given in Fig. (a). The equivalent static forces, calculated from Eq. (8.4.15) are shown in Fig. (b). Static analysis of the structure gives the story shears and story overturning moments in Figs. (c) and (d).



(a) u_{jo} (b) f_{jo} (c) V_{jo} (d) M_{jo}

Problem 8.13

1. *Determine deflections due to applied forces.*



2. *Determine shape vector:*

$$\psi = \langle 2/11 \quad 5/11 \quad 1 \rangle^T$$

3. *Determine generalized properties.*

$$\tilde{m} = \sum_{j=1}^{3} m_j \psi_j^2 = 0.739 w/g$$

$$\tilde{k} = \sum_{j=1}^{3} k_j (\psi_j - \psi_{j-1})^2 = 0.545k$$

$$\tilde{L} = \sum_{j=1}^{3} m_j \psi_j = 1.136m$$

$$\tilde{\Gamma} = \tilde{L}/\tilde{m} = 1.537$$

4. *Determine the natural period.*

$$\omega_n = \sqrt{\tilde{k}/\tilde{m}} = 0.858\sqrt{k/m}$$

From Problem 8.11:

$$k = 108.77 \text{ kip/in.}; \; m = 0.259 \text{ kip} - \text{sec}^2/\text{in.}$$

Substituting in above equation:

$$\omega_n = 17.58 \text{ rad/sec}$$

$$T_n = 2\pi/\omega_n = 0.357 \text{ sec}$$

5. *Determine the peak value of z(t).*

For $T_n = 0.357$sec and $\xi = 0.05$, the design spectrum gives:

$$A/g = 2.71 \times 0.25 = 0.677 \; ; \; D = \frac{A}{\omega_n^2} = 0.846 \text{ in}$$

$$z_o = \tilde{\Gamma} D = 1.3 \text{ in}$$

6. *Determine the peak responses.*

$$u_{jo} = \psi_j z_o \Rightarrow$$

$u_{10} = 0.236$; $u_{20} = 0.591$; $u_{30} = 1.3$; all in inches. (Fig. a).

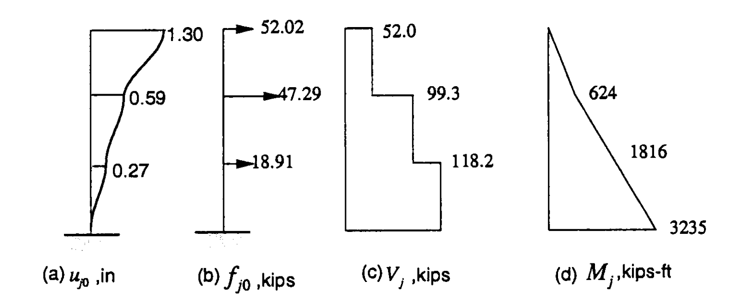
7. *Determine equivalent static forces.*

$$f_{jo} = \tilde{\Gamma} m_j \psi_j A = 401.65\, m_j \psi_j$$

$\Rightarrow f_{10} = 18.91$; $f_{20} = 47.29$; $f_{30} = 52.02$; all in kips (Fig. b).

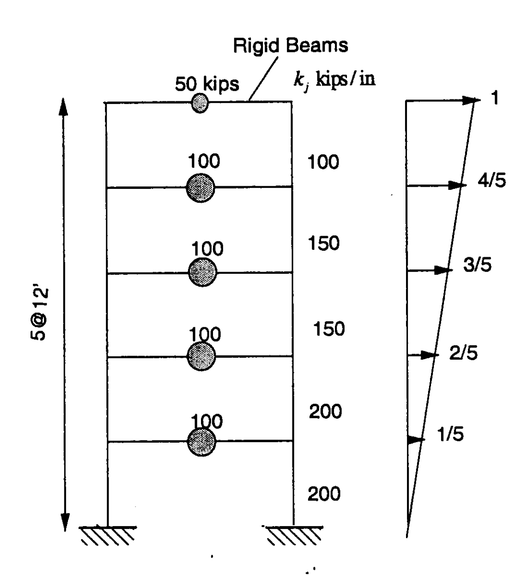
8. *Determine story forces.*

Static analysis of the structure due to external forces f_{j0} gives the story shears and floor overturning moments in Figs. (c) and (d).



(a) u_{j0}, in (b) f_{j0}, kips (c) V_j, kips (d) M_j, kips-ft

Problem 8.14



1. *Determine generalized properties.*

$$\tilde{m} = \sum_{j=1}^{5} m_j \psi_j^2 = 0.44 \text{ kip} - \text{sec}^2 / \text{in}$$

$$\tilde{k} = \sum_{j=1}^{5} k_j (\psi_j - \psi_{j-1})^2 = 32.0 \text{ kips / in}$$

$$\tilde{L} = \sum_{j=1}^{5} m_j \psi_j = 0.647 \text{ kip} - \text{sec}^2 / \text{in}$$

$$\tilde{\Gamma} = \tilde{L} / \tilde{m} = 1.471$$

2. *Formulate the equation of motion.*

$$\ddot{z} + \omega_n^2 z = -\tilde{\Gamma} \ddot{u}_g(t) = -1.471\, \ddot{u}_g(t) \;,$$

where z is the lateral displacement at the location where $\psi_j = 1$, in this case the top of the frame.

3. *Determine the natural vibration frequency and period.*

$$\omega_n = \sqrt{\tilde{k} / \tilde{m}} = 8.528 \text{ rad/sec}$$

$$T_n = 2\pi / \omega_n = 0.737 \text{ sec}$$

Note that the exact value of ω_n=8.262 rad/sec.

4. *Determine the peak value of z(t).*

For $T_n = 0.737$ and ζ=0.05, the design spectrum gives

$A/\text{g} = 0.25(1.80/0.737) = 0.61; D = A / \omega_n^2 = 3.244$ in.

$z_o = 1.471 D = 4.77$ in.

5. *Determine the peak values u_{j0} of floor displacements.*

$$u_{jo} = \psi_j z_o \qquad \psi_j = \frac{j}{5}$$

$\Rightarrow \; u_{10} = 0.95$; $u_{20} = 1.90$; u_{30}=2.85 ;

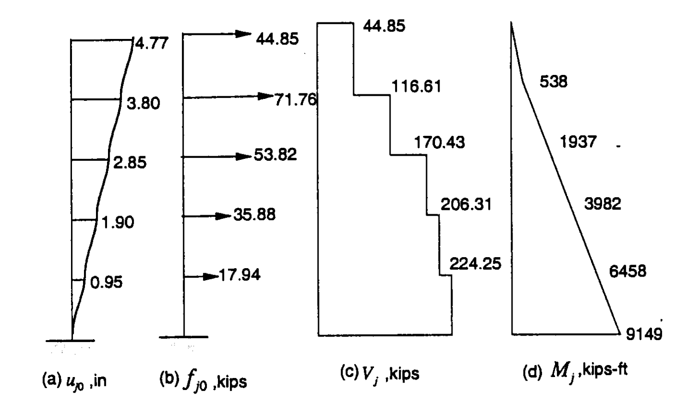
u_{40}=3.80 ; u_{50}=4.77 ; all in inches. (Fig. a)

6. *Determine equivalent static forces. (Fig. b).*

$$f_{j0} = \tilde{\Gamma} m_j \psi_j A = 346.6\, m_j \psi_j \text{ kips}$$

$\Rightarrow \; f_{10}$=17.94 ; f_{20}=35.88 ; f_{30}=53.82 ;

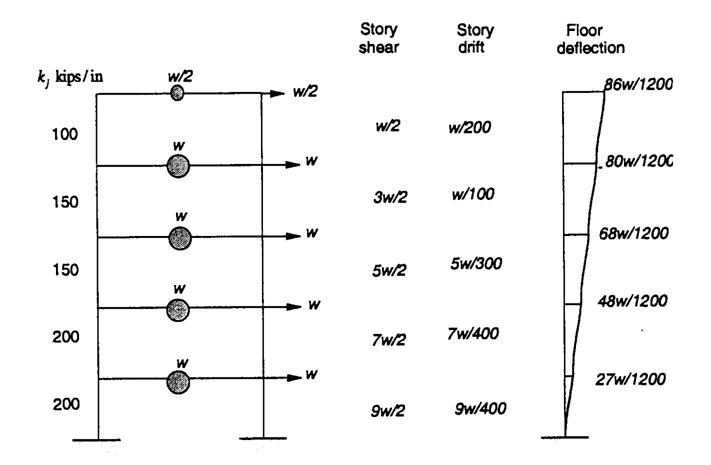
f_{40}=71.76 ; f_{50}=44.85 all in kips



7. *Determine story forces.*

Static analysis of the structure subjected to external forces f_{j0} gives the story shears and floor overturning moments in Figs. (c) and (d).

Problem 8.15



In the above figure, $w = 100$ kips

1. *Determine the deflections due to applied forces.*

The static deflections are determined by calculating the story shears and the resulting story drifts,and adding these drifts from the bottom to the top to obtain

$$\mathbf{u} = (w/1200)\langle 27 \quad 48 \quad 68 \quad 80 \quad 86\rangle^T$$

2. *Determine the shape vector.*

$$\psi = \langle 0.314 \quad 0.558 \quad 0.791 \quad 0.93 \quad 1\rangle^T$$

3. *Determine generalized properties.*

$$\tilde{m} = \sum_{j=1}^{5} m_j \psi_j^2 = 0.622 \text{ kip} - \text{sec}^2/\text{in.}$$

$$\tilde{K} = \sum_{j=1}^{5} k_j (\psi_j - \psi_{j-1})^2 = 43.15 \text{ kip/in.}$$

$$\tilde{L} = \sum_{j=1}^{5} m_j \psi_j = 0.801 \text{ kip} - \text{sec}^2/\text{in.}$$

$$\tilde{\Gamma} = \tilde{L}/\tilde{m} = 1.289$$

4. *Determine te natural vibration period.*

$$\omega_n = \sqrt{\tilde{k}/\tilde{m}} = 8.329 \text{ rad/sec}$$

$$T_n = 2\pi/\omega_n = 0.754 \text{ sec}$$

5. *Formulate the equation of motion.*

$$\ddot{z} + \omega_n^2 z = -\tilde{\Gamma}\ddot{u}_g(t) = -1.289\ddot{u}_g(t)$$

6. *Determine the peak value of z(t).*

For T_n=0.754 sec and ζ=0.05, the design spectrum gives:

$$\frac{A}{g} = \frac{1.8}{T_n} \times 0.25 = 0.597 \ ; \ D = \frac{A}{\omega_n^2} = 3.32 \text{ in}$$

7. *Determine the peak values u_{j0} of floor displacements.*

$$u_{jo} = \psi_j z_o$$

$\Rightarrow u_{10}$=1.34 $\quad u_{20}$=2.389 $\quad u_{30}$=3.385

u_{40}=3.98 $\quad u_{50}$=4.28 all in inches. (Fig. a).

8. *Determine the equivalent static forces. (Fig. b).*

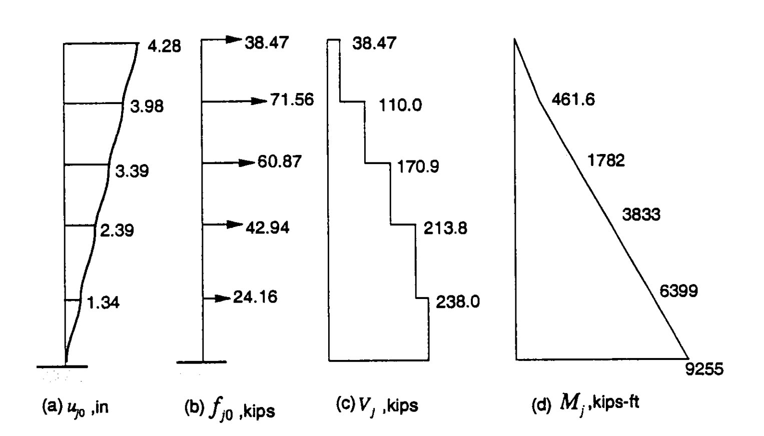
$$f_{j0} = \tilde{\Gamma} m_j \psi_j A = 297.03\, m_j \psi_j$$

$\Rightarrow f_{10}$=24.16 $\quad f_{20}$=42.94 $\quad f_{30}$=60.87

f_{40}=71.56 $\quad f_{50}$=38.47 all in kips

9. *Determine story forces.*

Static analysis of the structure subjected to external forces f_{j0} gives the story shears and floor overturning moments in Figs. (c) and (d).



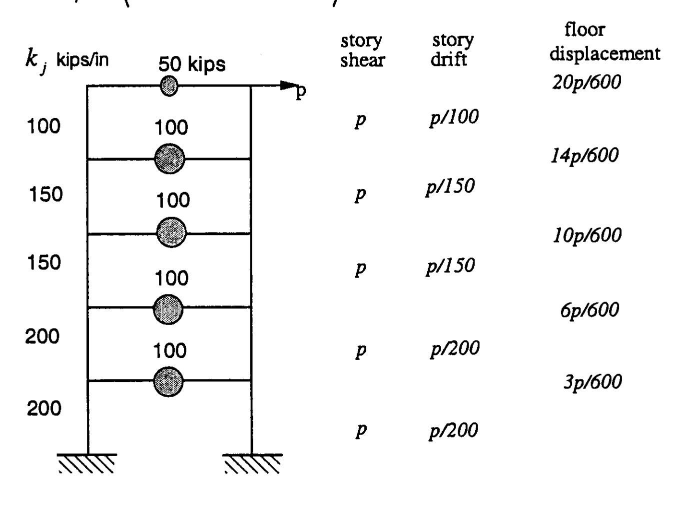
Problem 8.16

1. *Determine deflections due to applied force.*

$$\mathbf{u} = (p/600)\langle 3 \quad 6 \quad 10 \quad 14 \quad 20 \rangle^T$$

2. *Determine the shape vector.*

$$\psi = \langle 3 \quad 6 \quad 10 \quad 14 \quad 20 \rangle^T$$



3. *Determine generalized properties.*

$$\tilde{m} = \sum_{j=1}^{5} m_j \psi_j^2 = 140.2 \; ; \quad \tilde{k} = \sum_{j=1}^{5} k_j (\psi_j - \psi_{j-1})^2 = 12{,}000$$

$$\tilde{L} = \sum_{j=1}^{5} m_j \psi_j = 11.14 ; \quad \tilde{\Gamma} = \tilde{L} / \tilde{m} = 0.0795$$

4. *Determine the natural vibration period.*

$$\omega_n = \sqrt{\tilde{k} / \tilde{m}} = 9.25 \text{ rad/sec}$$

$$T_n = 2\pi / \omega_n = 0.679 \text{ sec}$$

5. *Formulate the equation of motion.*

$$\ddot{z} + \omega_n^2 z = -\tilde{\Gamma}\ddot{u}_g(t) = -0.0795\ddot{u}_g(t)$$

6. *Determine the peak value of z(t).*

For T_n=0.679 sec and ζ=0.05 ,the design spectrum gives:

$$\frac{A}{g} = \frac{1.8}{T_n} \times 0.25 = 0.663 \; ; D = \frac{A}{\omega_n^2} = 2.989 \text{ in.}$$

$$z_o = \tilde{\Gamma} D = 0.237 \text{ in.}$$

7. *Determine peak responses.*

$$u_{jo} = \psi_j z_o$$

$$\Rightarrow u_{10} = 0.713 \quad u_{20} = 1.426 \quad u_{30} = 2.37$$

$$u_{40} = 3.32 \quad u_{50} = 4.75 \quad \text{all in inches.}$$

8. *Determine equivalent static forces.*

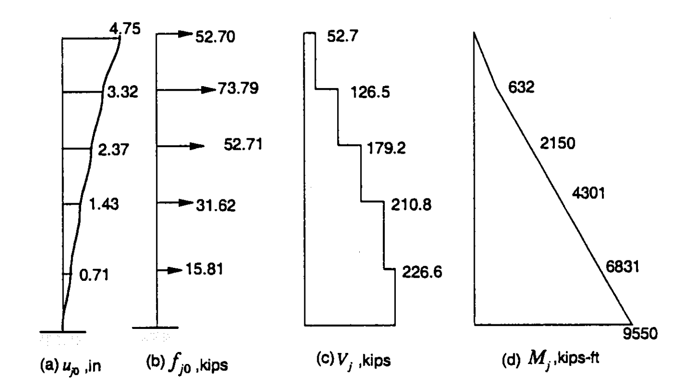
$$f_{jo} = \tilde{\Gamma} m_j \psi_j A = 20.345\, m_j \psi_j$$

$$\Rightarrow f_{10} = 15.81 \quad f_{20} = 31.62 \quad f_{30} = 52.71$$

$$f_{40} = 73.79 \quad f_{50} = 52.7 \quad \text{all in kips.}$$

9. *Determine story forces.*

Static analysis of the structure subjected to external forces f_{j0} gives the story shears and floor overturning moments in Figs. (c) and (d).



(a) u_{jo} ,in (b) f_{j0} ,kips (c) V_j ,kips (d) M_j ,kips-ft

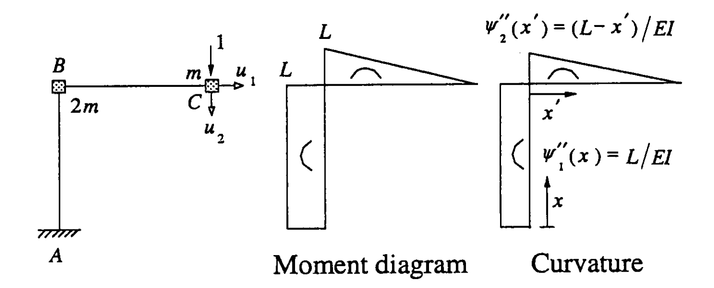
Problem 8.17

1. *Determine the shape function.*

$$\psi_1(x) = \int_0^x \int_0^x \psi_1''(\xi)\, d\xi\, d\xi;$$

$$\psi_1(0) = \psi_1'(0) = 0$$

$$\Rightarrow \psi_1(x) = \frac{L}{2EI} x^2$$



$$\psi_2(x') = \int_0^{x'} \int_0^{x'} \psi_2''(\xi)\, d\xi\, d\xi\, ;$$

$$\psi_2(0) = 0, \qquad \psi_2'(0) = \psi_1'(L) = \frac{L^2}{EI}$$

$$\Rightarrow \psi_2(x') = \frac{L^2}{EI} x' + \frac{L}{2EI}(x')^2 - \frac{1}{6EI}(x')^3$$

$$u_1 = \psi_1(L) = \frac{L^3}{2EI}; \qquad u_2 = \psi_2(L) = \frac{4L^3}{3EI}$$

2. *Determine the natural frequency.*

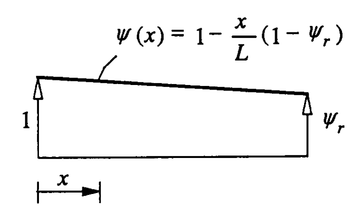
$$\tilde{m} = (2m + m)\, u_1^2 + m u_2^2 = \frac{91 m L^6}{36 E^2 I^2}$$

$$\tilde{k} = \int_0^L EI\left[\psi_1''(x)\right]^2 dx + \int_0^L EI\left[\psi_2''(x')\right]^2 dx' = \frac{4L^3}{3EI}$$

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = 0.726\sqrt{\frac{EI}{mL^3}}$$

Problem 8.18

1. *Determine a shape function.*



2. *Determine natural frequency.*

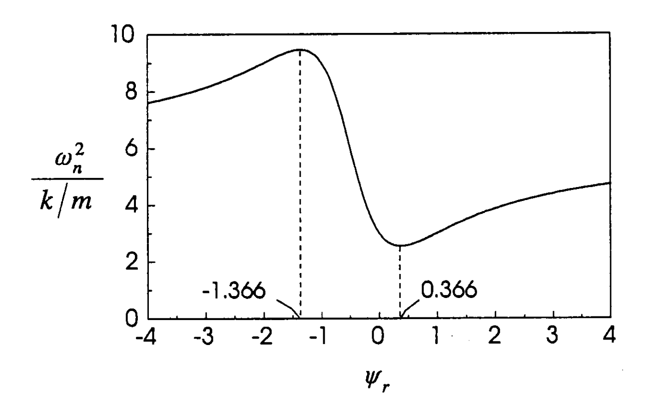
$$\tilde{k} = k(1)^2 + 2k\psi_r^2 = k(1 + 2\psi_r^2)$$

$$\tilde{m} = \int_0^L m(x)\left[\psi(x)\right]^2 dx$$

$$= \int_0^L \frac{m}{L}\left[1 - x(1-\psi_r)/L\right]^2 dx = m(1 + \psi_r + \psi_r^2)$$

$$\omega_n^2 = \frac{\tilde{k}}{\tilde{m}} = \frac{3k(1 + 2\psi_r^2)}{m(1 + \psi_r + \psi_r^2)}$$

A plot of ω_n^2 vs. ψ_r is shown.



3. *Find stationary values of ω_n^2.*

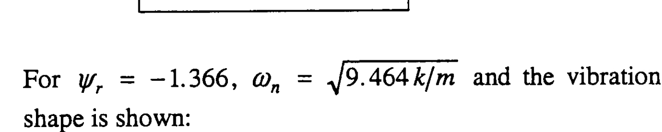
$$\frac{d\omega_n^2}{d\psi_r} = 0 \Rightarrow$$

$$\frac{4\psi_r(1 + \psi_r + \psi_r^2) - (1 + 2\psi_r^2)(1 + 2\psi_r)}{(1 + \psi_r + \psi_r^2)^2} = 0$$

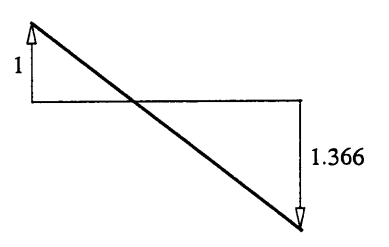
$$\Rightarrow 2\psi_r^2 + 2\psi_r - 1 = 0$$

$$\Rightarrow \psi_r = 0.366 \text{ or } -1.366$$

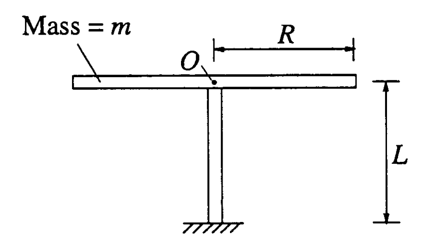
For $\psi_r = 0.366$, $\omega_n = \sqrt{2.536\,k/m}$ and the vibration shape is shown:



For $\psi_r = -1.366$, $\omega_n = \sqrt{9.464\,k/m}$ and the vibration shape is shown:



Problem 8.19



1. *Determine the shape function.*

Assume that the slab is rigid in flexure and column is axially rigid. The system has two degrees of freedom: lateral deflection $u(L)$ and rotation θ of the slab. To represent the system as a generalized SDF system, we assume a suitable deflected shape for the column:

$$\psi(x) = 1 - \cos\frac{\pi x}{2L}$$

Then

$u(x,t) = z_o\, \psi(x) \sin \omega_n t'$; $u_o(x) = z_o\, \psi(x)$

$\dot{u}(x,t) = \omega_n z_o\, \psi(x) \cos \omega_n t'$; $\dot{u}_o(x) = \omega_n z_o\, \psi(x)$

2. *Determine* E_{So} *and* E_{Ko}.

$$E_{So} = \int_0^L \frac{1}{2} EI(x) \left[u_o''(x)\right]^2 dx$$

$$= \frac{1}{2} z_o^2 \int_0^L EI(x) \left[\psi''(x)\right]^2 dx$$

$$= \frac{1}{2} z_o^2\, EI \left(\frac{\pi}{2L}\right)^4 \int_0^L \cos^2\left(\frac{\pi x}{2L}\right) dx = \frac{\pi^4 EI}{64 L^3} z_o^2$$

$$E_{Ko} = \frac{1}{2} m \left[\dot{u}_o(L)\right]^2 + \frac{1}{2} I_O \left[\dot{u}_o'(L)\right]^2$$

$$= \frac{1}{2} \omega_n^2\, z_o^2 \left[m\, \psi(L)^2 + I_O\, \psi'(L)^2\right]$$

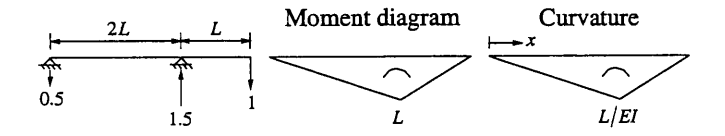
$$= \frac{1}{2} \omega_n^2\, z_o^2 \left[m + \frac{mR^2}{4}\left(\frac{\pi}{2L}\right)^2\right]$$

$$= \frac{1}{2}\, \omega_n^2 z_o^2 m \left[1 + \frac{\pi^2}{16}\left(\frac{R}{L}\right)^2\right]$$

3. *Determine* ω_n.

$$E_{Ko} = E_{So} \Rightarrow \omega_n = \sqrt{\frac{\pi^4 EI/32 L^3}{m\left[1 + (\pi^2/16)(R/L)^2\right]}}$$

Problem 8.20



1. *Determine the shape function.*

Calculate the deflection due to a unit force applied at the free end:

$$u''(x) = \begin{cases} \dfrac{x}{2EI} & 0 \le x \le 2L \\ \dfrac{3L - x}{EI} & 2L \le x \le 3L \end{cases}$$

Boundary conditions:

$u(0) = u(2L) = 0,\ u'(2L^-) = u'(2L^+) \Rightarrow$

$$u(x) = \begin{cases} \dfrac{1}{EI}\left(\dfrac{1}{12}x^3 - \dfrac{L^2}{3}x\right) & 0 \le x \le 2L \\ \dfrac{1}{EI}\left(-\dfrac{1}{6}x^3 + \dfrac{3L}{2}x^2 - \dfrac{10L^2}{3}x + 2L^3\right) & 2L \le x \le 3L \end{cases}$$

Assumed shape function $\psi(x) = u(x)$.

$$u(x,t) = z_o\ \psi(x) \sin \omega_n t';\ u_o(x) = z_o\ \psi(x)$$

$$\dot{u}(x,t) = \omega_n\ z_o\ \psi(x) \cos \omega_n t';\ \dot{u}_o(x) = \omega_n\ z_o\ \psi(x)$$

2. *Determine E_{So} and E_{Ko}.*

$$E_{So} = \frac{1}{2}\int_0^{3L} EI(x)\left[u_o''(x)\right]^2 dx$$

$$= \frac{z_o^2}{2EI}\left[\int_0^{2L}\left(\frac{x}{2}\right)^2 dx + \int_{2L}^{3L}(3L-x)^2\, dx\right]$$

$$= \frac{L^3}{2EI} z_o^2$$

$$E_{Ko} = \frac{1}{2}\int_0^{3L} m(x)\left[\dot{u}_o(x)\right]^2 dx$$

$$= \frac{1}{2}\frac{m}{(EI)^2}\omega_n^2 z_o^2\left[\int_0^{2L}\left(\frac{1}{12}x^3 - \frac{L^2}{3}x\right)^2 dx + \int_{2L}^{3L}\left(-\frac{1}{6}x^3 + \frac{3L}{2}x^2 - \frac{10L^2}{3}x + 2L^3\right)^2 dx\right]$$

$$= \frac{51mL^7}{280(EI)^2}\omega_n^2 z_o^2$$

3. *Determine ω_n.*

$$E_{Ko} = E_{So} \Rightarrow$$

$$\omega_n = \sqrt{\frac{L^3/2EI}{51mL^7/280(EI)^2}} = \frac{1.657}{L^2}\sqrt{\frac{EI}{m}}$$

Problem 8.21

The beam shown in Fig. P8.21 is statically indeterminate. We will first compute the deflection of the simply supported beam by releasing the two bents. Then, we'll solve the unknown reactions in the two bents and the total deflection. This is the flexibility method.

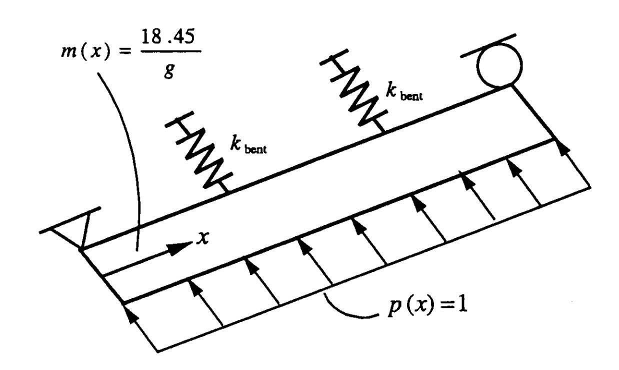


Fig. P8.21

1. *Determine the total deflection of the beam by the flexibility method.*

We will make the structure statically determinate by releasing the two bents. The resulting primary structure is a simply supported beam.

1.1 *Determine the deflection of the primary structure.*

The deflection $u_0(x)$ of a simply supported beam under uniform force $p(x) = 1$ is

$$u_0(x) = \frac{x}{24EI_y}(L^3 - 2Lx^2 + x^3) \qquad \text{(a)}$$

and the deflection at the midspan is

$$u_0(L/2) = \frac{5L^4}{384EI_y} \qquad \text{(b)}$$

1.2 *Determine the unknown reaction force F at the bent.*

Since the structure is symmetric about $x = L/2$, we can utilize the property of symmetry and thus reduce the problem to one-degree indeterminacy. It can be shown that the deflection of the beam due a unit force at the bent is given by

$$u_1(x) = \frac{x}{18EI_y}(-3x^2 + 2L^2) \qquad 0 \le x \le L/3 \qquad \text{(c)}$$

$$u_1(x) = \frac{L}{18EI_y}(-3x^2 + 3Lx - \frac{L^2}{9}) \qquad \text{(d)}$$
$$L/3 \le x \le L/2$$

Thus, at $x = L/3$ the compatibility condition states

$$u_0(L/3) + Fu_1(L/3) + F/k_{\text{bent}} = 0 \qquad \text{(e)}$$

where the first term is computed from Eq. (a) and the second term is from Eq. (c) or (d).

Substituting L = 375 ft, E = 3,000 ksi, I_y = 65,550 ft^4, x = L/3, and k_{bent} = 12,940 kip/ft into Eqs. (a) and (c) and after some simplifications, the unknown reaction force F in the bent is

$$F = -\frac{u_0(L/3)}{u_1(L/3) + 1/k_{\text{bent}}}$$

$$F = -58.65 \text{ kips}$$

1.3 *Determine the total deflection.*

The total deflection is

$$u(x) = u_0(x) + Fu_1(x) \qquad \text{(f)}$$

It is convenient to express $F = -58.65$ kips $= -0.1564L$ kips. Substituting Eqs. (a), (c), and (d) in Eq. (f) gives the total deflection:

$$u(x) = \frac{x}{EI_y}(0.04167x^3 - 0.05727Lx^2 + 0.02429L^3)$$
$$0 \le x \le L/3 \qquad \text{(g)}$$

$$u(x) = \frac{10^{-2}}{EI_y}(4.167x^4 - 8.333Lx^3 + 2.6065L^2x^2$$
$$+ 1.5602L^3x + 0.09654L^4) \qquad L/3 \le x \le L/2 \qquad \text{(h)}$$

2. *Compute natural frequency.*

From Eq. 8.6.1, specialized for $p(x) = 1$ and $m(x) = m$, the natural vibration frequency is:

$$\omega_n^2 = \frac{\int_0^L u(x)dx}{\int_0^L m[u(x)]^2 dx} \qquad \text{(i)}$$

Substituting Eqs. (g) and (h) into Eq. (i) and evaluating the integrals using MathCAD gives

$$\omega_n = \sqrt{\frac{1.255}{0.00296}} = 20.612 \text{ rad/sec.}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.305 \text{ sec.}$$

Problem 8.22

1. *Determine the mid-span deflection* u_o *of the beam due to* $p(x) = 1$.

Substituting L = 375 ft, E = 3,000 ksi, I_y = 65,550 ft^4, and $x = L/2$ into Eq. (h) of Problem 8.21, the mid-span deflection is

$$u_o = 5.217 \times 10^{-3} \text{ ft} \tag{a}$$

2. *Compute natural frequency.*

The deflection is given by

$$u(x) = u_o \sin(\pi x / L) \tag{b}$$

with u_o given by Eq. (a). From Eq. 8.6.1, specialized for $p(x) = 1$ and $m(x) = m$, the natural vibration frequency is:

$$\omega_n^2 = \frac{\int_0^L u(x)dx}{\int_0^L m[u(x)]^2 dx} \tag{c}$$

Substituting Eq. (b) into Eq. (c) with $m = 18.45/g$ and evaluating the integrals gives

$$\omega_n^2 = \frac{\int_0^L \sin\frac{\pi x}{L} dx}{u_o \int_0^L m \sin^2 \frac{\pi x}{L} dx} = \frac{4}{\pi m u_o}$$

$$\omega_n = \sqrt{\frac{4g}{18.45\pi \cdot 5.217 \times 10^{-3}}} = 20.639 \text{ rad/sec.}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.304 \text{ sec.}$$

Problem 8.23

1. *Determine the mid-span deflection* u_o *of the beam due to* $p(x) = 1$.

Substituting L = 375 ft, E = 3,000 ksi, I_y = 65,550 ft^4, and $x = L/2$ into Eq. (h) of Problem 8.21, the mid-span deflection is

$$u_o = 5.217 \times 10^{-3} \text{ ft}$$

2. *Determine the deflection u(x).*

The deflection is given by

$$u(x) = u_o \psi(x) \tag{a}$$

where $\psi(x)$ will be taken as the deflected shape of a simply supported beam without bents. Dividing Eq. (a) of Problem 8.21 by Eq. (b), also of Problem 8.21, gives

$$\psi(x) = \frac{16}{5}\left[\frac{x}{L} - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right] \tag{b}$$

Substituting $\psi(x)$ in Eq. (a) gives

$$u(x) = \frac{16u_o}{5}\left[\frac{x}{L} - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right] \tag{c}$$

3. *Compute natural frequency.*

From Eq. 8.6.1, specialized for $p(x) = 1$ and $m(x) = m$, the natural vibration frequency is:

$$\omega_n^2 = \frac{\int_0^L u(x)dx}{\int_0^L m[u(x)]^2 dx} \tag{d}$$

Substituting Eq. (c) into Eq. (d) with $m = 18.45/g$ and evaluating the integrals gives

$$\omega_n = \sqrt{\frac{1.252}{0.00295}} = 20.614 \text{ rad/sec.}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.305 \text{ sec.}$$

Problem 8.24

Including the torsional stiffness of the bents adds another degree of freedom to the system, namely, the rotational degree of freedom. Thus, the beam is now indeterminate to two degrees. We will use the classical flexibility method in solving this problem.

1. *Determine the total deflection of the beam by the flexibility method.*

Note that the deflection $u_0(x)$ of the simply support beam (the primary structure) as well as the deflection $u_1(x)$ of the beam due to unit lateral force at the bend (first released structure) were given by Eqs. (a), (c) and (d) of Problem 8.21, respectively. Next, we need to determine the deflection $u_2(x)$ of the beam due to unit rotational moment at the bend.

It can be easily shown that the deflection of the beam due to unit moment at the bent is given by

$$u_2(x) = \frac{Lx}{6EI_y} \qquad 0 \le x \le L/3 \qquad \text{(a)}$$

$$u_2(x) = \frac{1}{2EI_y}\left[\frac{Lx}{3} - \left(x - \frac{L}{3}\right)^2\right] \qquad L/3 < x \le L/2 \qquad \text{(b)}$$

The compatibility conditions at the location of the bents ($x = L/3$) state

$$\Delta_{10} + F_1\left(\Delta_{11} + 1/k_{\text{bent}}\right) + F_2\Delta_{12} = 0 \qquad \text{(c)}$$

$$\Delta_{20} + F_1\Delta_{21} + F_2\left(\Delta_{22} + 1/k_\theta\right) = 0 \qquad \text{(d)}$$

where k_θ is the torsional stiffness of the bent, $k_\theta = 1.941\times10^6$ kips-ft assuming that the distance between the adjacent columns in the bent is 30 ft.

Substituting L = 375 ft, E = 3,000 ksi, I_y = 65,550 ft^4 and $x = L/3$ into Eqs (a) and (c) of Problem 8.21 and Eqs. (a) and (b) above, we have

$$\Delta_{10} = u_0(L/3) = 7.903\times10^{-3} \text{ ft}$$

$$\Delta_{20} = u'_0(L/3) = 3.736\times10^{-5} \text{ rad}$$

$$\Delta_{11} = u_1(L/3) = 5.748\times10^{-5} \text{ ft}$$

$$\Delta_{21} = u'_1(L/3) = 2.759\times10^{-7} \text{ rad}$$

$$\Delta_{12} = u_2(L/3) = 2.759\times10^{-7} \text{ ft}$$

$$\Delta_{22} = u'_2(L/3) = 2.207\times10^{-9} \text{ rad}$$

Thus, solving Eqs. (c) and (d) simultaneously gives

$$F_1 = -58.56 \text{ kips.}$$

$$F_2 = -40.98 \text{ kips-ft.}$$

Finally, the total deflection is

$$u(x) = u_0(x) + F_1u_1(x) + F_2u_2(x) \qquad \text{(e)}$$

where $u_0(x)$ is defined by Eq. (a) of Problem 8.21; $u_1(x)$ is defined by Eqs. (c) and (d) of Problem 8.21; and $u_2(x)$ is given by Eqs. (a) and (b) above.

2. *Compute natural frequency.*

From Eq. 8.6.1, specialized for $p(x) = 1$ and $m(x) = m$, the natural vibration frequency is:

$$\omega_n^2 = \frac{\int_0^L u(x)dx}{\int_0^L m[u(x)]^2 dx} \qquad \text{(f)}$$

Substituting Eq. (e) into Eq. (f) and evaluating the integrals using MathCAD gives

$$\omega_n = \sqrt{\frac{1.254}{0.002946}} = 20.628 \text{ rad/sec.}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.305 \text{ sec.}$$

Problem 8.25

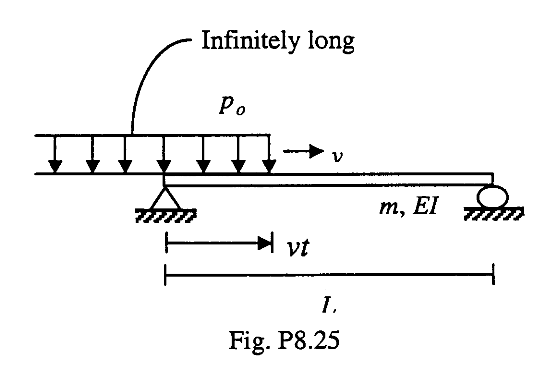


Fig. P8.25

1. *Determine the generalized mass, generalized stiffness, and natural frequency.*

$$\psi(x) = \sin\frac{\pi x}{L} \qquad \psi''(x) = -\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}$$

$$\tilde{m} = \int_0^L m\sin^2\frac{\pi x}{L}dx = \frac{mL}{2} \tag{a}$$

$$\tilde{k} = \int_0^L EI\left(\frac{\pi^2}{L^2}\right)^2 \sin^2\frac{\pi x}{L}dx = \frac{\pi^4 EI}{2L^3} \tag{b}$$

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{m}} \tag{c}$$

2. *Determine the generalized force.*

The front of the load $p(x,t)$ traveling with a velocity v takes time $t_d = L/v$ to cross the bridge. At any time t its position is as shown in Fig. P8.25. Thus the moving load can be described mathematically as

$$p(x,t) = \begin{cases} p_o & 0 \le x \le vt \quad 0 \le t \le t_d \\ 0 & vt < x \le L \quad 0 \le t \le t_d \\ p_o & 0 \le x \le L \quad t \ge t_d \end{cases} \tag{d}$$

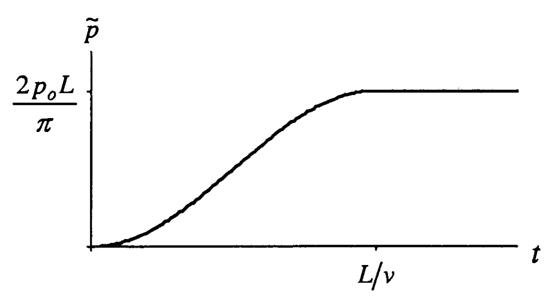
From Eq. (8.3.26) the generalized force is

$$\tilde{p}(t) = \int_0^L p(x,t)\,\psi(x)\,dx$$

$$= \begin{cases} \int_0^{vt} p_o \sin(\pi x/L)dx & 0 \le t \le t_d \\ \int_0^L p_o \sin(\pi x/L)dx & t \ge t_d \end{cases}$$

$$= \begin{cases} \frac{p_o L}{\pi}\left[1-\cos\frac{\pi v}{L}t\right] & 0 \le t \le t_d \\ \frac{2p_o L}{\pi} & t \ge t_d \end{cases} \tag{e}$$

This generalized force is plotted next:



Observe that this excitation has some similarity to Fig. 4.5.1b.

3. *Solve the equation of motion.*

$$\tilde{m}\ddot{z} + \tilde{k}z = \tilde{p}(t) \tag{f}$$

Response for $0 \le t \le t_d$.

The particular solution to Eq. (f) can be obtained by superposing the steady-state responses to the constant and the cosine term on the right-hand side of Eq. (e). The steady-state response to the constant term is $\tilde{p}_o/\tilde{k}$, where $\tilde{p}_o = p_o L/\pi$, and that for the cosine term is adapted from Eqs. (3.2.3) and (3.2.26), noting that $\zeta = 0$ and replacing p_o and k by $\tilde{p}_o$ and $\tilde{k}$ respectively. The complete solution is obtained by adding the complementary solution, given by Eq. (3.1.4), to the particular solution, and determining the constants A and B by imposing zero initial conditions. The result is

$$z(t) = \frac{2p_o}{\pi m}\frac{1}{\omega_n^2}\left[1 - \frac{\omega_n^2}{\omega_n^2-(\pi v/L)^2}\cos\frac{\pi v}{L}t + \frac{(\pi v/L)^2}{\omega_n^2-(\pi v/L)^2}\cos\omega_n t\right] \quad t \le L/v \tag{g}$$

Equation (g) is valid if $\omega_n \neq \pi v/L$; otherwise the particular solution to the cosine term should formulated similar to that in Eq. (3.1.12), noting that the forcing function is a cosine function instead of a sine function.

Response for $t \geq t_d$.

The motion is described by Eq. (4.5.3) with z instead of u, t_d instead of t_r, and $z(t_d)$ and $\dot{z}(t_d)$ determined from Eq. (g):

$$z(t_d) = \frac{2p_o}{\pi m}\frac{1}{\omega_n^2}\left[1 + \frac{\omega_n^2}{\omega_n^2 - (\pi v/L)^2} + \frac{(\pi v/L)^2}{\omega_n^2 - (\pi v/L)^2}\cos\omega_n t_d\right] \tag{h}$$

$$\dot{z}(t_d) = -\frac{2p_o}{\pi m}\frac{1}{\omega_n}\frac{(\pi v/L)^2}{\omega_n^2 - (\pi v/L)^2}\sin\omega_n t_d \tag{i}$$

Substituting these in Eq. (4.5.3), using trigonometric identities, and manipulating the mathematical quantities, we obtain

$$z(t) = \frac{2p_o}{\pi m}\frac{1}{\omega_n^2}\left[2 - \left\{1 - \frac{\omega_n^2}{\omega_n^2 - (\pi v/L)^2}\right\}\cos\omega_n(t - t_d) + \frac{(\pi v/L)^2}{\omega_n^2 - (\pi v/L)^2}\cos\omega_n t\right] \qquad t \geq L/v \tag{j}$$

The generalized coordinate response is given by Eq. (g) while the front of the load is on the bridge span and by Eq. (j) after the front has crossed the span.

4. *Determine the deflection at midspan.*

$$u(x,t) = z(t)\psi(x) = z(t)\sin\frac{\pi x}{L} \tag{k}$$

At midspan, $x = L/2$ and

$$u\left(\frac{L}{2}, t\right) = z(t)$$

Thus the deflection at midspan is also given by Eqs. (g) and (j).

Problem 8.26

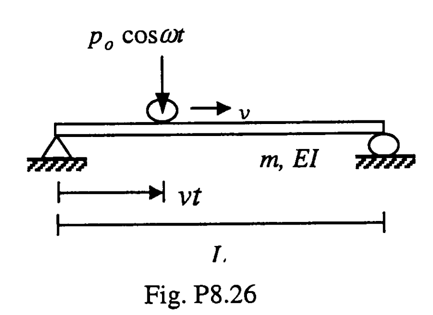


Fig. P8.26

1. *Determine the generalized mass, generalized stiffness, and natural frequency.*

$$\psi(x) = \sin\frac{\pi}{L} \qquad \psi''(x) = -\frac{\pi^2}{L^2}\sin\frac{\pi x}{L}$$

$$\tilde{m} = \int_0^L m \sin^2\frac{\pi x}{L}dx = \frac{mL}{2} \qquad \text{(a)}$$

$$\tilde{k} = \int_0^L EI\left(\frac{\pi^2}{L^2}\right)^2 \sin^2\frac{\pi x}{L}dx = \frac{\pi^4 EI}{2L^3} \qquad \text{(b)}$$

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{m}} \qquad \text{(c)}$$

2. *Determine the generalized force.*

A load $p(x,t)$ traveling with a velocity v takes time $t_d = L/v$ to cross the bridge. At any time t its position is as shown in Fig. P8.26. Thus the moving load can be described mathematically as

$$p(x,t) = \begin{cases} p_o \cos\omega t\, \delta(x - vt) & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(d)}$$

where $\delta(x - vt)$ is the Dirac delta function centered at $x = vt$; it is a mathematical description of the travelling concentrated load. From Eq. (8.3.26) the generalized force is

$$\tilde{p}(t) = \int_0^L p(x,t)\,\psi(x)\,dx$$

$$= \begin{cases} \int_0^L p_o \cos\omega t\, \delta(x - vt)\sin(\pi x/L)dx & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases}$$

$$= \begin{cases} p_o \cos\omega t \sin(\pi x/L) & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases}$$

$$= \begin{cases} p_o \cos\omega t \sin(\pi t/t_d) & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases}$$

This can be re-written as

$$\tilde{p}(t) = \begin{cases} \dfrac{p_o}{2}[\sin(\omega + \pi t/t_d) - \sin(\omega - \pi t/t_d)] & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(e)}$$

3. *Solve the equation of motion.*

$$\tilde{m}\ddot{z} + \tilde{k}z = \tilde{p}(t) \qquad \text{(f)}$$

Forced vibration phase

The response $z(t)$ can be obtained by superposing the responses due to the two sine terms in the right-hand side of Eq. (e). The individual responses are adapted from Eq. (3.1.6b) by changing the notation from $u(t)$ to $z(t)$, substituting for ω with $\omega + \pi t/t_d$ in one case and with $\omega - \pi t/t_d$ in the other, and noting that

$$t_d = \frac{L}{v} \qquad (z_{st})_o = \frac{p_o}{2\tilde{k}} = \frac{p_o}{mL\omega_n^2}$$

The result is

$$z(t) = \frac{p_o}{mL}\left[\frac{1}{\omega_n^2 - (\omega + \pi v/L)^2}\left\{\sin\left(\omega + \frac{\pi v}{L}\right)t - \left(\frac{\omega + \pi v/L}{\omega_n}\right)\sin\omega_n t\right\} - \frac{1}{\omega_n^2 - (\omega - \pi v/L)^2}\left\{\sin\left(\omega - \frac{\pi v}{L}\right)t - \left(\frac{\omega - \pi v/L}{\omega_n}\right)\sin\omega_n t\right\}\right]$$

$$t \le L/v \qquad \text{(g)}$$

Based on Eq. (3.1.6b), Eq. (g) is valid if $\omega_n \neq \omega + \pi v/L$ and $\omega_n \neq \omega - \pi v/L$; otherwise Eq. (3.1.13a) should be used instead of Eq. (3.1.6b).

Free vibration phase.

The motion is described by Eq. (4.7.3) with $z(t)$ instead of $u(t)$, and $z(t_d)$ and $\dot{z}(t_d)$ determined from Eq. (g) :

$$z(t_d) = \frac{p_o}{mL}\left[\left\{\frac{1}{\omega_n^2 - (\omega - \pi v/L)^2} - \frac{1}{\omega_n^2 - (\omega + \pi v/L)^2}\right\}\sin \omega L/v + \frac{1}{\omega_n}\left\{\frac{(\omega - \pi v/L)}{\omega_n^2 - (\omega - \pi v/L)^2} - \frac{(\omega + \pi v/L)}{\omega_n^2 - (\omega + \pi v/L)^2}\right\}\sin \omega_n L/v\right] \tag{h}$$

$$\dot{z}(t_d) = \frac{p_o}{mL}\left[\left\{\frac{(\omega - \pi v/L)}{\omega_n^2 - (\omega - \pi v/L)^2} - \frac{(\omega + \pi v/L)}{\omega_n^2 - (\omega + \pi v/L)^2}\right\} \times \left\{\cos\frac{\omega_n L}{v} + \cos\frac{\omega L}{v}\right\}\right] \tag{i}$$

Substituting these in Eq. (4.7.3), using trigonometric identities, and manipulating the mathematical quantities, we obtain

$$z(t) = \frac{p_o}{mL}\left[\left\{\frac{1}{\omega_n^2 - (\omega - \pi v/L)^2} - \frac{1}{\omega_n^2 - (\omega + \pi v/L)^2}\right\} \times \sin\frac{\omega L}{v}\cos\omega_n\left(t - \frac{L}{v}\right) + \frac{1}{\omega_n}\left\{\frac{(\omega - \pi v/L)}{\omega_n^2 - (\omega - \pi v/L)^2} - \frac{(\omega + \pi v/L)}{\omega_n^2 - (\omega + \pi v/L)^2}\right\} \times \left\{\cos\frac{\omega L}{v}\sin\omega_n\left(t - \frac{L}{v}\right) + \sin\omega_n t\right\}\right] \quad t \geq L/v \tag{j}$$

The generalized coordinate response is given by Eq. (g) while the moving load is on the bridge span and by Eq. (j) after the load has crossed the span.

4. *Determine the deflection at midspan.*

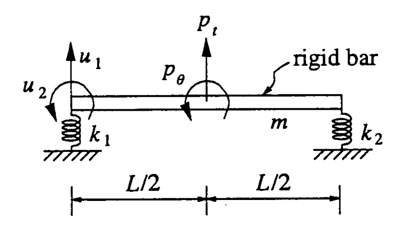
$$u(x,t) = z(t)\psi(x) = z(t)\sin\frac{\pi x}{L} \tag{k}$$

At midspan, $x = L/2$ and

$$u\left(\frac{L}{2}, t\right) = z(t)$$

Thus the deflection at midspan is also given by Eqs. (g) and (j).

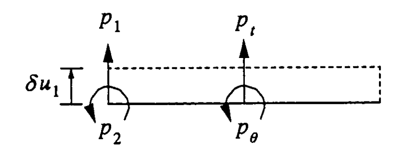
Problem 9.1



1. *Determine the force vector.*

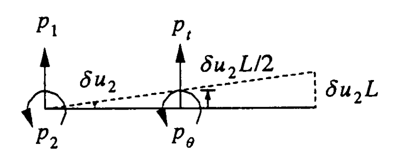
Introduce a virtual displacements δu_1 along DOF 1. The work done by the applied forces p_t and p_θ, and by the equivalent forces p_1 and p_2 is

$$\begin{aligned} \delta W &= p_t(\delta u_1) + p_\theta(0) \\ &= p_1(\delta u_1) + p_2(0) \end{aligned} \tag{a}$$



Similarly, introduce a virtual displacement δu_2 along DOF 2. The work done by the applied and equivalent forces is

$$\begin{aligned} \delta W &= p_t\left(\frac{\delta u_2 L}{2}\right) + p_\theta(\delta u_2) \\ &= p_1(0) + p_2(\delta u_2) \end{aligned} \tag{b}$$

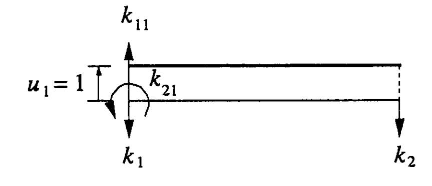


From the above equations, the vector of equivalent forces is

$$\mathbf{p} = \begin{Bmatrix} p_t \\ \dfrac{p_t L}{2} + p_\theta \end{Bmatrix} \tag{c}$$

2. *Determine the stiffness matrix.*

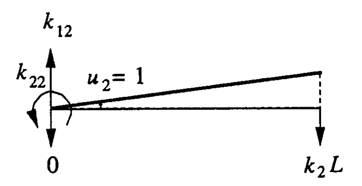
Apply a unit displacement $u_1 = 1$ with $u_2 = 0$ and identify the resulting forces and the stiffness influence coefficients, k_{11} and k_{21}.



By statics,

$$k_{11} = k_1 + k_2 \qquad k_{21} = k_2 L$$

Similarly, apply a unit displacement $u_2 = 1$ with $u_1 = 0$ and identify the resulting forces and the stiffness influence coefficients, k_{12} and k_{22}.



By statics,

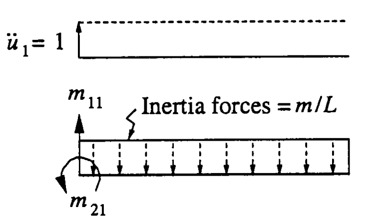
$$k_{12} = k_2 L \qquad k_{22} = k_2 L^2$$

Thus the stiffness matrix is

$$\mathbf{k} = \begin{bmatrix} k_1 + k_2 & k_2 L \\ k_2 L & k_2 L^2 \end{bmatrix} \tag{d}$$

3. *Determine the mass matrix.*

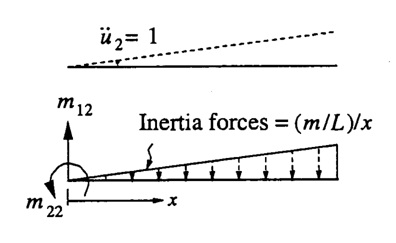
Impart a unit acceleration $\ddot{u}_1 = 1$ with $\ddot{u}_2 = 0$, determine the distribution of acceleration and the associated inertia forces, and identify the influence coefficients, m_{11} and m_{21}.



By statics,

$$m_{11} = m \qquad m_{21} = \frac{mL}{2}$$

Similarly, impart a unit acceleration $\ddot{u}_2 = 1$ with $\ddot{u}_1 = 0$, determine the distribution of acceleration and the associated inertia forces, and identify the influence coefficients, m_{12} and m_{22}.



By statics,

$$m_{12} = \frac{mL}{2} \qquad m_{22} = \frac{mL^2}{3}$$

Thus the mass matrix is

$$\mathbf{m} = \begin{bmatrix} m & \frac{mL}{2} \\ \frac{mL}{2} & \frac{mL^2}{3} \end{bmatrix} \qquad \text{(e)}$$

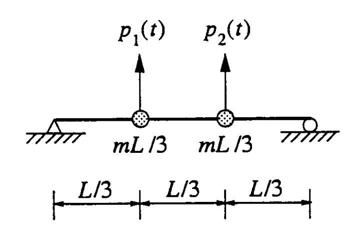
4. *Write the equations of motion.*

Substituting Eqs. (c), (d) and (e) in Eq. (9.2.12), with $\mathbf{c} = \mathbf{0}$, gives

$$\begin{bmatrix} m & \frac{mL}{2} \\ \frac{mL}{2} & \frac{mL^2}{3} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 L \\ k_2 L & k_2 L^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_t \\ \frac{p_t L}{2} + p_\theta \end{Bmatrix} \qquad \text{(f)}$$

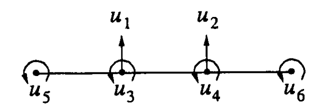
These two differential equations are coupled because of mass coupling and stiffness coupling.

Problem 9.2



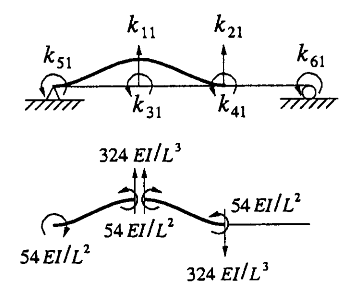
Part a

The elastic properties of the beam (neglecting axial deformation) are represented by six DOFs: two translational displacements and four rotational displacements.



$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_t \\ \mathbf{u}_0 \end{Bmatrix} \quad \mathbf{u}_t = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \mathbf{u}_0 = \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

The coefficients of the stiffness matrix corresponding to these DOF are computed following Example 9.4. For instance, to obtain the first column of the stiffness matrix, apply a unit displacement $u_1 = 1$ while the other displacements $u_j = 0$, $j = 2, 3, \cdots, 6$. Identify the resulting elastic forces and the stiffness influence coefficients.

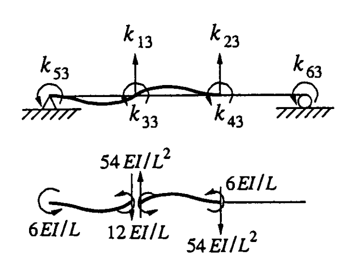


By statics,

$$k_{11} = \frac{648EI}{L^3} \qquad k_{21} = \frac{-324EI}{L^3} \qquad k_{31} = 0$$

$$k_{41} = \frac{54EI}{L^2} \qquad k_{51} = \frac{-54EI}{L^2} \qquad k_{61} = 0$$

Similarly to obtain the third column, apply $u_3 = 1$ with all other $u_j = 0$ and identify the resulting elactic forces and the stiffness influence coefficients.



Other elements of the stiffness matrix are obtained similarly. Apply a unit displacement $u_i = 1$ while $u_j = 0$, $j \neq i$. Identify the resulting elastic forces and by statics obtain the stiffness coefficients k_{ij}.

The complete stiffness matrix is

$$\mathbf{k} = \frac{EI}{L^3}\left[\begin{array}{cc|cccc} 648 & -324 & 0 & 54L & -54L & 0 \\ -324 & 648 & -54L & 0 & 0 & 54L \\ \hline 0 & -54L & 24L^2 & 6L^2 & 6L^2 & 0 \\ 54L & 0 & 6L^2 & 24L^2 & 0 & 6L^2 \\ -54L & 0 & 6L^2 & 0 & 12L^2 & 0 \\ 0 & 54L & 0 & 6L^2 & 0 & 12L^2 \end{array}\right] \tag{a}$$

The stiffness matrix is partitioned:

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{0t} & \mathbf{k}_{00} \end{bmatrix} \tag{b}$$

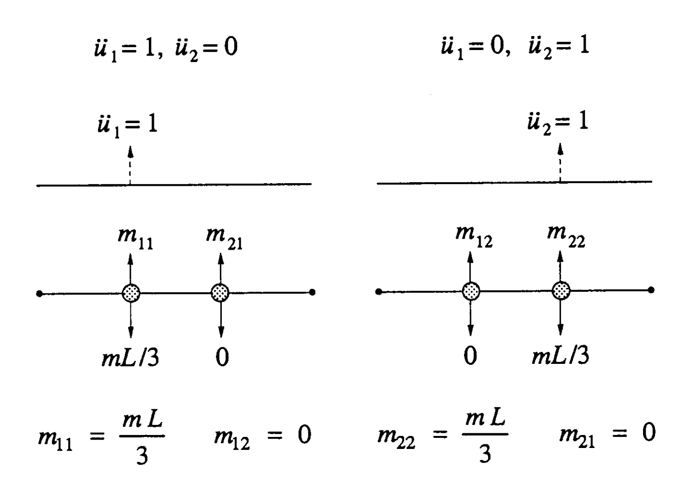
where the subscript t identifies the translational displacements, u_1 and u_2, and the subscript 0 identifies the rotational displacements, u_3, u_4, u_5 and u_6.

Part b

The DOF representing the inertial properties are the two translational displacements u_1 and u_2 associated with the concentrated masses.

$$\mathbf{u} = \mathbf{u}_t = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

To obtain the coefficients of the mass matrix for these DOF, apply first a unit acceleration $\ddot{u}_1 = 1$, while $\ddot{u}_2 = 0$. Next apply a unit acceleration $\ddot{u}_2 = 1$, while $\ddot{u}_1 = 0$. Determine the associated inertia forces and identify the mass coefficients:



$$m_{11} = \frac{mL}{3} \qquad m_{12} = 0 \qquad m_{22} = \frac{mL}{3} \qquad m_{21} = 0$$

Thus the mass matrix is

$$\mathbf{m} = \frac{mL}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{c}$$

Part c

The condensed stiffness matrix for the two vertical DOF is

$$\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{t0}\,\mathbf{k}_{00}^{-1}\,\mathbf{k}_{0t} \quad \Rightarrow$$

$$\hat{\mathbf{k}}_{tt} = \frac{162EI}{5L^3}\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix} \tag{d}$$

The force vector is given by

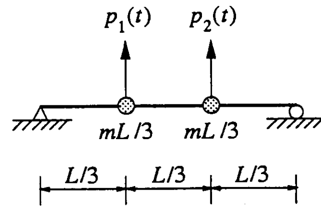
$$\mathbf{p}(t) = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \tag{e}$$

Substituting Eqs. (c), (d) and (e) in Eq. (9.2.12) with $\mathbf{c} = \mathbf{0}$, gives the equation governing the translational motion of the beam:

$$\frac{mL}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{162EI}{5L^3}\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

(f)

Problem 9.3



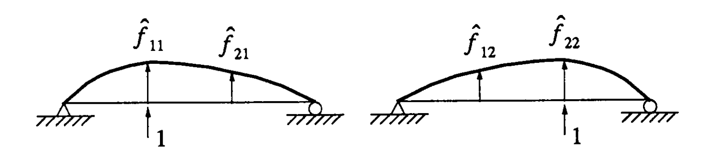
1. *Determine the stiffness matrix.*

The flexibility matrix is calculated first and inverted to obtain the stiffness matrix. The flexibility influence coefficient $\hat{f}_{ij}$ is the displacement in DOF i due to unit load applied along DOF j. The deflections due to unit load at node 1 are computed by standard procedures of structural analysis to obtain two of the influence coefficients:

$$\hat{f}_{11} = \frac{4L^3}{243EI} \qquad \hat{f}_{21} = \frac{7L^3}{486EI} \tag{a}$$

The deflections due to unit load at node 2 are computed to obtain the other two influence coefficients:

$$\hat{f}_{12} = \frac{7L^3}{486EI} \qquad \hat{f}_{22} = \frac{4L^3}{243EI} \tag{b}$$



Thus the flexibility matrix is

$$\hat{\mathbf{f}} = \frac{L^3}{486EI}\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} \tag{c}$$

The stiffness matrix is obtained by inverting $\hat{\mathbf{f}}$:

$$\mathbf{k} = \hat{\mathbf{f}}^{-1} = \frac{162EI}{5L^3}\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix} \tag{d}$$

2. *Determine the mass matrix.*

The mass matrix for the translational DOF is same as in Problem 9.2:

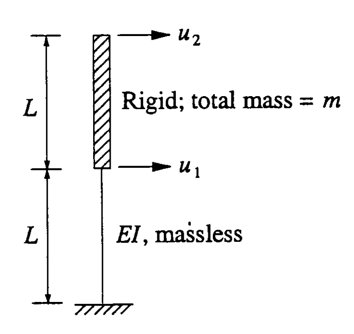
$$\mathbf{m} = \frac{mL}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{e}$$

3. *Write the equations of motion.*

Using Eqs. (d) and (e), the equation governing the translational motion of the beam is the same as in Problem 9.2:

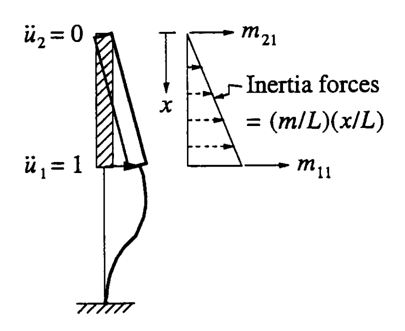
$$\frac{mL}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{162EI}{5L^3}\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \tag{f}$$

Problem 9.4



1. *Determine the mass matrix.*

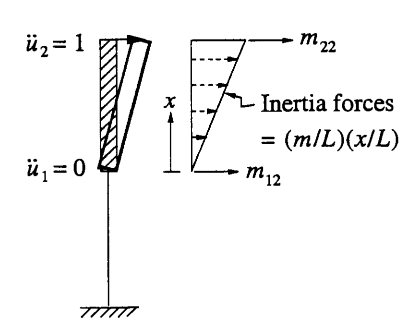
Impart a unit acceleration $\ddot{u}_1 = 1$, with $\ddot{u}_2 = 0$, determine the acceleration distribution and the associated inertia forces.



By statics,

$$m_{11} = \frac{m}{3} \qquad m_{21} = \frac{m}{6} \tag{a}$$

Similarly, impart a unit acceleration $\ddot{u}_2 = 1$, with $\ddot{u}_1 = 0$, determine the acceleration distribution and the associated inertia forces.



By statics,

$$m_{12} = \frac{m}{6} \qquad m_{22} = \frac{m}{3} \tag{b}$$

Thus the mass matrix is

$$\mathbf{m} = \frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{c}$$

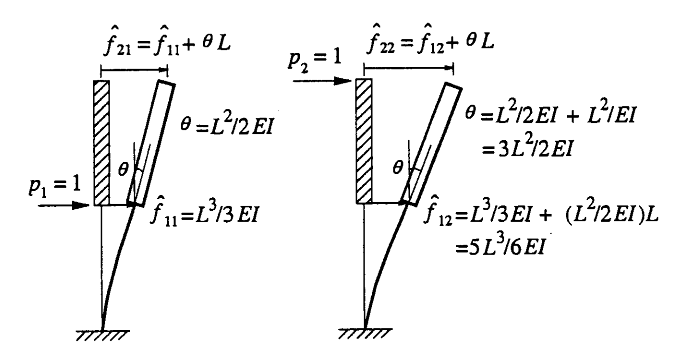
2. *Determine the flexibility matrix.*

Apply a unit force $p_1 = 1$ along DOF 1 with $p_2 = 0$ along DOF 2. The first two displacements or influence coefficients due to this force are computed following standard procedures of structural analysis:

$$\hat{f}_{11} = \frac{L^3}{3EI} \qquad \hat{f}_{21} = \frac{L^3}{3EI} + \left(\frac{L^2}{2EI}\right)L = \frac{5L^3}{6EI} \tag{d}$$

Similarly, apply a unit force $p_2 = 1$, with $p_1 = 0$. The other two displacements or influence coefficients due to this force are

$$\hat{f}_{12} = \frac{5L^3}{6EI} \qquad \hat{f}_{22} = \frac{5L^3}{6EI} + \left(\frac{3L^2}{2EI}\right)L = \frac{7L^3}{3EI} \tag{e}$$



Thus the flexibility matrix is

$$\hat{\mathbf{f}} = \frac{L^3}{6EI}\begin{bmatrix} 2 & 5 \\ 5 & 14 \end{bmatrix} \tag{f}$$

The stiffness matrix is computed by inverting the flexibility matrix $\hat{\mathbf{f}}$:

$$\mathbf{k} = \hat{\mathbf{f}}^{-1} = \frac{EI}{L^3}\begin{bmatrix} 28 & -10 \\ -10 & 4 \end{bmatrix} \tag{g}$$

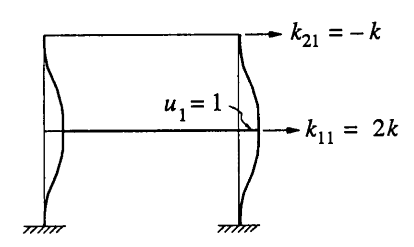
Problem 9.5

1. *Determine the stiffness matrix.*

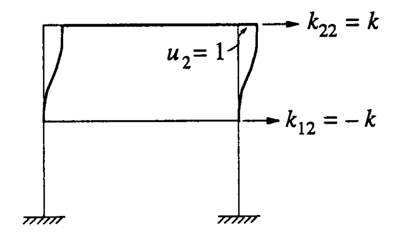
The story stiffness is

$$k = 2\left(\frac{12EI}{h^3}\right) = \frac{24EI}{h^3}$$

Apply $u_1 = 1$, $u_2 = 0$ and determine k_{i1}:



Apply $u_2 = 1$, $u_1 = 0$ and determine k_{i2}:



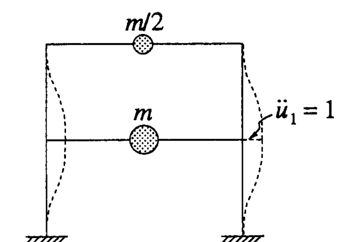
Thus, the stiffness matrix is

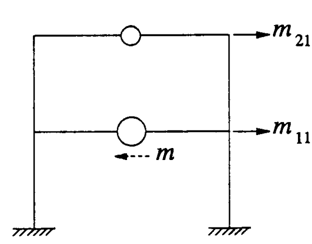
$$\mathbf{k} = k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

2. *Determine the mass matrix.*

Apply $\ddot{u}_1 = 1$, $\ddot{u}_2 = 0 \Rightarrow$

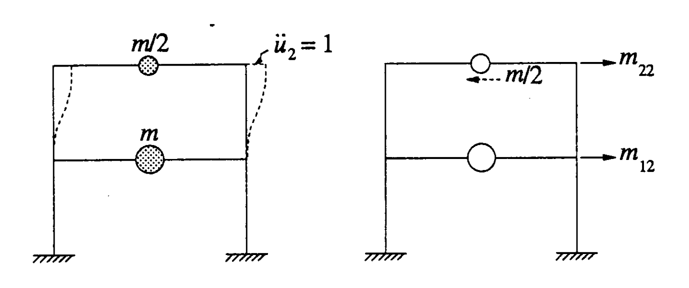
$m_{11} = m \qquad m_{21} = 0$





Apply $\ddot{u}_2 = 1$, $\ddot{u}_1 = 0 \Rightarrow$

$m_{12} = 0 \qquad m_{22} = \frac{m}{2}$



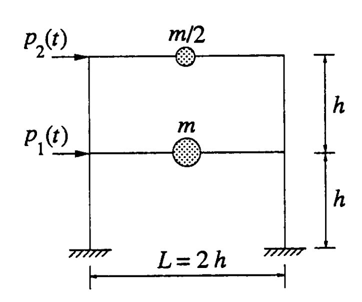
Thus, the mass matrix is

$$\mathbf{m} = m\begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix}$$

3. *Write the equations of motion.*

$$m\begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

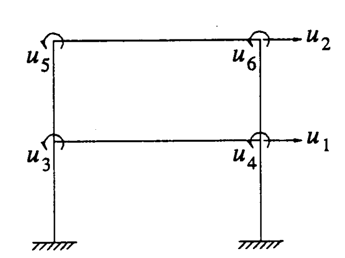
Problem 9.6



Part a

The elastic properties of the shear frame (neglecting axial deformation) are represented by six DOFs: two horizontal displacements and four rotational displacements.

The coefficients of the stiffness matrix corresponding to these DOF are computed following Example 9.7.



$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_t \\ \mathbf{u}_0 \end{Bmatrix} \quad \mathbf{u}_t = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \mathbf{u}_0 = \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

The complete stiffness matrix is

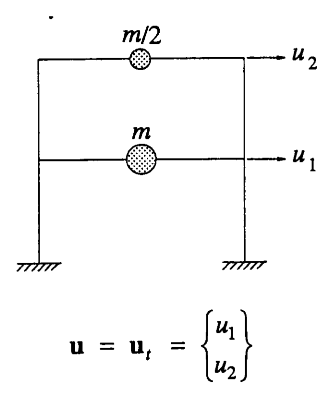
$$\mathbf{k} = \frac{EI}{h^3} \left[\begin{array}{cc|cccc} 48 & -24 & 0 & 0 & -6h & -6h \\ -24 & 24 & 6h & 6h & 6h & 6h \\ \hline 0 & 6h & 10h^2 & 1h^2 & 2h^2 & 0 \\ 0 & 6h & 1h^2 & 10h^2 & 0 & 2h^2 \\ -6h & 6h & 2h^2 & 0 & 6h^2 & 1h^2 \\ -6h & 6h & 0 & 2h^2 & 1h^2 & 6h^2 \end{array} \right] \quad \text{(a)}$$

This matrix can be written in partitioned form as follows:

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{0t} & \mathbf{k}_{00} \end{bmatrix} \quad \text{(b)}$$

Part b

The DOFs representing the inertial properties are the two translational displacements u_1 and u_2.



$$\mathbf{u} = \mathbf{u}_t = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

The mass matrix is

$$\mathbf{m} = m \begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix} \quad \text{(c)}$$

Part c

The condensed stiffness matrix is

$$\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{t0}\,\mathbf{k}_{00}^{-1}\,\mathbf{k}_{0t}$$

$$= \frac{EI}{h^3} \begin{bmatrix} 37.15 & -15.12 \\ -15.12 & 10.19 \end{bmatrix} \quad \text{(d)}$$

Substituting Eqs. (c) and (d) in Eq. (9.2.12) gives

$$m \begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{EI}{h^3} \begin{bmatrix} 37.15 & -15.12 \\ -15.12 & 10.19 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

Problem 9.7

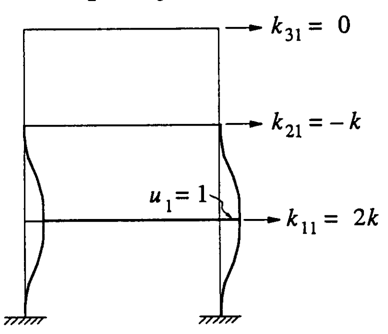
Since the beams are rigid in flexure and axial deformation is neglected in columns, three DOFs associated with each story represent the properties of this three-story shear building. The corresponding story masses and story stiffness are:

$$m_1 = m \qquad m_2 = m \qquad m_3 = \frac{m}{2} \tag{a}$$

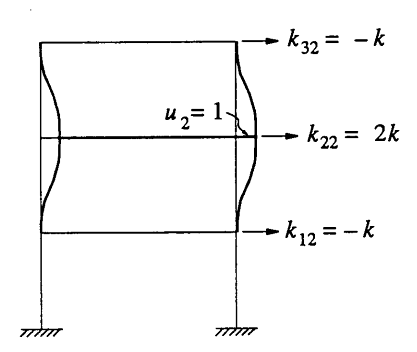
$$k_1 = k_2 = k_3 = 2\frac{12EI}{h^3} = \frac{24EI}{h^3} \equiv k \tag{b}$$

1. *Determine the stiffness matrix.*

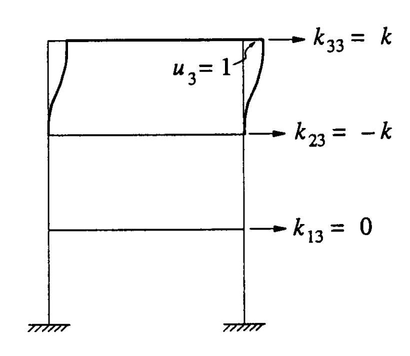
Apply $u_1 = 1, u_2 = u_3 = 0$ and determine k_{i1}:



Apply $u_2 = 1, u_1 = u_3 = 0$ and determine k_{i2}:



Apply $u_3 = 1, u_1 = u_2 = 0$ and determine k_{i3}:



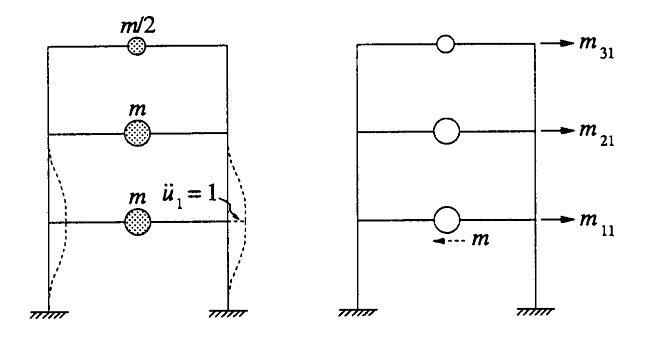
The stiffness matrix is

$$\mathbf{k} = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{c}$$

2. *Determine the mass matrix.*

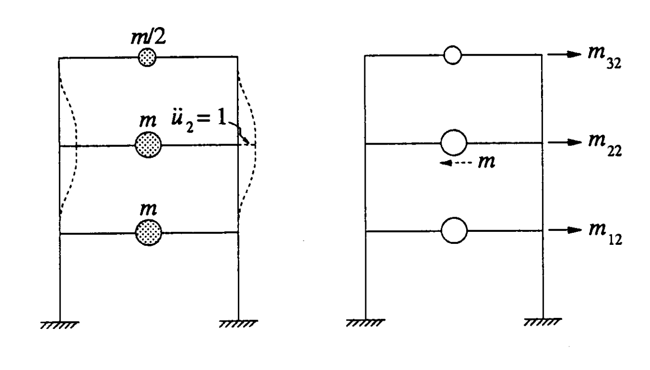
Apply $\ddot{u}_1 = 1, \ddot{u}_2 = \ddot{u}_3 = 0 \Rightarrow$

$m_{11} = m \qquad m_{21} = m_{31} = 0$



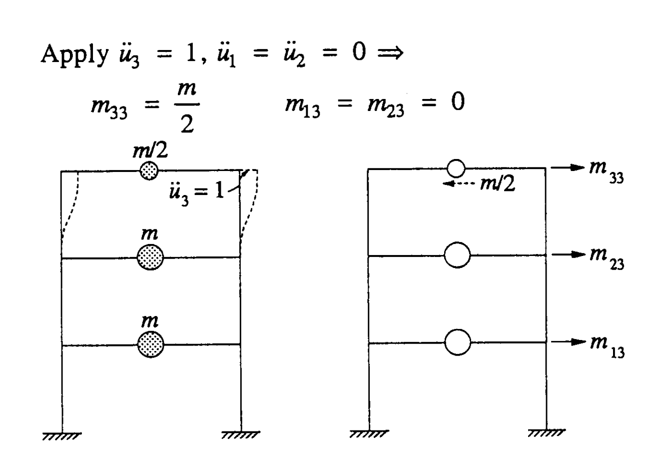
Apply $\ddot{u}_2 = 1, \ddot{u}_1 = \ddot{u}_3 = 0 \Rightarrow$

$m_{22} = m \qquad m_{12} = m_{32} = 0$



Apply $\ddot{u}_3 = 1, \ddot{u}_1 = \ddot{u}_2 = 0 \Rightarrow$

$m_{33} = \dfrac{m}{2} \qquad m_{13} = m_{23} = 0$



The mass matrix is

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix} \tag{d}$$

3. *Write the equations of motion.*

$$m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{Bmatrix} \tag{e}$$

Problem 9.8

Since the beams are rigid in flexure and axial deformation is neglected in columns, the three floor displacements are the three DOFs. The floor masses and story stiffnesses are:

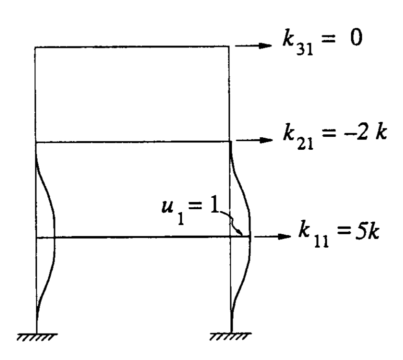
$$m_1 = m \qquad m_2 = m \qquad m_3 = m/2 \tag{a}$$

$$k_3 = 2\frac{12(EI/3)}{h^3} = \frac{8EI}{h^3} \equiv k$$

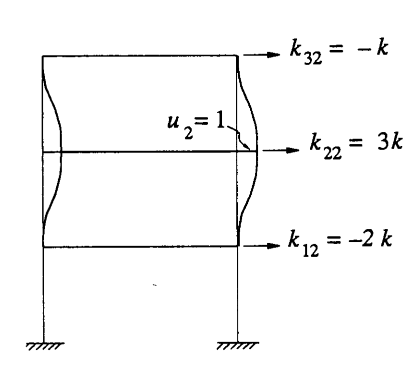
$$k_1 = 3k \qquad k_2 = 2k \tag{b}$$

1. *Determine the stiffness matrix.*

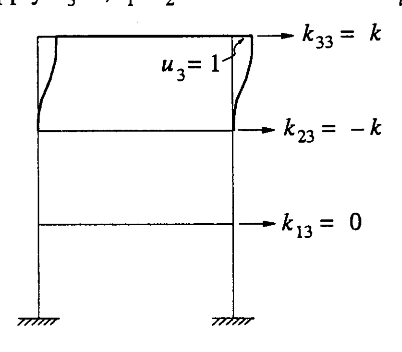
Apply $u_1=1, u_2=u_3=0$ and determine k_{i1}:



Apply $u_2=1, u_1=u_3=0$ and determine k_{i2}:



Apply $u_3=1, u_1=u_2=0$ and determine k_{i3}:



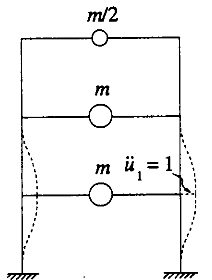
The stiffness matrix is

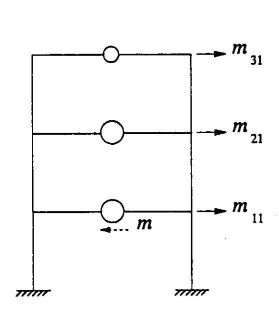
$$\mathbf{k} = k\begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{c}$$

2. *Determine the mass matrix.*

Apply $\ddot{u}_1=1, \ddot{u}_2=\ddot{u}_3=0 \Rightarrow$

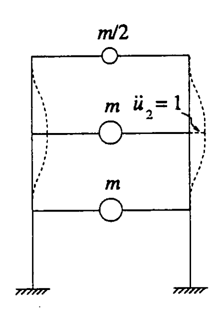
$m_{11}=m \qquad m_{21}=m_{31}=0$

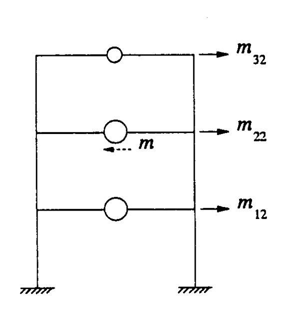




Apply $\ddot{u}_2=1, \ddot{u}_1=\ddot{u}_3=0 \Rightarrow$

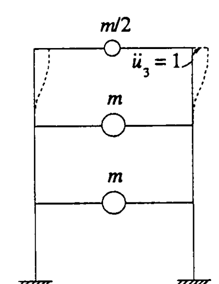
$m_{22}=m \qquad m_{12}=m_{32}=0$

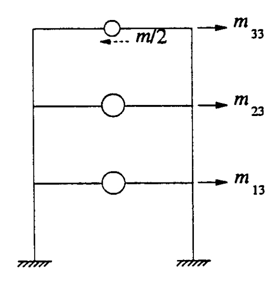




Apply $\ddot{u}_3=1, \ddot{u}_1=\ddot{u}_2=0 \Rightarrow$

$m_{33}=m/2 \qquad m_{13}=m_{23}=0$





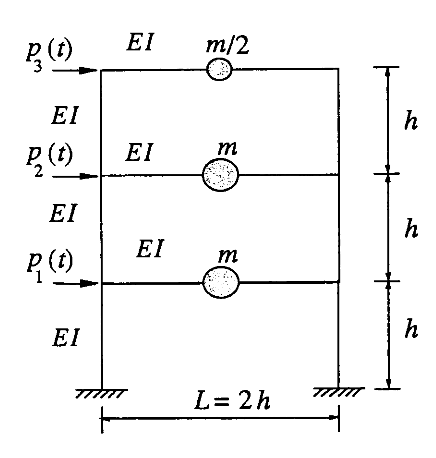
The mass matrix is

$$\mathbf{m} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \qquad \text{(d)}$$

3. *Write the equations of motion.*

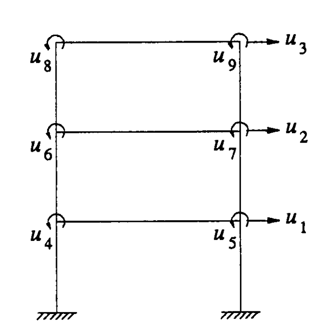
$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \mathrm{k} \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{Bmatrix} \qquad \text{(e)}$$

Problem 9.9



Part a

The elastic properties of the frame (neglecting axial deformation) are represented by 9 DOFs: three horizontal displacements and six rotational displacements.



$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_t \\ \mathbf{u}_0 \end{Bmatrix} \quad \mathbf{u}_t = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \mathbf{u}_0 = \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}$$

The coefficients of the stiffness matrix corresponding to these DOFs are computed following Example 9.7. The complete stiffness matrix is

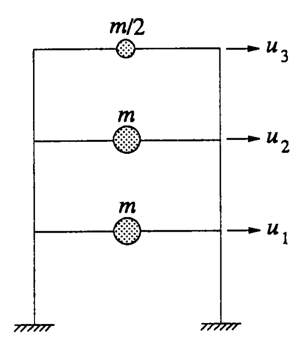
$$\mathbf{k} = \frac{EI}{h^3} \left[\begin{array}{ccc|cccccc} 48 & -24 & 0 & 0 & 0 & -6h & -6h & 0 & 0 \\ -24 & 48 & -24 & 6h & 6h & 0 & 0 & -6h & -6h \\ 0 & -24 & 24 & 0 & 0 & 6h & 6h & 6h & 6h \\ \hline 0 & 6h & 0 & 10h^2 & 1h^2 & 2h^2 & 0 & 0 & 0 \\ 0 & 6h & 0 & 1h^2 & 10h^2 & 0 & 2h^2 & 0 & 0 \\ -6h & 0 & 6h & 2h^2 & 0 & 10h^2 & 1h^2 & 2h^2 & 0 \\ -6h & 0 & 6h & 0 & 2h^2 & 1h^2 & 10h^2 & 0 & 2h^2 \\ 0 & -6h & 6h & 0 & 0 & 2h^2 & 0 & 6h^2 & 1h^2 \\ 0 & -6h & 6h & 0 & 0 & 0 & 2h^2 & 1h^2 & 6h^2 \end{array} \right] \quad \text{(a)}$$

The stiffness matrix can be written in partitioned form as follows:

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{0t} & \mathbf{k}_{00} \end{bmatrix} \quad \text{(b)}$$

Part b

The DOFs representing the inertial properties are the three translational displacements, u_1, u_2 and u_3.



$$\mathbf{u} = \mathbf{u}_t = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The mass matrix is

$$\mathbf{m} = m \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix} \quad \text{(c)}$$

Part c

The condensed stiffness matrix for the three lateral DOFs is

$$\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{t0}\,\mathbf{k}_{00}^{-1}\,\mathbf{k}_{0t}$$

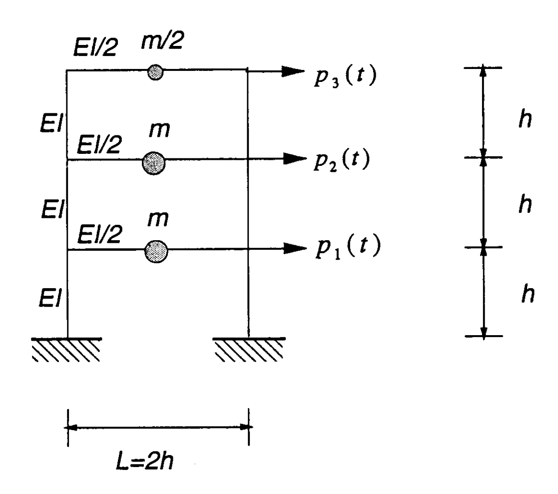
$$= \frac{EI}{h^3} \begin{bmatrix} 40.85 & -23.26 & 5.11 \\ -23.26 & 31.09 & -14.25 \\ 5.11 & -14.25 & 10.06 \end{bmatrix} \quad \text{(d)}$$

The equation governing the translational motion of the building is

$$m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \frac{EI}{h^3}\begin{bmatrix} 40.85 & -23.26 & 5.11 \\ -23.26 & 31.09 & -14.25 \\ 5.11 & -14.25 & 10.06 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{Bmatrix}$$

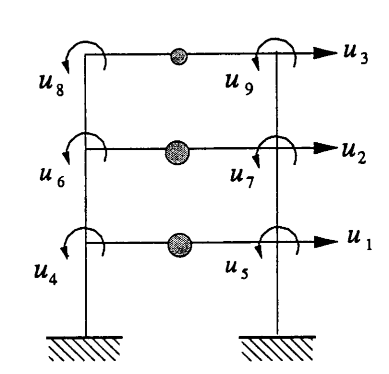
(e)

Problem 9.10



Part a

The elastic properties of the frame (neglecting axial deformation) are represented by 9 DOFs: three horizontal displacements and six rotational displacements.



$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_t \\ \mathbf{u}_0 \end{Bmatrix} \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \mathbf{u}_0 = \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}$$

The coefficients of the stiffness matrix corresponding to these DOFs are computed following Example 9.7. The complete stiffness matrix is

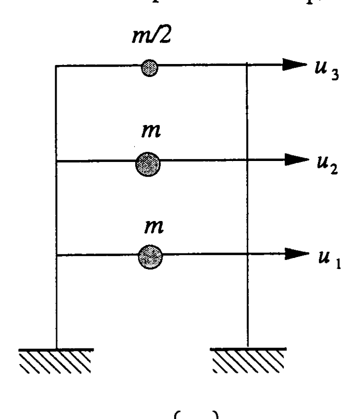
$$\mathbf{k} = \frac{EI}{h^3}\begin{bmatrix} 48 & -24 & 0 & 0 & 0 & -6h & -6h & 0 & 0 \\ & 48 & -24 & 6h & 6h & 0 & 0 & -6h & -6h \\ & & 24 & 0 & 0 & 6h & 6h & 6h & 6h \\ & & & 9h^2 & \frac{1}{2}h^2 & 2h^2 & 0 & 0 & 0 \\ & & & & 9h^2 & 0 & 2h^2 & 0 & 0 \\ & & & & & 9h^2 & \frac{1}{2}h^2 & 2h^2 & 0 \\ & & \text{Symm} & & & & 9h^2 & 0 & 2h^2 \\ & & & & & & & 5h^2 & \frac{1}{2}h^2 \\ & & & & & & & & 2h^2 \end{bmatrix} \tag{a}$$

The stiffness matrix can be written in partitioned form as follows:

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{to} \\ \mathbf{k}_{ot} & \mathbf{k}_{oo} \end{bmatrix} \tag{b}$$

Part b

The DOFs representing the inertial properties are the three translational displacements, u_1, u_2 and u_3.



$$\mathbf{u} = \mathbf{u}_t = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The mass matrix is

$$\mathbf{m} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{c}$$

Part c

The condensed stiffness matrix for the three lateral DOFs is

$$\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{to}\mathbf{k}_{oo}^{-1}\mathbf{k}_{ot}$$

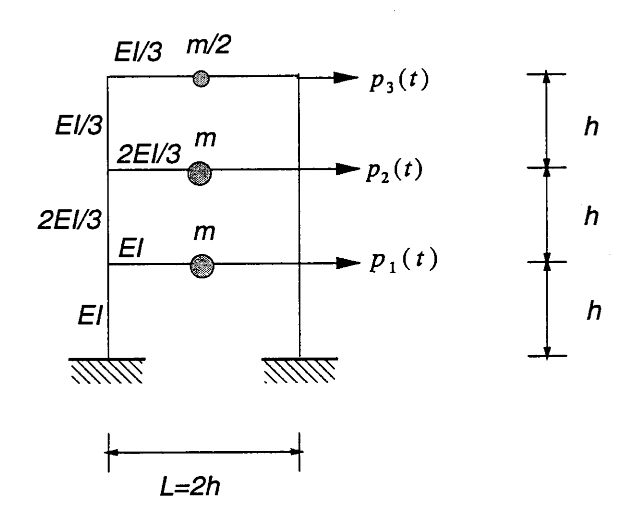
$$= \frac{EI}{h^3}\begin{bmatrix} 39.38 & -22.68 & 5.486 \\ & 27.13 & -11.75 \\ Symm & & 7.418 \end{bmatrix} \qquad \text{(d)}$$

The equations governing the translational motion of the building are

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \frac{EI}{h^3}\begin{bmatrix} 39.38 & -22.68 & 5.486 \\ & 27.13 & -11.75 \\ Symm & & 7.418 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{Bmatrix}$$

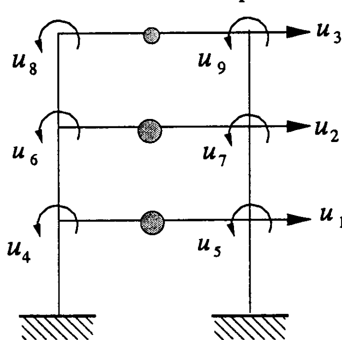
(e)

Problem 9.11



Part a

The elastic properties of the frame (neglecting axial deformation) are represented by 9 DOFs: three horizontal displacements and six rotational displacements.



$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_t \\ \mathbf{u}_0 \end{Bmatrix} \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \mathbf{u}_0 = \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}$$

The coefficients of the stiffness matrix corresponding to these DOFs are computed following Example 9.7. The complete stiffness matrix is

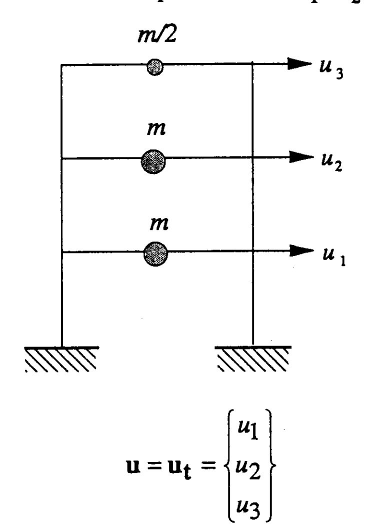
$$\mathbf{k} = \frac{EI}{h^3} \begin{bmatrix} 40 & -16 & 0 & 2h & 2h & -4h & -4h & 0 & 0 \\ & 24 & -8 & 4h & 4h & 2h & 2h & -2h & -2h \\ & & 8 & 0 & 0 & 2h & 2h & 2h & 2h \\ & & & \frac{26}{3}h^2 & h^2 & \frac{4}{3}h^2 & 0 & 0 & 0 \\ & & & & \frac{26}{3}h^2 & 0 & \frac{4}{3}h^2 & 0 & 0 \\ & & & & & \frac{16}{3}h^2 & \frac{4}{3}h^2 & \frac{2}{3}h^2 & 0 \\ & & \text{Symm} & & & & \frac{16}{3}h^2 & 0 & \frac{2}{3}h^2 \\ & & & & & & & 2h^2 & \frac{1}{3}h^2 \\ & & & & & & & & 2h^2 \end{bmatrix} \tag{a}$$

The stiffness matrix can be written in partitioned form as follows:

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{to} \\ \mathbf{k}_{ot} & \mathbf{k}_{oo} \end{bmatrix} \tag{b}$$

Part b

The DOFs representing the inertial properties are the three translational displacements, u_1, u_2 and u_3.



$$\mathbf{u} = \mathbf{u}_t = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The mass matrix is

$$\mathbf{m} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{c}$$

Part c

The condensed stiffness matrix for the three lateral DOFs is

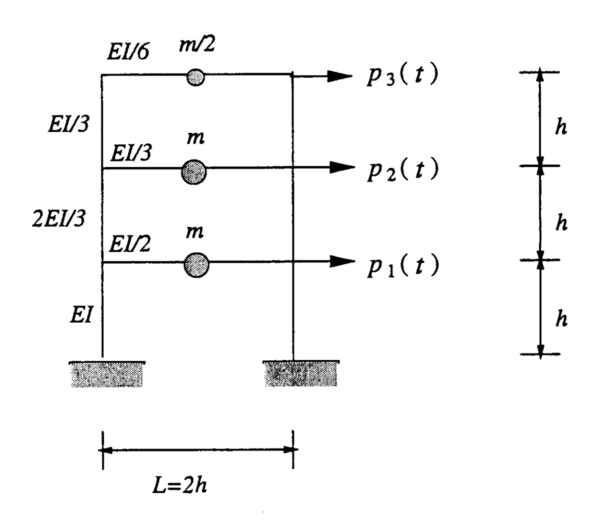
$$\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{to}\mathbf{k}_{oo}^{-1}\mathbf{k}_{ot}$$
$$= \frac{EI}{h^3}\begin{bmatrix} 33.36 & -14.91 & 1.942 \\ & 15.96 & -5.489 \\ Symm & & 3.923 \end{bmatrix} \qquad \text{(d)}$$

The equation governing the translational motion of the building is

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \frac{EI}{h^3}\begin{bmatrix} 33.36 & -14.91 & 1.942 \\ & 15.96 & -5.489 \\ Symm & & 3.923 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{Bmatrix}$$

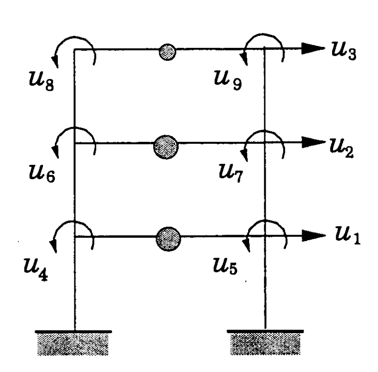
(e)

Problem 9.12



Part a

The elastic properties of the frame (neglecting axial deformation) are represented by 9 DOFs: three horizontal displacements and six rotational displacements.



$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_t \\ \mathbf{u}_0 \end{Bmatrix} \qquad \mathbf{u}_t = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \mathbf{u}_0 = \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}$$

The coefficients of the stiffness matrix corresponding to these DOFs are computed following Example 9.7. The complete stiffness matrix is

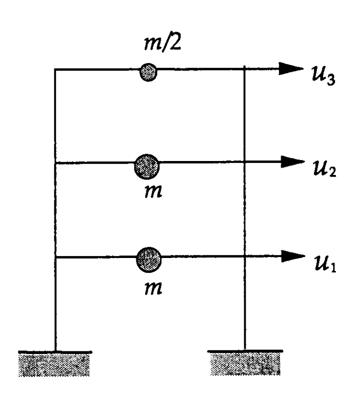
$$\mathbf{k} = \frac{EI}{h^3}\begin{bmatrix} 40 & -16 & 0 & 2h & 2h & -4h & -4h & 0 & 0 \\ & 24 & -8 & 4h & 4h & 2h & 2h & -2h & -2h \\ & & 8 & 0 & 0 & 2h & 2h & 2h & 2h \\ & & & \frac{23}{3}h^2 & \frac{1}{2}h^2 & \frac{4}{3}h^2 & 0 & 0 & 0 \\ & & & & \frac{23}{3}h^2 & 0 & \frac{4}{3}h^2 & 0 & 0 \\ & & & & & \frac{14}{3}h^2 & \frac{1}{3}h^2 & \frac{2}{3}h^2 & 0 \\ & & \text{Symm} & & & & \frac{14}{3}h^2 & 0 & \frac{2}{3}h^2 \\ & & & & & & & \frac{5}{3}h^2 & \frac{1}{6}h^2 \\ & & & & & & & & \frac{5}{3}h^2 \end{bmatrix} \tag{a}$$

The stiffness matrix can be written in partitioned form as follows:

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{to} \\ \mathbf{k}_{ot} & \mathbf{k}_{oo} \end{bmatrix} \tag{b}$$

Part b

The DOFs representing the inertial properties are the three translational displacements, u_1, u_2 and u_3.



$$\mathbf{u} = \mathbf{u_t} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The mass matrix is

$$\mathbf{m} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{c}$$

Part c

The condensed stiffness matrix for the three lateral DOFs is

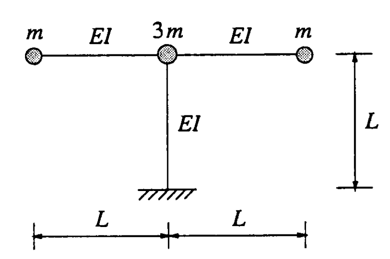
$$\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{to}\mathbf{k}_{oo}^{-1}\mathbf{k}_{ot}$$

$$= \frac{EI}{h^3}\begin{bmatrix} 30.77 & -14.01 & 2.43 \\ & 13.82 & -4.80 \\ Symm & & 2.92 \end{bmatrix} \qquad \text{(d)}$$

The equation governing the translational motion of the building is

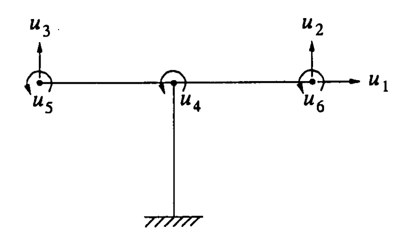
$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \frac{EI}{h^3}\begin{bmatrix} 30.77 & -14.01 & 2.43 \\ & 13.82 & -4.80 \\ Symm & & 2.92 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{Bmatrix} \qquad \text{(e)}$$

Problem 9.13



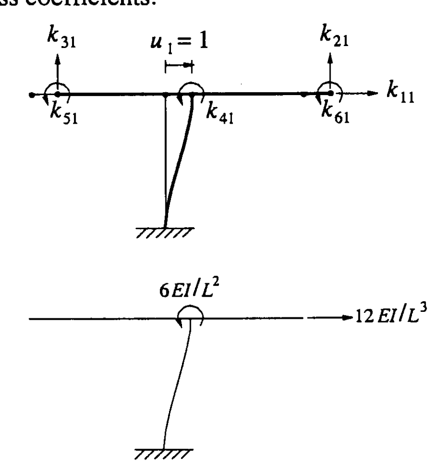
Part a

The elastic properties of the umbrella (neglecting axial deformation of the elements) are represented by six DOFs: three translational displacements and three rotations.



$$\mathbf{u}=\begin{Bmatrix}\mathbf{u}_t\\ \mathbf{u}_0\end{Bmatrix} \quad \mathbf{u}_t=\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix} \quad \mathbf{u}_0=\begin{Bmatrix}u_4\\ u_5\\ u_6\end{Bmatrix}$$

The coefficients of the stiffness matrix corresponding to these DOFs are computed following Example 9.4. For instance, to obtain the first column of the stiffness matrix, apply a unit displacement $u_1 = 1$ while the other displacements are zero, i.e., $u_j = 0, j = 2, 3, \cdots, 6$; identify the resulting elastic forces, and by statics obtain the stiffness coefficients:



$$k_{11} = \frac{12EI}{L^3} \qquad k_{21} = 0 \qquad k_{31} = 0$$
$$k_{41} = \frac{6EI}{L^2} \qquad k_{51} = 0 \qquad k_{61} = 0$$

Other columns of $\mathbf{k}$ are determined similarly. The complete stiffness matrix is

$$\mathbf{k} = \frac{EI}{L^3}\left[\begin{array}{ccc|ccc} 12 & 0 & 0 & 6L & 0 & 0 \\ 0 & 12 & 0 & -6L & 0 & -6L \\ 0 & 0 & 12 & 6L & 6L & 0 \\ \hline 6L & -6L & 6L & 12L^2 & 2L^2 & 2L^2 \\ 0 & 0 & 6L & 2L^2 & 4L^2 & 0 \\ 0 & -6L & 0 & 2L^2 & 0 & 4L^2 \end{array}\right] \qquad \text{(a)}$$

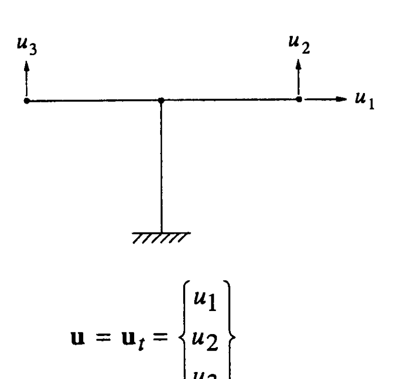
The stiffness matrix can be written in partitioned form as follows:

$$\mathbf{k} = \begin{bmatrix}\mathbf{k}_{tt} & \mathbf{k}_{t0}\\ \mathbf{k}_{0t} & \mathbf{k}_{00}\end{bmatrix} \qquad \text{(b)}$$

where the subscript t identifies the translational displacements, u_1, u_2 and u_3, and the subscript 0 identifies the rotational displacements u_4, u_5 and u_6.

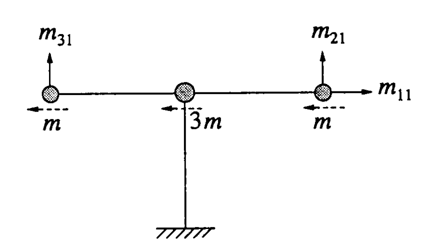
Part b

The DOFs representing the inertial properties are the three translational displacements u_1, u_2 and u_3.



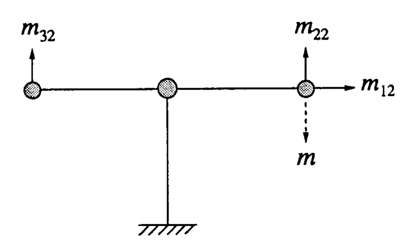
$$\mathbf{u} = \mathbf{u}_t = \begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}$$

To obtain the coefficients of the mass matrix for these DOF, first apply a unit acceleration $\ddot{u}_1 = 1$, while $\ddot{u}_2 = 0$ and $\ddot{u}_3 = 0$.



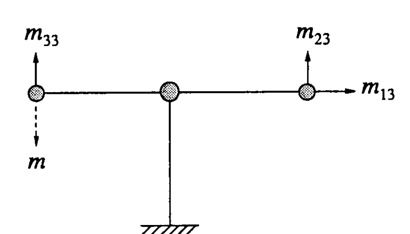
$m_{11} = 5m \qquad m_{21} = 0 \qquad m_{31} = 0$

Next apply a unit acceleration $\ddot{u}_2 = 1$, while $\ddot{u}_1 = 0$ and $\ddot{u}_3 = 0$.



$m_{12} = 0 \qquad m_{22} = m \qquad m_{32} = 0$

Finally, apply a unit acceleration $\ddot{u}_3 = 1$, while $\ddot{u}_1 = 0$ and $\ddot{u}_2 = 0$.



$m_{13} = 0 \qquad m_{23} = 0 \qquad m_{33} = m$

Thus, the mass matrix is

$$\mathbf{m} = m\begin{bmatrix} 5 & & \\ & 1 & \\ & & 1 \end{bmatrix} \tag{c}$$

Part c

The condensed stiffness matrix for the three translational DOFs is

$$\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{t0}\,\mathbf{k}_{00}^{-1}\,\mathbf{k}_{0t}$$

$$= \frac{3EI}{10L^3}\begin{bmatrix} 28 & 6 & -6 \\ 6 & 7 & 3 \\ -6 & 3 & 7 \end{bmatrix} \tag{d}$$

The equations governing the translational DOFs are

$$m\begin{bmatrix} 5 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1^t \\ \ddot{u}_2^t \\ \ddot{u}_3^t \end{Bmatrix} + \frac{3EI}{10L^3}\begin{bmatrix} 28 & 6 & -6 \\ 6 & 7 & 3 \\ -6 & 3 & 7 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{e}$$

(i) If the excitation is horizontal ground motion $u_{gx}(t)$, the total and relative displacements are related as follows:

$$\begin{Bmatrix} u_1^t \\ u_2^t \\ u_3^t \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} u_{gx} \tag{f}$$

Substituting Eq. (f) in Eq. (e) gives

$$m\begin{bmatrix} 5 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \frac{3EI}{10L^3}\begin{bmatrix} 28 & 6 & -6 \\ 6 & 7 & 3 \\ -6 & 3 & 7 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = -\begin{Bmatrix} 5m \\ 0 \\ 0 \end{Bmatrix} \ddot{u}_{gx}(t) \tag{g}$$

(ii) If the excitation is vertical ground motion $u_{gy}(t)$,

$$\begin{Bmatrix} u_1^t \\ u_2^t \\ u_3^t \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} u_{gy} \tag{h}$$

Substituting Eq. (h) in Eq. (e) gives a matrix equation with its left hand side same as Eq. (g) and the right side is

$$\mathbf{p}_{\text{eff}}(t) = -\begin{Bmatrix} 0 \\ m \\ m \end{Bmatrix} \ddot{u}_{gy}(t) \tag{i}$$

(iii) If the excitation is ground motion $u_{gbd}(t)$ in the direction *b-d*,

$$\begin{Bmatrix} u_1^t \\ u_2^t \\ u_3^t \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix} u_{gbd} \tag{j}$$

Substituting Eq. (j) into Eq. (e) gives a matrix equation with its left hand side the same as Eq. (g) and the right hand side is

$$\mathbf{p}_{\text{eff}}(t) = -\begin{Bmatrix} 5m/\sqrt{2} \\ m/\sqrt{2} \\ m/\sqrt{2} \end{Bmatrix} \ddot{u}_{gbd}(t) \qquad \text{(k)}$$

(iv) If the excitation is ground motion $u_{gbc}(t)$ in the direction b-c,

$$\begin{Bmatrix} u_1^t \\ u_2^t \\ u_3^t \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix} u_{gbc} \qquad \text{(l)}$$

Substituting Eq. (l) into Eq. (e) gives a matrix equation with its left hand side the same as Eq. (g) and the right hand side is

$$\mathbf{p}_{\text{eff}}(t) = -\begin{Bmatrix} -5m/\sqrt{2} \\ m/\sqrt{2} \\ m/\sqrt{2} \end{Bmatrix} \ddot{u}_{gbc}(t) \qquad \text{(m)}$$

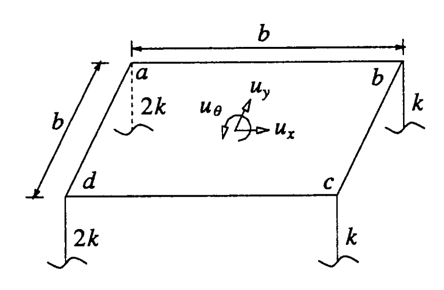
(v) If the excitation is rocking ground motion defined by counter-clockwise rotation $u_{g\theta}$ (in radians) in the plane of the structure,

$$\begin{Bmatrix} u_1^t \\ u_2^t \\ u_3^t \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} -L \\ L \\ -L \end{Bmatrix} u_{g\theta} \qquad \text{(n)}$$

Substituting Eq. (n) into Eq. (e) gives a matrix equation with its left hand side the same as Eq. (g) and the right hand side is

$$\mathbf{p}_{\text{eff}}(t) = -\begin{Bmatrix} -5mL \\ mL \\ -mL \end{Bmatrix} \ddot{u}_{g\theta}(t) \qquad \text{(o)}$$

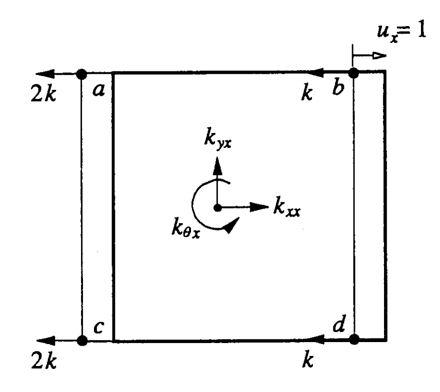
Problem 9.14



Part a

1. *Formulate the stiffness matrix.*

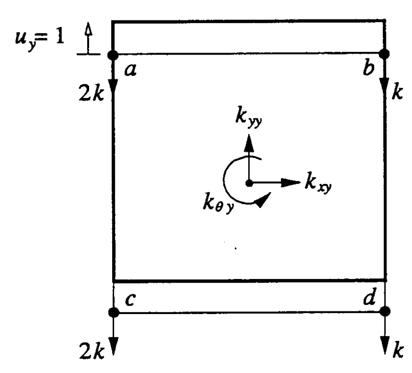
Apply $u_x = 1, u_y = u_\theta = 0$:



By statics,

$$k_{xx} = 6k \qquad k_{yx} = k_{\theta x} = 0$$

Apply $u_y = 1, u_x = u_\theta = 0$:



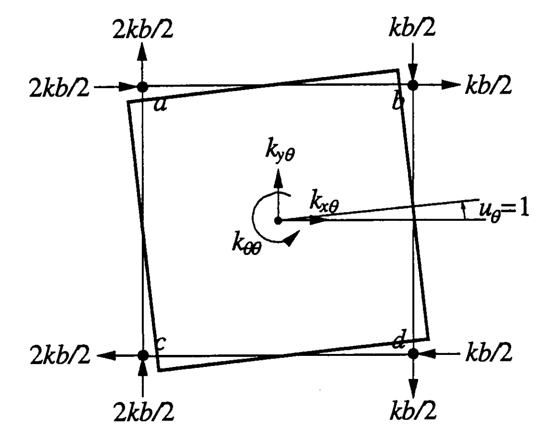
By statics,

$$k_{xy} = 0$$

$$k_{yy} = 6k$$

$$k_{\theta y} = 2\left(k\frac{b}{2} - 2k\frac{b}{2}\right) = -kb$$

Apply $u_\theta = 1, u_x = u_y = 0$:



By statics,

$$k_{x\theta} = 0$$

$$k_{y\theta} = 2\left(k\frac{b}{2} - 2k\frac{b}{2}\right) = -kb$$

$$k_{\theta\theta} = \left(3k\frac{b}{2}\right)\frac{b}{2} + \left(3k\frac{b}{2}\right)\frac{b}{2} + \left(2k\frac{b}{2}\right)\frac{b}{2} + \left(4k\frac{b}{2}\right)\frac{b}{2} = 3kb^2$$

Hence the stiffness matrix is

$$\mathbf{k} = k\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & -b \\ 0 & -b & 3b^2 \end{bmatrix}$$

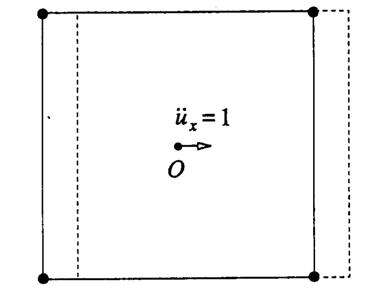
where

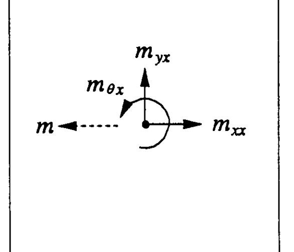
$$k = \frac{12EI}{h^3}$$

2. *Formulate the mass matrix.*

Apply $\ddot{u}_x = 1, \ddot{u}_y = \ddot{u}_\theta = 0 \Rightarrow$

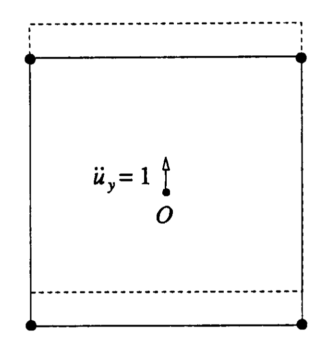
$$m_{xx} = m \qquad m_{yx} = m_{\theta x} = 0$$

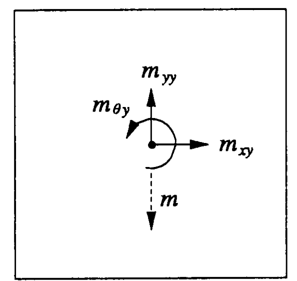




Apply $\ddot{u}_y = 1, \ddot{u}_x = \ddot{u}_\theta = 0 \Rightarrow$

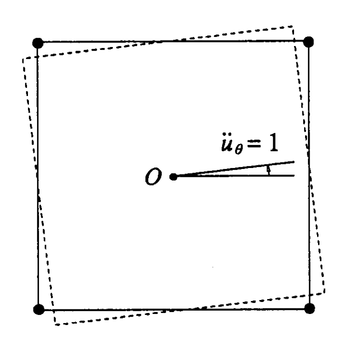
$m_{yy} = m \qquad m_{xy} = m_{\theta y} = 0$

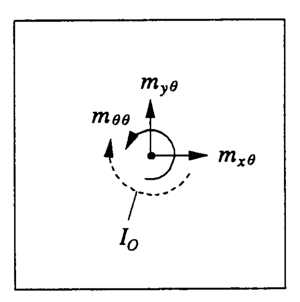




Apply $\ddot{u}_\theta = 1, \ddot{u}_x = \ddot{u}_y = 0 \Rightarrow$

$m_{\theta\theta} = I_O \qquad m_{x\theta} = m_{y\theta} = 0$





Hence the mass matrix is

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & r^2 \end{bmatrix}$$

where

$$r^2 = \frac{I_O}{m} = \frac{1}{m} m \frac{b^2 + b^2}{12} = \frac{b^2}{6}$$

***Part b**: Formulate the equations of motion.*

$$\mathbf{m}\ddot{\mathbf{u}}^t + \mathbf{k}\mathbf{u} = \mathbf{0} \qquad \text{(a)}$$

(i) *Ground motion in the x-direction.*

$$\begin{Bmatrix} \ddot{u}_x^t \\ \ddot{u}_y^t \\ \ddot{u}_\theta^t \end{Bmatrix} = \begin{Bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_\theta \end{Bmatrix} + \begin{Bmatrix} \ddot{u}_{gx} \\ 0 \\ 0 \end{Bmatrix}$$

$$m\begin{bmatrix} 1 & & \\ & 1 & \\ & & b^2/6 \end{bmatrix}\ddot{\mathbf{u}} + k\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & -b \\ 0 & -b & 3b^2 \end{bmatrix}\mathbf{u} = -m\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}\ddot{u}_{gx}(t) \qquad \text{(b)}$$

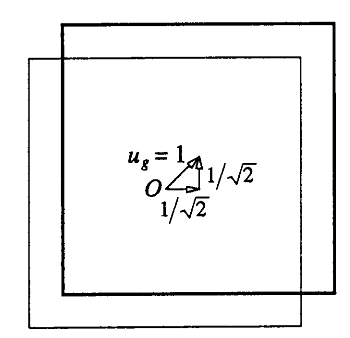
(ii) *Ground motion in the y-direction.*

$$\begin{Bmatrix} \ddot{u}_x^t \\ \ddot{u}_y^t \\ \ddot{u}_\theta^t \end{Bmatrix} = \begin{Bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_\theta \end{Bmatrix} + \begin{Bmatrix} 0 \\ \ddot{u}_{gy} \\ 0 \end{Bmatrix}$$

Substituting Eq. (b) in Eq. (a) gives an equation with the left side same as Eq. (b) but the right side is

$$\mathbf{p}_{\text{eff}}(t) = -m\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}\ddot{u}_{gy}(t)$$

(iii) *Ground motion in the d-b direction.*



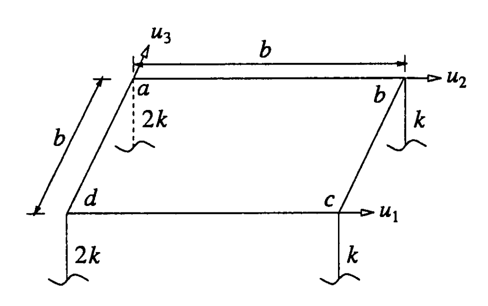
The influence vector is

$$\iota = \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{Bmatrix}$$

The equations of motion have the same left side as Eq. (b) but the right side is

$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\,\iota\,\ddot{\mathbf{u}}_g(t) = -m\begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{Bmatrix}\ddot{u}_g(t)$$

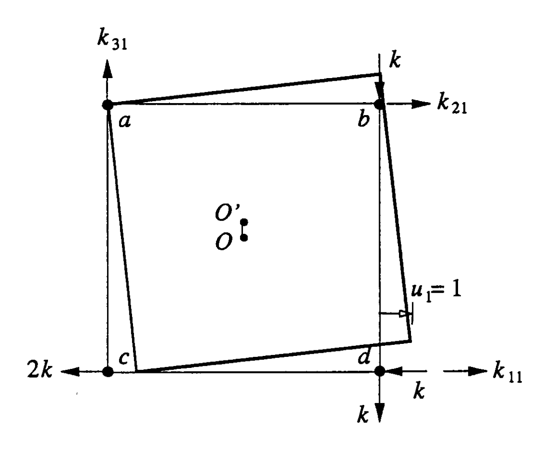
Problem 9.15



Part a

1. *Formulate the stiffness matrix.*

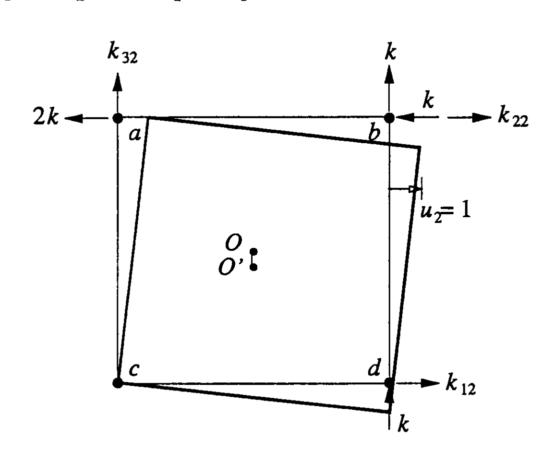
Apply $u_1 = 1, u_2 = u_3 = 0$:



By statics,

$$k_{11} = 5k \qquad k_{21} = -2k \qquad k_{31} = 2k$$

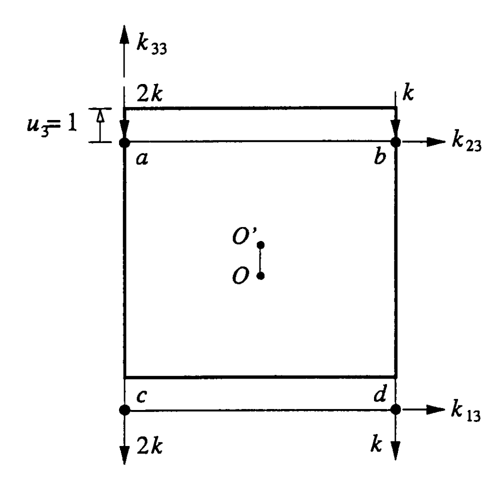
Apply $u_2 = 1, u_1 = u_3 = 0$:



By statics,

$$k_{12} = -2k \qquad k_{22} = 5k \qquad k_{32} = -2k$$

Apply $u_3 = 1, u_1 = u_2 = 0$:



By statics,

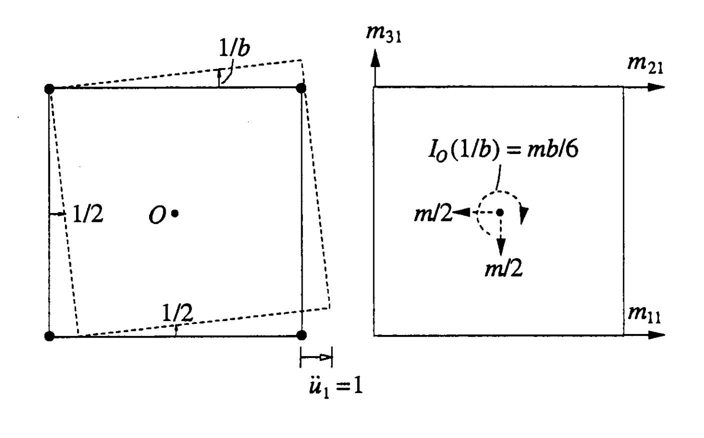
$$k_{13} = 2k \qquad k_{23} = -2k \qquad k_{33} = 6k$$

Hence the stiffness matrix is

$$\mathbf{k} = k\begin{bmatrix} 5 & -2 & 2 \\ -2 & 5 & -2 \\ 2 & -2 & 6 \end{bmatrix}$$

2. *Formulate the mass matrix.*

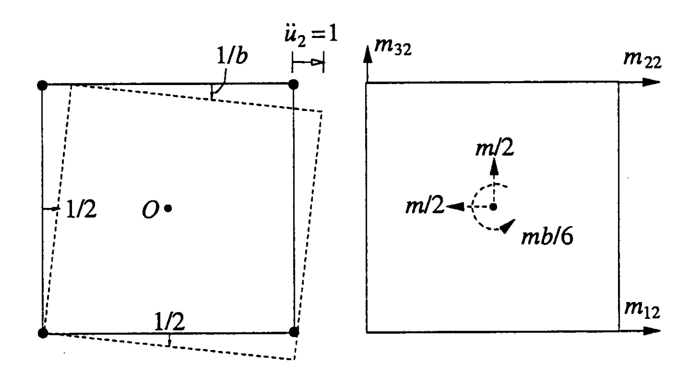
Apply $\ddot{u}_1 = 1, \ddot{u}_2 = \ddot{u}_3 = 0$:



By statics,

$$m_{11} = \frac{2m}{3} \qquad m_{21} = -\frac{m}{6} \qquad m_{31} = \frac{m}{2}$$

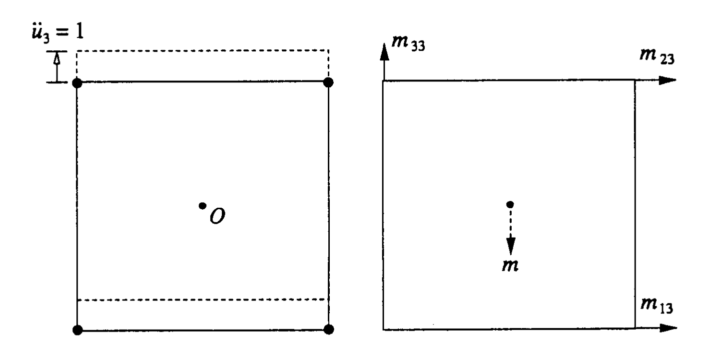
Apply $\ddot{u}_2 = 1$, $\ddot{u}_1 = \ddot{u}_3 = 0$:



By statics,

$$m_{12} = -\frac{m}{6} \qquad m_{22} = \frac{2m}{3} \qquad m_{32} = -\frac{m}{2}$$

Apply $\ddot{u}_3 = 1$, $\ddot{u}_1 = \ddot{u}_2 = 0$:



By statics,

$$m_{13} = \frac{m}{2} \qquad m_{23} = -\frac{m}{2} \qquad m_{33} = m$$

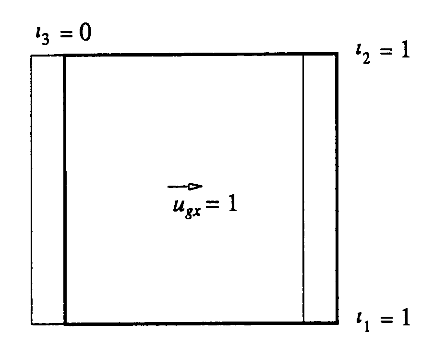
Hence the mass matrix is

$$\mathbf{m} = m\begin{bmatrix} 2/3 & -1/6 & 1/2 \\ -1/6 & 2/3 & -1/2 \\ 1/2 & -1/2 & 1 \end{bmatrix}$$

***Part b**: Formulate the equations of motion.*

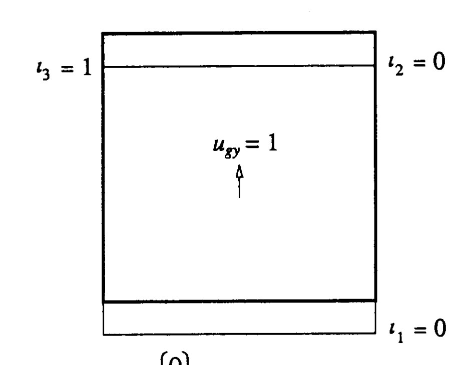
$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \underbrace{-\mathbf{m}\,\boldsymbol{\iota}\,\ddot{u}_g(t)}_{\mathbf{p}_{\text{eff}}(t)} \qquad \text{(a)}$$

(i) *Ground motion in the x-direction.*



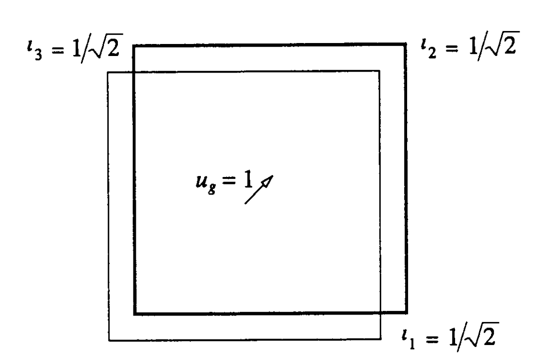
$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}\ddot{u}_{gx}(t) = -m\begin{Bmatrix} 1/2 \\ 1/2 \\ 0 \end{Bmatrix}\ddot{u}_{gx}(t)$$

(ii) *Ground motion in the y-direction.*



$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}\ddot{u}_{gy}(t) = -m\begin{Bmatrix} 1/2 \\ -1/2 \\ 1 \end{Bmatrix}\ddot{u}_{gy}(t)$$

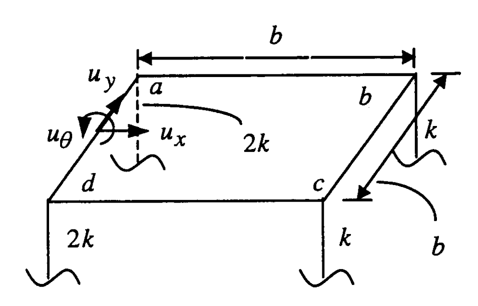
(iii) *Ground motion in the d-b direction.*



$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix}\ddot{u}_g(t)$$

$$= -m\begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{Bmatrix}\ddot{u}_g(t)$$

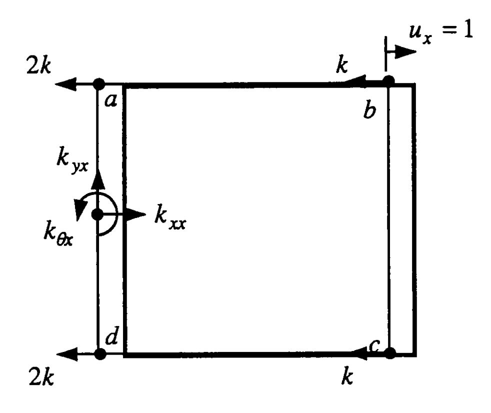
Problem 9.16



Part a

1. *Formulate the stiffness matrix.*

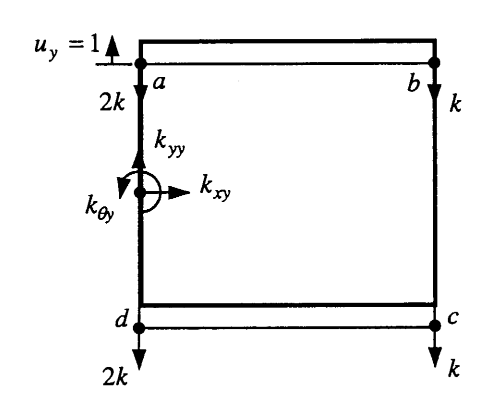
Apply $u_x = 1,\ u_y = u_\theta = 0$:



By statics,

$$k_{xx} = 6k \qquad k_{yx} = k_{\theta x} = 0$$

Apply $u_y = 1,\ u_x = u_\theta = 0$:



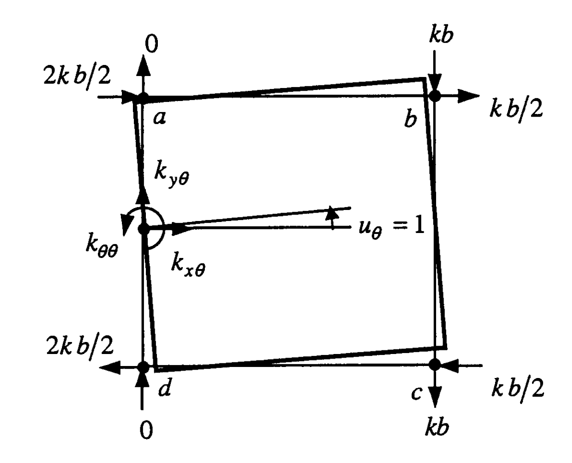
By statics,

$$k_{xy} = 0$$

$$k_{yy} = 6k$$

$$k_{\theta y} = 2kb$$

Apply $u_\theta = 1,\ u_x = u_y = 0$:



By statics,

$$k_{x\theta} = 0$$

$$k_{y\theta} = 2kb$$

$$k_{\theta\theta} = \left(3k\frac{b}{2}\right)\frac{b}{2} + \left(3k\frac{b}{2}\right)\frac{b}{2} + (2kb)b$$

$$= \frac{7}{2}kb^2$$

Hence the stiffness matrix is

$$\mathbf{k} = k\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 2b \\ 0 & 2b & 7b^2/2 \end{bmatrix}$$

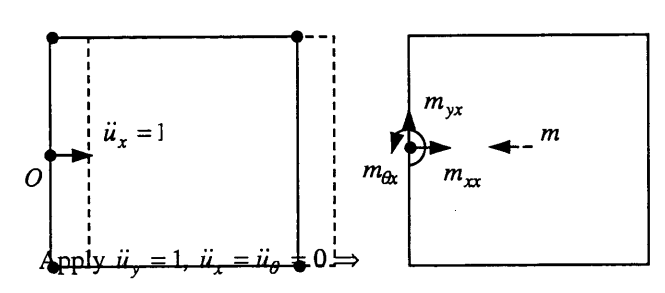
where

$$k = 12\frac{EI}{h^3}$$

2. *Formulate the mass matrix.*

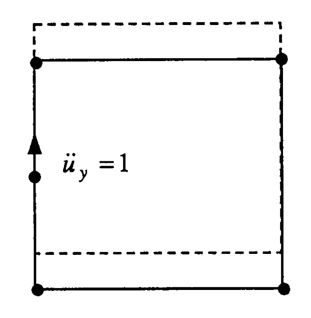
Apply $\ddot{u}_x = 1,\ \ddot{u}_y = \ddot{u}_\theta = 0 \Rightarrow$

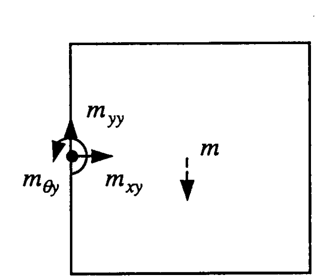
$$m_{xx} = m \qquad m_{yx} = m_{\theta x} = 0$$



Apply $\ddot{u}_y = 1,\ \ddot{u}_x = \ddot{u}_\theta = 0 \Rightarrow$

$$m_{xy} = 0 \qquad m_{yy} = m \qquad m_{\theta y} = m\frac{b}{2}$$



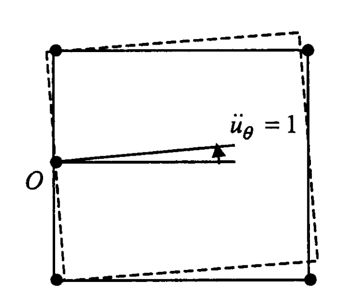


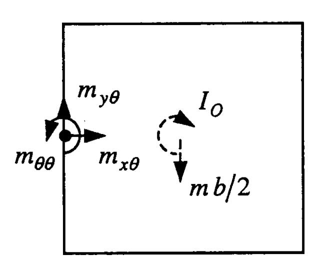
Apply $\ddot{u}_\theta = 1,\ \ddot{u}_x = \ddot{u}_y = 0 \Rightarrow$

$$m_{x\theta} = 0$$

$$m_{y\theta} = \frac{mb}{2}$$

$$m_{\theta\theta} = I_O + m\frac{b}{2}\frac{b}{2} = m\left(\frac{b^2 + b^2}{12}\right) + m\frac{b^2}{4} = \frac{5mb^2}{12}$$





Hence the mass matrix is

$$\mathbf{m} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b/2 \\ 0 & b/2 & 5b^2/12 \end{bmatrix}$$

Part b: *Formulate the equations of motion.*

$$\mathbf{m}\ddot{\mathbf{u}}^t + \mathbf{k}\mathbf{u} = \mathbf{0} \tag{a}$$

(i) Ground motion in the x-direction.

$$\begin{Bmatrix} \ddot{u}_x^t \\ \ddot{u}_y^t \\ \ddot{u}_\theta^t \end{Bmatrix} = \begin{Bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_\theta \end{Bmatrix} + \begin{Bmatrix} \ddot{u}_{gx} \\ 0 \\ 0 \end{Bmatrix}$$

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b/2 \\ 0 & b/2 & 5b^2/12 \end{bmatrix}\ddot{\mathbf{u}} + k\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 2b \\ 0 & 2b & 7b^2/2 \end{bmatrix}\mathbf{u} = -m\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}\ddot{u}_{gx}(t) \tag{b}$$

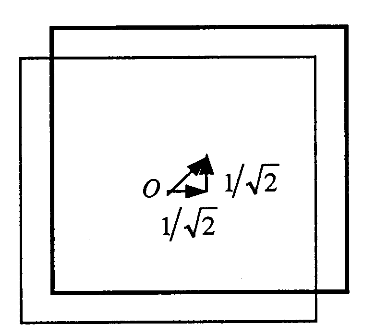
(ii) Ground motion in the y-direction.

$$\begin{Bmatrix} \ddot{u}_x^t \\ \ddot{u}_y^t \\ \ddot{u}_\theta^t \end{Bmatrix} = \begin{Bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_\theta \end{Bmatrix} + \begin{Bmatrix} 0 \\ \ddot{u}_{gy} \\ 0 \end{Bmatrix} \tag{c}$$

Substituting Eq. (c) in Eq. (a) gives an equation with the left hand side the same as Eq. (b) but the right side is

$$\mathbf{p}_{\text{eff}}(t) = -m\begin{Bmatrix} 0 \\ 1 \\ b/2 \end{Bmatrix}\ddot{u}_{gy}(t)$$

(iii) Ground motion in the d-b direction.



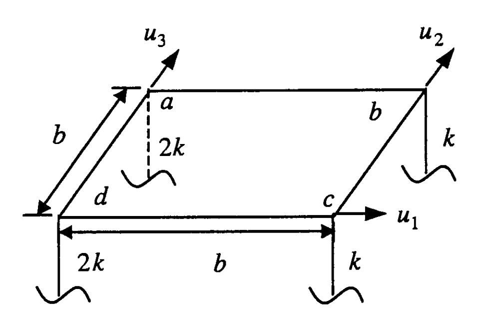
The influence vector is

$$\boldsymbol{\iota} = \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{Bmatrix}$$

The equations of motion have the same left side as Eq. (b) but the right side is

$$\mathbf{p}_{eff}(t) = -\mathbf{m}\,\boldsymbol{\iota}\,\ddot{u}_g(t) = -m\begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ b/2\sqrt{2} \end{Bmatrix}\ddot{u}_g(t)$$

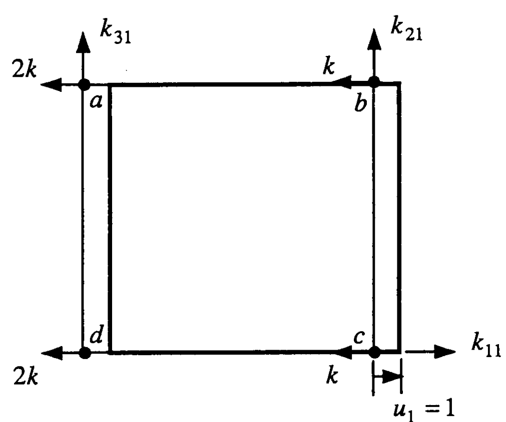
Problem 9.17



Part a

1. *Formulate the stiffness matrix.*

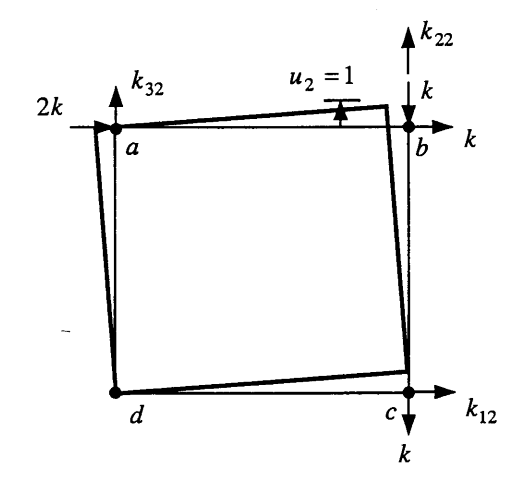
Apply $u_1 = 1,\ u_2 = u_3 = 0$:



By statics,

$$k_{11} = 6k \qquad k_{21} = -3k \qquad k_{31} = 3k$$

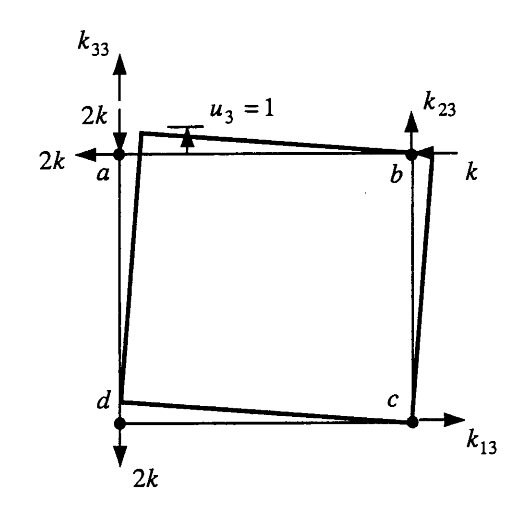
Apply $u_2 = 1,\ u_1 = u_3 = 0$:



By statics,

$$k_{12} = -3k \qquad k_{22} = 5k \qquad k_{32} = -3k$$

Apply $u_3 = 1,\ u_1 = u_2 = 0$:



By statics,

$$k_{13} = 3k \qquad k_{23} = -3k \qquad k_{33} = 7k$$

Hence the stiffness matrix is

$$\mathbf{k} = k\begin{bmatrix} 6 & -3 & 3 \\ -3 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

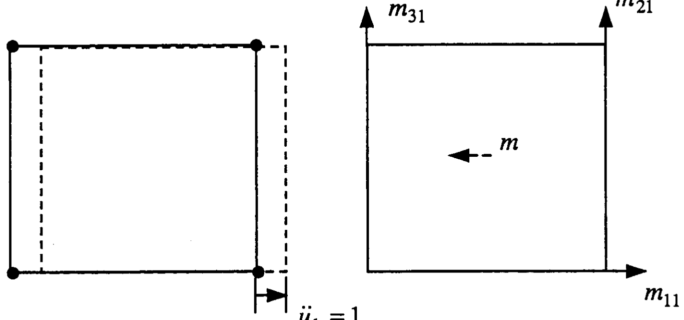
where

$$k = 12\frac{EI}{h^3}$$

2. *Formulate the mass matrix.*

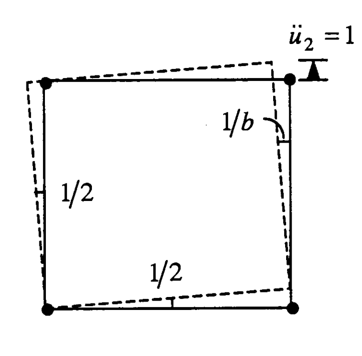
Apply $\ddot{u}_1 = 1,\ \ddot{u}_2 = \ddot{u}_3 = 0 \Rightarrow$

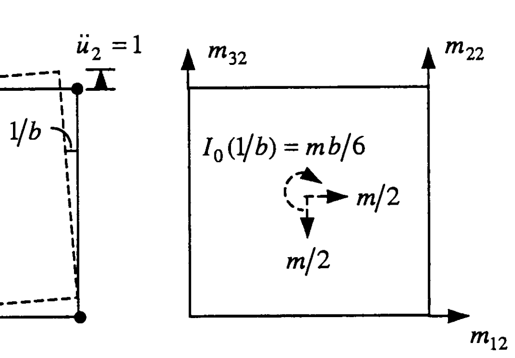
$$m_{11} = m \qquad m_{21} = -\frac{m}{2} \qquad m_{31} = \frac{m}{2}$$



Apply $\ddot{u}_1 = 1,\ \ddot{u}_2 = \ddot{u}_3 = 0 \Rightarrow$

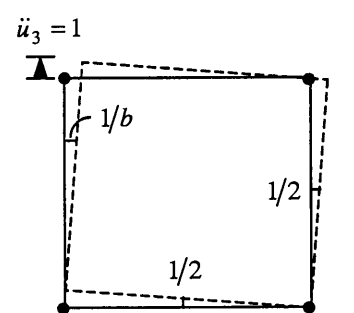
$$m_{12} = -\frac{m}{2} \qquad m_{22} = \frac{2m}{3} \qquad m_{32} = -\frac{m}{6}$$

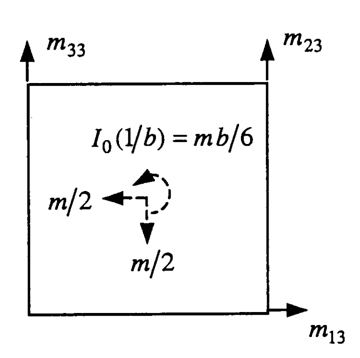




Apply $\ddot{u}_3 = 1,\ \ddot{u}_1 = \ddot{u}_2 = 0 \Rightarrow$

$$m_{13} = \frac{m}{2} \qquad m_{23} = -\frac{m}{6} \qquad m_{33} = \frac{2m}{3}$$





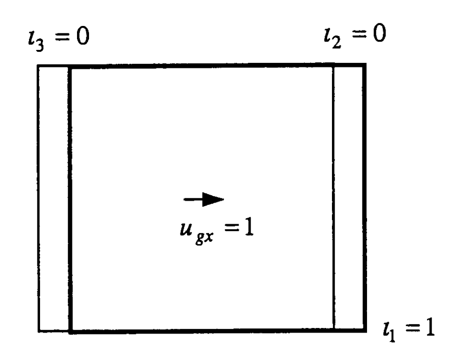
Hence the mass matrix is

$$\mathbf{m} = m\begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 2/3 & -1/6 \\ 1/2 & -1/6 & 2/3 \end{bmatrix}$$

Part b: Formulate the equations of motion.

$$\mathbf{m\ddot{u}} + \mathbf{ku} = \underbrace{-\mathbf{m}\,\boldsymbol{\iota}\,\ddot{u}_g(t)}_{\mathbf{p}_{\text{eff}}(t)} \qquad \text{(a)}$$

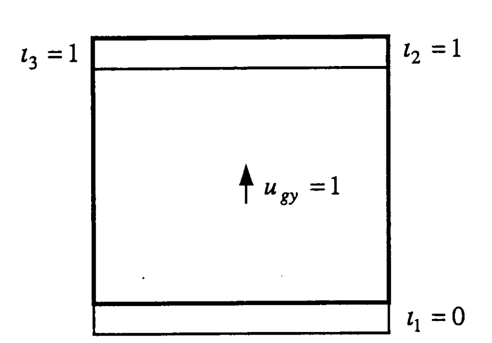
i) Ground motion in the x-direction.



$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}\ddot{u}_{gx}(t)$$

$$= -m\begin{Bmatrix} 1 \\ -1/2 \\ 1/2 \end{Bmatrix}\ddot{u}_{gx}(t)$$

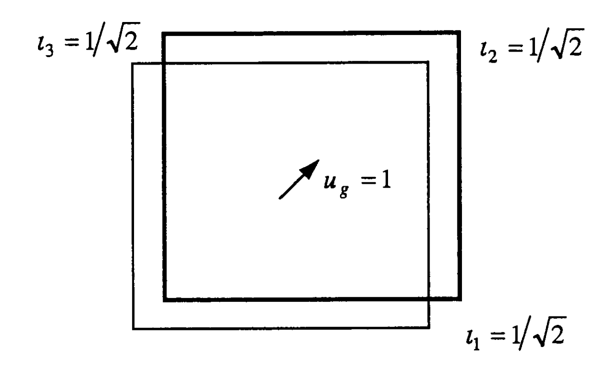
(ii) Ground motion in the y-direction.



$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}\ddot{u}_{gy}(t)$$

$$= -m\begin{Bmatrix} 0 \\ 1/2 \\ 1/2 \end{Bmatrix}\ddot{u}_{gy}(t)$$

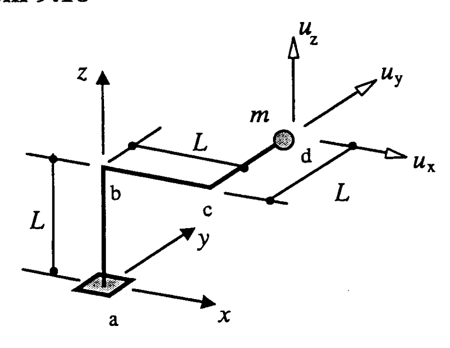
(iii) Ground motion in the d-b direction.



$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix}\ddot{u}_g(t)$$

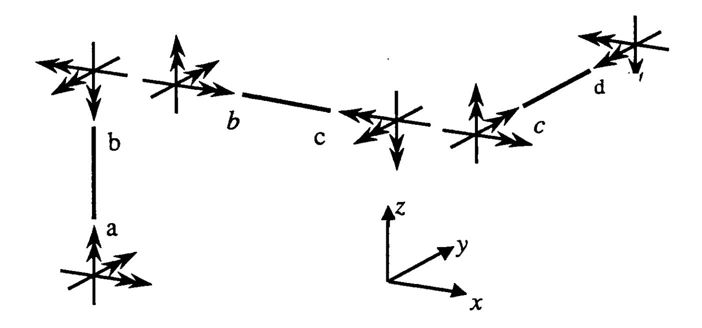
$$= -m\begin{Bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{Bmatrix}\ddot{u}_g(t)$$

Problem 9.18



1. *Sign Convention.*

The assumed positive sense of the internal bending moments and torques acting on each element are as shown:



2. *Determine mass matrix.*

The mass matrix, **m**, for this structure is

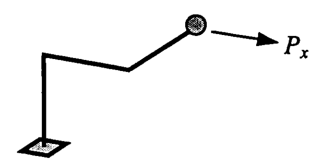
$$\mathbf{m} = \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix}$$

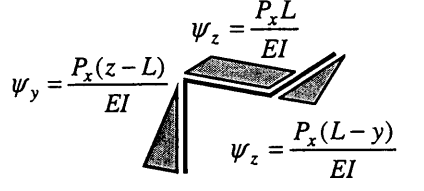
3. *Determine flexibility matrix.*

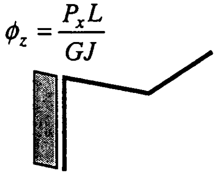
Compute the flexibility matrix, $\hat{\mathbf{f}}$, in terms of the DOF u_x, u_y and u_z using the principle of virtual forces.

3.1 Establish the curvatures and rates of twist in the elements due to real forces P_x, P_y and P_z applied at the mass in the x, y and z directions respectively:

(i) Apply real force P_x

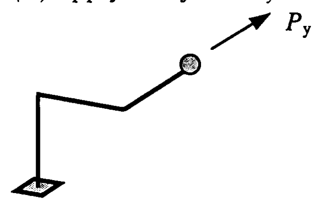


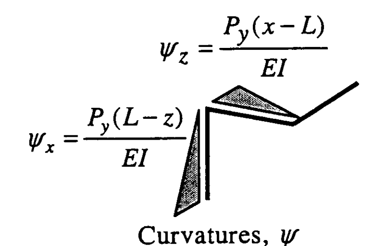


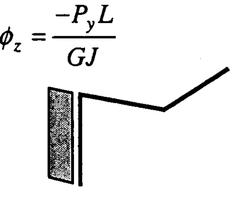


Curvatures, ψ Rate of twist, ϕ

(ii) Apply real force P_y

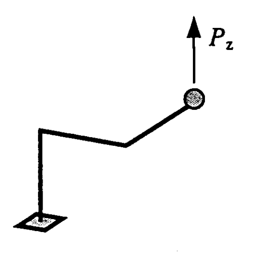


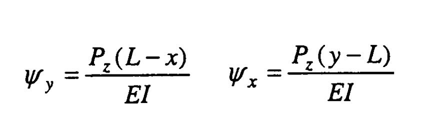


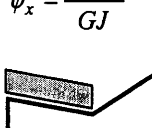


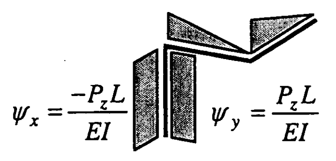
Curvatures, ψ Rate of twist, ϕ

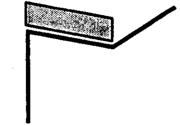
(iii) Apply real force P_z





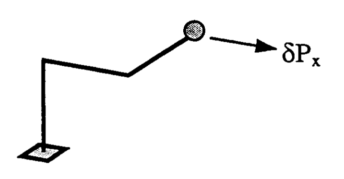


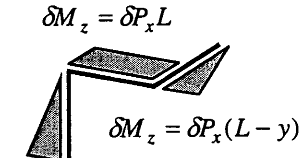


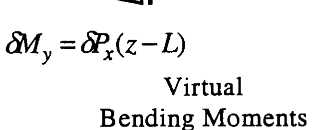


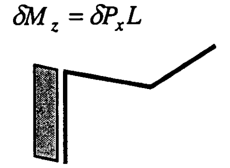
Curvatures, ψ Rate of twist, ϕ

3.2 Determine deflection u_x due to forces P_x, P_y and P_z. Apply virtual force δP_x in the direction of DOF u_x









Equate external and internal virtual work due to δP_x

$$\delta P_x u_x = \int_0^L \delta P_x (L-y) \frac{P_x (L-y)}{EI} dy$$

$$+ \int_0^L \delta P_x L \left[\frac{P_x L + P_y (x-L)}{EI} \right] dx$$

$$+ \int_0^L \delta P_x (z-L) \left[\frac{P_x (z-L) + P_z L}{EI} \right] dz$$

$$+ \int_0^L \delta P_x L \left[\frac{P_x L - P_y L}{GJ} \right] dx$$

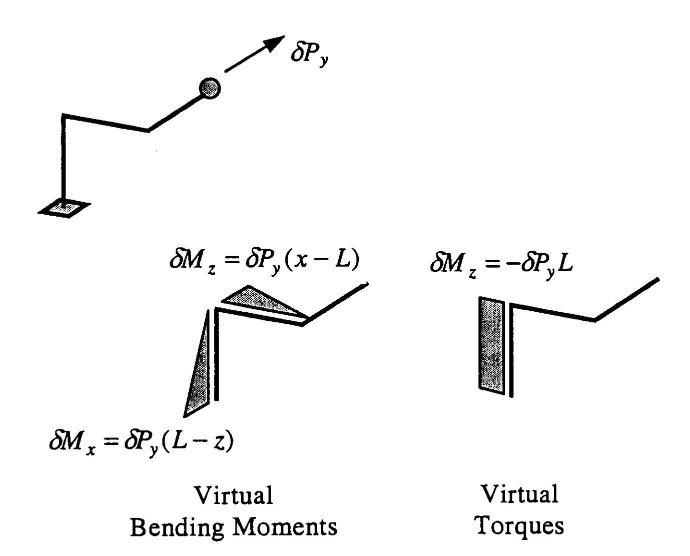
Hence,

$$u_x = P_x \left[\frac{5L^3}{3EI} + \frac{L^3}{GJ} \right] + P_y \left[-\frac{L^3}{2EI} - \frac{L^3}{GJ} \right] + P_z \left[-\frac{L^3}{2EI} \right]$$

and

$$\hat{f}_{xx} = \frac{5L^3}{3EI} + \frac{L^3}{GJ} \qquad \hat{f}_{xy} = -\frac{L^3}{2EI} - \frac{L^3}{GJ} \qquad \hat{f}_{xz} = -\frac{L^3}{2EI}$$

3.3 Determine deflection u_y due to forces P_x, P_y and P_z. Apply virtual force δP_y in the direction of DOF u_y



Equate external and internal virtual work due to δP_y

$$\delta P_y u_y = \int_0^L \delta P_y (x-L) \left[\frac{P_y (x-L) + P_x L}{EI} \right] dx$$

$$+ \int_0^L \delta P_y (L-z) \left[\frac{P_y (L-z) - P_z L}{EI} \right] dz$$

$$+ \int_0^L -\delta P_y L \left[\frac{-P_y L + P_x L}{GJ} \right] dz$$

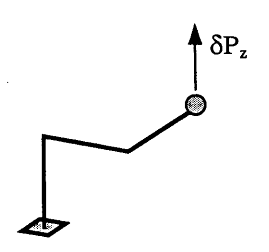
Hence,

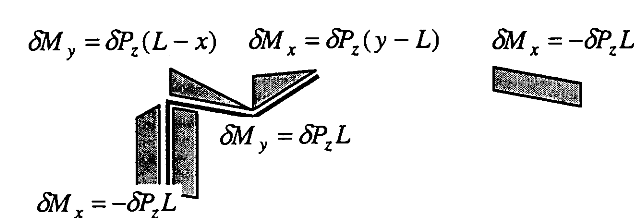
$$u_y = P_x \left[-\frac{L^3}{2EI} - \frac{L^3}{GJ} \right] + P_y \left[\frac{2L^3}{3EI} + \frac{L^3}{GJ} \right] + P_z \left[-\frac{L^3}{2EI} \right]$$

and

$$\hat{f}_{yx} = -\frac{L^3}{2EI} - \frac{L^3}{GJ} \qquad \hat{f}_{yy} = \frac{2L^3}{3EI} + \frac{L^3}{GJ} \qquad \hat{f}_{yz} = -\frac{L^3}{2EI}$$

3.4 Determine deflection u_z due to forces P_x, P_y and P_z. Apply virtual force δP_z in the direction of DOF u_z.





Virtual Bending Moments

Virtual Torques

Equate external and internal virtual work due to δP_z

$$\delta P_z u_z = \int_0^L \delta P_z (y-L)\left[\frac{P_z(y-L)}{EI}\right]dy$$

$$+\int_0^L \delta P_z (L-x)\left[\frac{P_z(L-x)}{EI}\right]dx$$

$$+\int_0^L -\delta P_z L\left[\frac{-P_z L + P_y(L-z)}{EI}\right]dz$$

$$+\int_0^L \delta P_z L\left[\frac{P_z L + P_x(z-L)}{EI}\right]dz$$

$$+\int_0^L -\delta P_z L\left[\frac{-P_z L}{GJ}\right]dx$$

Hence,

$$u_z = P_x\left[-\frac{L^3}{2EI}\right] + P_y\left[-\frac{L^3}{2EI}\right] + P_z\left[\frac{8L^3}{3EI}+\frac{L^3}{GJ}\right]$$

and

$$\hat{f}_{zx} = -\frac{L^3}{2EI} \qquad \hat{f}_{zy} = -\frac{L^3}{2EI} \qquad \hat{f}_{zz} = \frac{8L^3}{3EI}+\frac{L^3}{GJ}$$

Therefore the flexibility matrix, $\hat{\mathbf{f}}$, is

$$\hat{\mathbf{f}} = \begin{bmatrix} \frac{5L^3}{3EI}+\frac{L^3}{GJ} & -\frac{L^3}{2EI}-\frac{L^3}{GJ} & -\frac{L^3}{2EI} \\ -\frac{L^3}{2EI}-\frac{L^3}{GJ} & \frac{2L^3}{3EI}+\frac{L^3}{GJ} & -\frac{L^3}{2EI} \\ -\frac{L^3}{2EI} & -\frac{L^3}{2EI} & \frac{8L^3}{3EI}+\frac{L^3}{GJ} \end{bmatrix}$$

4. *Determine stiffness matrix.*

The stiffness matrix, **k**, is obtained by inverting $\hat{\mathbf{f}}$. For the case when $GJ = 4/5\ EI$:

$$\hat{\mathbf{f}} = \frac{L^3}{12EI}\begin{bmatrix} 35 & -21 & -6 \\ -21 & 23 & -6 \\ -6 & -6 & 47 \end{bmatrix}$$

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 0.9283 & 0.9088 & 0.2345 \\ 0.9088 & 1.4294 & 0.2985 \\ 0.2345 & 0.2985 & 0.3234 \end{bmatrix}$$

5. *Formulate the equations of motion* $\mathbf{u} = [\ u_x,\ u_y,\ u_z\ ]^T$

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\iota\ddot{u}_g$$

where **m** and **k** are given above and influence vector ι = static displacements of DOF due to $u_g = 1$.

(i) ground motion in x-direction : $\iota = [\ 1\ \ 0\ \ 0\ ]^T$

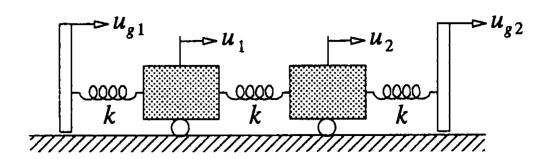
(ii) ground motion in y-direction : $\iota = [\ 0\ \ 1\ \ 0\ ]^T$

(iii) ground motion in z-direction : $\iota = [\ 0\ \ 0\ \ 1\ ]^T$

(iv) ground motion in direction a-d:

$$\iota = \frac{1}{\sqrt{3}}[1\ \ 1\ \ 1]^T$$

Problem 9.19



1. *Determine the stiffness matrix..*

$$\begin{bmatrix} \mathbf{k} & \mathbf{k}_g \\ \mathbf{k}_g^T & \mathbf{k}_{gg} \end{bmatrix} = k \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_{g1} & u_{g2} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \left[\begin{array}{cc|cc} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ \hline -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \end{array}$$

where

$$\mathbf{k} = k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

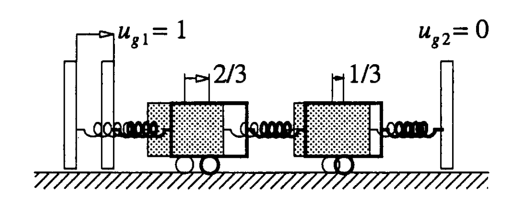
$$\mathbf{k}_g = k\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

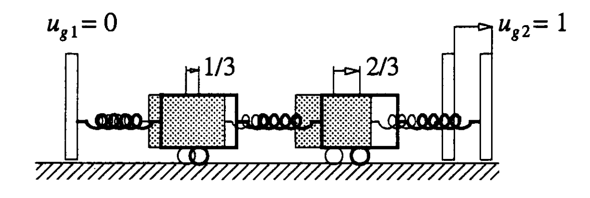
$$\mathbf{k}_{gg} = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. *Determine the mass matrix..*

$$\mathbf{m} = m\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

3. *Determine the influence matrix..*





$$\boldsymbol{\iota} = -\mathbf{k}^{-1}\mathbf{k}_g = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

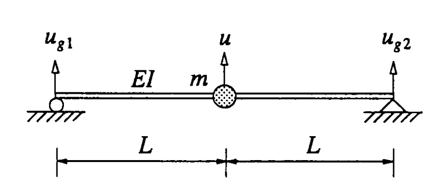
4. *Write the equations of motion..*

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}_{\text{eff}}(t)$$

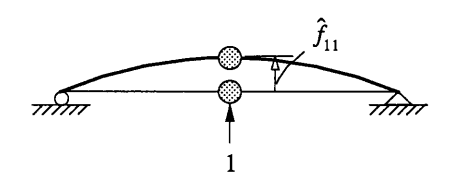
where

$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\,\boldsymbol{\iota}\,\ddot{\mathbf{u}}_g(t) = -m\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{g1}(t) \\ \ddot{u}_{g2}(t) \end{Bmatrix}$$

Problem 9.20



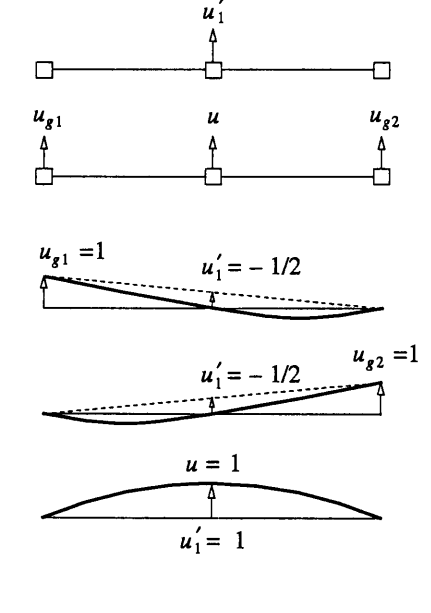
1. *Formulate the stiffness matrix..*



$$\hat{f}_{11} = \frac{(2L)^3}{48EI} = \frac{L^3}{6EI}$$

$$\hat{k}_{11} = \frac{6EI}{L^3}$$

Coordinate transformation:



$$u_1' = \langle 1 \quad -1/2 \quad -1/2 \rangle \begin{Bmatrix} u \\ u_{g1} \\ u_{g2} \end{Bmatrix}$$

Stiffness matrix in DOFs u, u_{g1} and u_{g2}:

$$\begin{bmatrix} \mathbf{k} & \mathbf{k}_g \\ \mathbf{k}_g^T & \mathbf{k}_{gg} \end{bmatrix} = \begin{Bmatrix} 1 \\ -1/2 \\ -1/2 \end{Bmatrix} \frac{6EI}{L^3} \langle 1 \quad -1/2 \quad -1/2 \rangle$$

$$= \frac{6EI}{L^3} \left[\begin{array}{c|cc} 1 & -1/2 & -1/2 \\ \hline -1/2 & 1/4 & 1/4 \\ -1/2 & 1/4 & 1/4 \end{array} \right]$$

$$\mathbf{k} = \left[\frac{6EI}{L^3} \right]$$

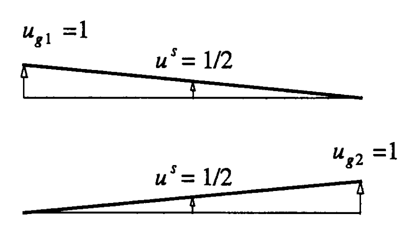
$$\mathbf{k}_g = \frac{6EI}{L^3} \langle -1/2 \quad -1/2 \rangle$$

$$\mathbf{k}_{gg} = \frac{6EI}{L^3} \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$$

2. *Write the mass matrix..*

$$\mathbf{m} = m$$

3. *Determine the influence matrix..*



By kinematics

$$\iota = \langle 1/2 \quad 1/2 \rangle$$

Alternatively,

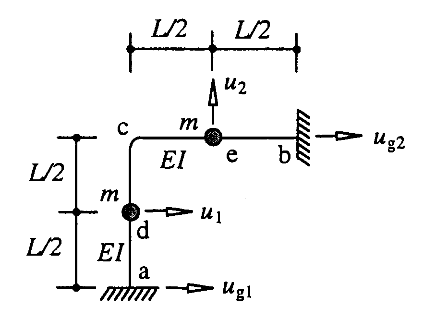
$$\iota = -\mathbf{k}^{-1}\mathbf{k}_g = -\frac{L^3}{6EI}\frac{6EI}{L^3}\langle -1/2 \quad -1/2 \rangle$$

$$= \langle 1/2 \quad 1/2 \rangle$$

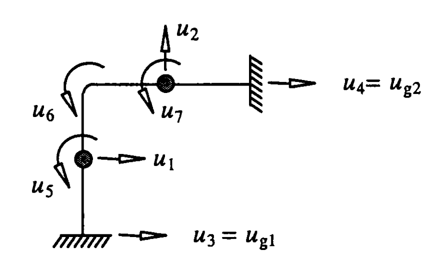
4. *Write the equation of motion.*

$$m\ddot{u} + \frac{6EI}{L^3}u = -m\langle 1/2 \quad 1/2 \rangle \begin{Bmatrix} \ddot{u}_{g1}(t) \\ \ddot{u}_{g2}(t) \end{Bmatrix}$$

Problem 9.21



1. *Define the degrees of freedom.*



2. *Formulate stiffness matrix..*

Construct the element stiffness matrices

Element a-d

$$\hat{\mathbf{k}}_{a-d} = \frac{4EI}{L^3}\begin{bmatrix} 24 & -24 & 6L \\ -24 & 24 & -6L \\ 6L & -6L & 2L^2 \end{bmatrix} \quad (u_1, u_3, u_5)$$

with $u_3 = u_{g1}$.

Element d-c

$$\hat{\mathbf{k}}_{d-c} = \frac{4EI}{L^3}\begin{bmatrix} 24 & -24 & -6L & -6L \\ -24 & 24 & 6L & 6L \\ -6L & 6L & 2L^2 & L^2 \\ -6L & 6L & L^2 & 2L^2 \end{bmatrix} \quad (u_1, u_4, u_5, u_6)$$

with $u_4 = u_{g2}$.

Element c-e

$$\hat{\mathbf{k}}_{c-e} = \frac{4EI}{L^3}\begin{bmatrix} 24 & -6L & -6L \\ -6L & 2L^2 & L^2 \\ -6L & L^2 & 2L^2 \end{bmatrix} \quad (u_2, u_6, u_7)$$

Element e-b

$$\hat{\mathbf{k}}_{e-b} = \frac{4EI}{L^3}\begin{bmatrix} 24 & 6L \\ 6L & 2L^2 \end{bmatrix} \quad (u_2, u_7)$$

Add element stiffness matrices into global stiffness matrix

$$\left[\begin{array}{c|c} \mathbf{k}_{tt} & \mathbf{k}_{t\theta} \\ \hline \mathbf{k}_{\theta t} & \mathbf{k}_{\theta\theta} \end{array}\right] = \frac{4EI}{L^3}\left[\begin{array}{cccc|ccc} 48 & 0 & -24 & -24 & 0 & -6L & 0 \\ 0 & 48 & 0 & 0 & 0 & -6L & 0 \\ -24 & 0 & 24 & 0 & -6L & 0 & 0 \\ -24 & 0 & 0 & 24 & 6L & 6L & 0 \\ \hline 0 & 0 & -6L & 6L & 4L^2 & L^2 & 0 \\ -6L & -6L & 0 & 6L & L^2 & 4L^2 & L^2 \\ 0 & 0 & 0 & 0 & 0 & L^2 & 4L^2 \end{array}\right] \quad (u_1, u_2, u_3, u_4, u_5, u_6, u_7)$$

Condense out the rotational DOF $[u_5, u_6, u_7]^T$

$$\left[\begin{array}{c|c} \mathbf{k} & \mathbf{k}_g \\ \hline \mathbf{k}_g^T & \mathbf{k}_{gg} \end{array}\right] = \mathbf{k}_{tt} - \mathbf{k}_{t\theta}\mathbf{k}_{\theta\theta}^{-1}\mathbf{k}_{\theta t}$$

$$= \frac{6EI}{7L^3}\left[\begin{array}{cc|cc} 176 & -48 & -100 & -76 \\ -48 & 176 & 12 & 36 \\ \hline -100 & 12 & 67 & 33 \\ -76 & 36 & 33 & 43 \end{array}\right] \quad (u_1, u_2, u_{g1}, u_{g2})$$

3. *Write the mass matrix.*

$$\mathbf{m} = \begin{bmatrix} m & \\ & m \end{bmatrix} \quad (u_1, u_2)$$

4. *Determine the influence matrix.*

$$\boldsymbol{\iota} = -\mathrm{k}^{-1}\mathrm{k}_g = \frac{1}{448}\begin{bmatrix} 266 & 182 \\ 42 & -42 \end{bmatrix}$$

Note that each column of $\boldsymbol{\iota}$ can be interpreted as the displacement $\{u_1, u_2\}^T$ due to a unit displacement of one of the supports. The first column corresponds to $u_{g1} = 1$ and the second column corresponds to $u_{g2} = 1$.

5. *Write the equations of motion.*

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\boldsymbol{\iota}\ddot{\mathbf{u}}_g(t) \qquad \text{(a)}$$

where $\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ and $\ddot{\mathbf{u}}_g(t) = \begin{Bmatrix} \ddot{u}_{g1}(t) \\ \ddot{u}_{g2}(t) \end{Bmatrix}$

6. *For the case of identical ground motion.*

$$\ddot{u}_{g2}(t) = \ddot{u}_{g1}(t), \quad \text{i.e.} \quad \ddot{\mathbf{u}}_g(t) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\ddot{u}_{g1}(t)$$

Therefore, the equations of motion become

$$\mathrm{m}\ddot{\mathrm{u}} + \mathrm{k}\mathrm{u} = -\mathrm{m}\boldsymbol{\iota}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\ddot{u}_g(t) = -\mathrm{m}\hat{\boldsymbol{\iota}}\ddot{u}_{g1} \qquad \text{(b)}$$

where $\hat{\boldsymbol{\iota}} = \boldsymbol{\iota}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

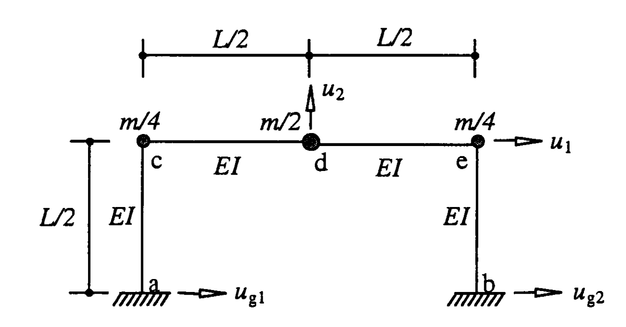
Note that this influence vector can be interpreted as the displacement $\{u_1, u_2\}^T$ due to a simultaneous unit displacement of u_{g1} and u_{g2}.

Observe that

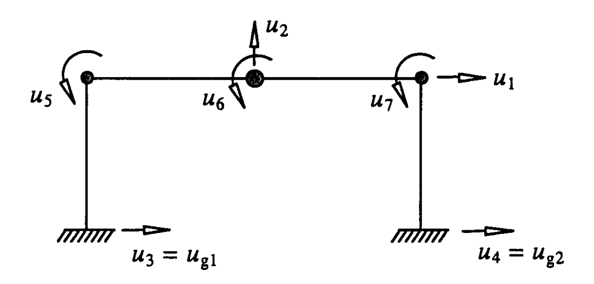
$$\mathbf{p}_{eff}(t) = -\begin{Bmatrix} m \\ 0 \end{Bmatrix}\ddot{u}_{g1}(t),$$

implying that the effective force in the vertical DOF is zero because the horizontal support motions are identical. When different, even horizontal support motions create effective force in the vertical DOF; see Eq. (a).

Problem 9.22



1. *Define the degrees of freedom.*



2. *Formulate stiffness matrix.*

Construct the element stiffness matrices

Element a-c

$$\hat{\mathbf{k}}_{a-c} = \frac{4EI}{L^3}\begin{bmatrix} 24 & -24 & 6L \\ -24 & 24 & -6L \\ 6L & -6L & 2L^2 \end{bmatrix} \quad (u_1, u_3, u_5)$$

Element b-e

$$\hat{\mathbf{k}}_{b-e} = \frac{4EI}{L^3}\begin{bmatrix} 24 & -24 & 6L \\ -24 & 24 & -6L \\ 6L & -6L & 2L^2 \end{bmatrix} \quad (u_1, u_4, u_7)$$

Element c-d

$$\hat{\mathbf{k}}_{c-d} = \frac{4EI}{L^3}\begin{bmatrix} 24 & -6L & -6L \\ -6L & 2L^2 & L^2 \\ -6L & L^2 & 2L^2 \end{bmatrix} \quad (u_2, u_5, u_6)$$

Element d-e

$$\hat{\mathbf{k}}_{d-e} = \frac{4EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 2L^2 & L^2 \\ 6L & L^2 & 2L^2 \end{bmatrix} \quad (u_2, u_6, u_7)$$

Add element stiffness matrices into global stiffness matrix

$$\begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t\theta} \\ \mathbf{k}_{\theta t} & \mathbf{k}_{\theta\theta} \end{bmatrix} = \frac{4EI}{L^3}\left[\begin{array}{cccc|ccc} 48 & 0 & -24 & -24 & 6L & 0 & 6L \\ 0 & 48 & 0 & 0 & -6L & 0 & 6L \\ -24 & 0 & 24 & 0 & -6L & 0 & 0 \\ -24 & 0 & 0 & 24 & 0 & 0 & -6L \\ \hline 6L & -6L & -6L & 0 & 4L^2 & L^2 & 0 \\ 0 & 0 & 0 & 0 & L^2 & 4L^2 & L^2 \\ 6L & 6L & 0 & -6L & 0 & L^2 & 4L^2 \end{array}\right] \quad (u_1, u_2, u_3, u_4, u_5, u_6, u_7)$$

Condense out the rotational DOF $[u_5, u_6, u_7]^T$

$$\begin{bmatrix} \mathbf{k} & \mathbf{k}_g \\ \mathbf{k}_g^T & \mathbf{k}_{gg} \end{bmatrix} = \mathbf{k}_{tt} - \mathbf{k}_{t\theta}\mathbf{k}_{\theta\theta}^{-1}\mathbf{k}_{\theta t}$$

$$= \frac{6EI}{7L^3}\left[\begin{array}{cc|cc} 128 & 0 & -64 & -64 \\ 0 & 140 & -42 & 42 \\ \hline -64 & -42 & 67 & -3 \\ -64 & 42 & -3 & 67 \end{array}\right] \quad (u_1, u_2, u_{g1}, u_{g2})$$

3. *Write the mass matrix.*

$$\mathbf{m} = \begin{bmatrix} m & \\ & m/2 \end{bmatrix} \quad (u_1, u_2)$$

4. *Determine the influence matrix.*

$$\boldsymbol{\iota} = -\mathbf{k}^{-1}\mathbf{k}_g = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & -0.3 \end{bmatrix}$$

Note that each column of $\boldsymbol{\iota}$ can be interpreted as the displacement $\{u_1, u_2\}^T$ due to a unit displacement of one of the supports. The first column corresponds to $u_{g1} = 1$ and the second column corresponds to $u_{g2} = 1$.

5. *Write the equations of motion.*

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\boldsymbol{\iota}\,\ddot{\mathbf{u}}_g(t) \qquad \text{(a)}$$

where $\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ and $\ddot{\mathbf{u}}_g(t) = \begin{Bmatrix} u_{g1}(t) \\ u_{g2}(t) \end{Bmatrix}$

6. *For the case of identical ground motion.*

$$\ddot{u}_{g2}(t) = \ddot{u}_{g1}(t), \quad \textbf{i.e.} \quad \ddot{\mathbf{u}}_g(t) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ddot{u}_{g1}(t)$$

Therefore, the equations of motion become

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\boldsymbol{\iota}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\ddot{u}_{g1}(t) = -\mathbf{m}\hat{\boldsymbol{\iota}}\,\ddot{u}_{g1}(t) \qquad \text{(b)}$$

where $\hat{\boldsymbol{\iota}} = \boldsymbol{\iota}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

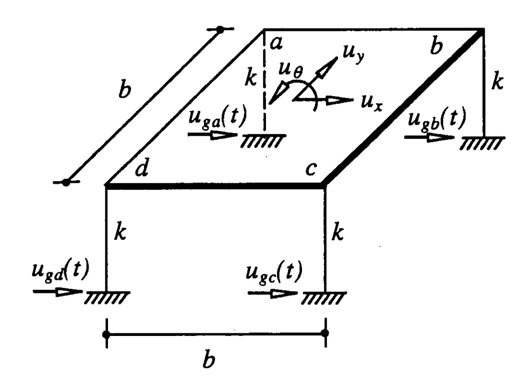
Note that this influence matrix can be interpreted as the displacement $\{u_1, u_2\}^T$ due to a simultaneous unit displacement of u_{g1} and u_{g2}.

Observe that

$$\mathbf{p}_{eff}(t) = -\begin{Bmatrix} m \\ 0 \end{Bmatrix}\ddot{u}_{g1}(t),$$

implying that the effective force in the vertical DOF is zero because the horizontal support motions are identical. When different, even horizontal support motions create effective force in the vertical DOF; see Eq. (a).

Problem 9.23



1. *Define the degrees of freedom.*

$$\left[\mathbf{u}^T \mid \mathbf{u}_g^T\right] = \left[u_x \;\; u_y \;\; u_\theta \mid u_{ga} \;\; u_{gb} \;\; u_{gc} \;\; u_{gd}\right]^T$$

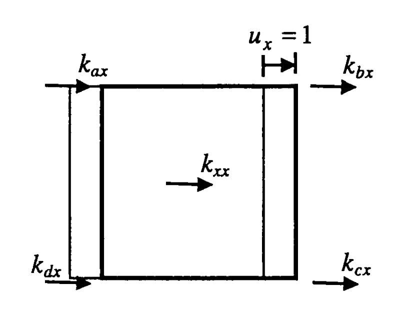
2. *Formulate mass matrix.*

From Problem 9.10

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & \dfrac{b^2}{6}\end{bmatrix}$$

2. *Formulate stiffness matrix.*

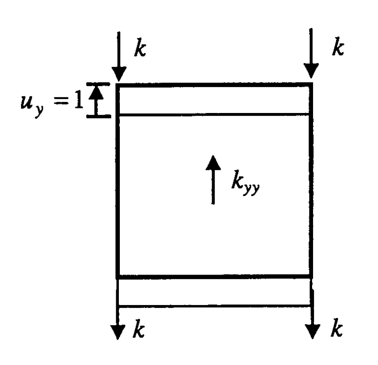
Apply $\left[\mathbf{u}^T \mid \mathbf{u}_g^T\right] = \left[1 \;\; 0 \;\; 0 \mid 0 \;\; 0 \;\; 0 \;\; 0\right]^T$



$k_{xx} = 4k$ $\quad k_{yx} = 0$ $\quad k_{\theta x} = 0$

$k_{ax} = -k$ $\quad k_{bx} = -k$ $\quad k_{cx} = -k$ $\quad k_{dx} = -k$

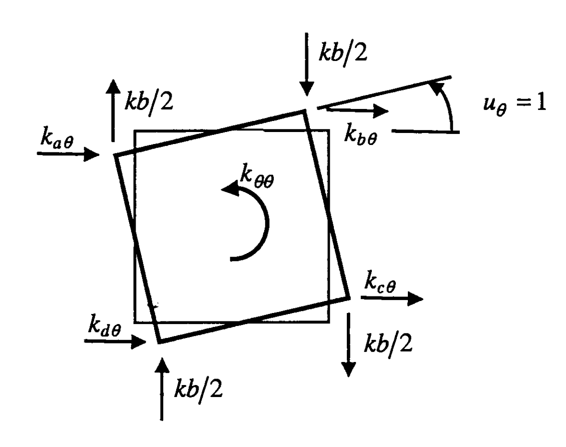
Apply $\left[\mathbf{u}^T \mid \mathbf{u}_g^T\right] = \left[0 \;\; 1 \;\; 0 \mid 0 \;\; 0 \;\; 0 \;\; 0\right]^T$



$k_{xy} = 0$ $\quad k_{yy} = 4k$ $\quad k_{\theta y} = 0$

$k_{ay} = 0$ $\quad k_{by} = 0$ $\quad k_{cy} = 0$ $\quad k_{dy} = 0$

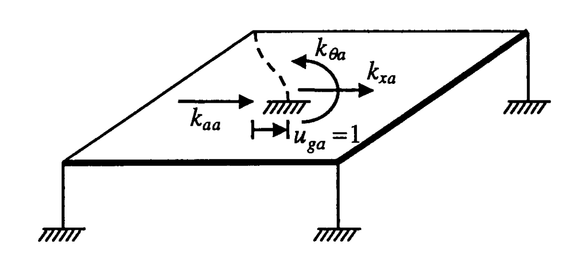
Apply $\left[\mathbf{u}^T \mid \mathbf{u}_g^T\right] = \left[0 \;\; 0 \;\; 1 \mid 0 \;\; 0 \;\; 0 \;\; 0\right]^T$



$k_{x\theta} = 0$ $\quad k_{y\theta} = 0$ $\quad k_{\theta\theta} = 2kb^2$

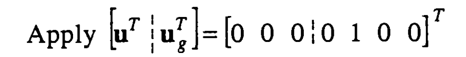
$k_{a\theta} = kb/2$ $\quad k_{b\theta} = kb/2$ $\quad k_{c\theta} = -kb/2$ $\quad k_{d\theta} = -kb/2$

Apply $\left[\mathbf{u}^T \mid \mathbf{u}_g^T\right] = \left[0 \;\; 0 \;\; 0 \mid 1 \;\; 0 \;\; 0 \;\; 0\right]^T$

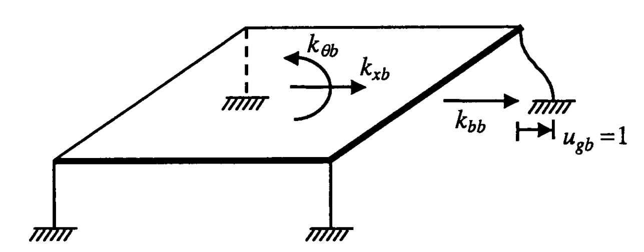


$k_{xa} = -k$ $\quad k_{ya} = 0$ $\quad k_{\theta a} = kb/2$

$k_{aa} = k$ $\quad k_{ba} = 0$ $\quad k_{ca} = 0$ $\quad k_{da} = 0$



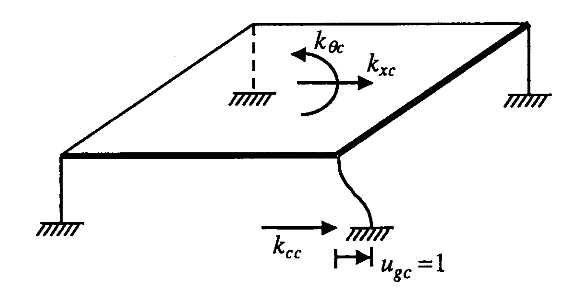
Apply $\left[\mathbf{u}^T \,\vdots\, \mathbf{u}_g^T\right] = \left[0 \;\; 0 \;\; 0 \,\vdots\, 0 \;\; 1 \;\; 0 \;\; 0\right]^T$



$k_{xb} = -k \qquad k_{yb} = 0 \qquad k_{\theta b} = kb/2$

$k_{ab} = 0 \qquad k_{bb} = k \qquad k_{cb} = 0 \qquad k_{db} = 0$

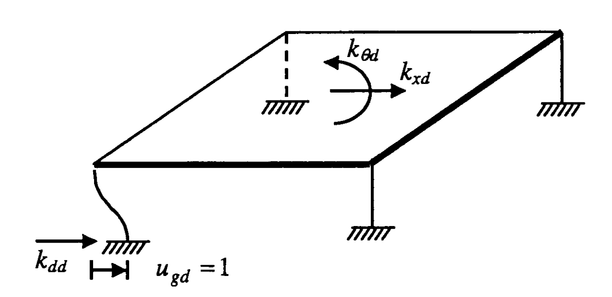
Apply $\left[\mathbf{u}^T \,\vdots\, \mathbf{u}_g^T\right] = \left[0 \;\; 0 \;\; 0 \,\vdots\, 0 \;\; 0 \;\; 1 \;\; 0\right]^T$



$k_{xc} = -k \qquad k_{yc} = 0 \qquad k_{\theta c} = -kb/2$

$k_{ac} = 0 \qquad k_{bc} = 0 \qquad k_{cc} = k \qquad k_{dc} = 0$

Apply $\left[\mathbf{u}^T \,\vdots\, \mathbf{u}_g^T\right] = \left[0 \;\; 0 \;\; 0 \,\vdots\, 0 \;\; 0 \;\; 0 \;\; 1\right]^T$



$k_{xd} = -k \qquad k_{yd} = 0 \qquad k_{\theta d} = -kb/2$

$k_{ad} = 0 \qquad k_{bd} = 0 \qquad k_{cd} = 0 \qquad k_{dd} = k$

Assemble the stiffness influence coefficients

$$\begin{bmatrix} \mathbf{k} & \mathbf{k}_g \\ \mathbf{k}_g^T & \mathbf{k}_{gg} \end{bmatrix} = k \left[\begin{array}{ccc|cccc} 4 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2b^2 & b/2 & b/2 & -b/2 & -b/2 \\ \hline -1 & 0 & b/2 & 1 & 0 & 0 & 0 \\ -1 & 0 & b/2 & 0 & 1 & 0 & 0 \\ -1 & 0 & -b/2 & 0 & 0 & 1 & 0 \\ -1 & 0 & -b/2 & 0 & 0 & 0 & 1 \end{array}\right]$$

4. *Determine the influence matrix.*

$$\boldsymbol{\iota} = -\mathbf{k}^{-1}\mathbf{k}_g = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1/b & -1/b & 1/b & 1/b \end{bmatrix} \qquad \text{(a)}$$

Note that each column of ι can be interpreted as the displacement $[u_x \;\; u_y \;\; u_\theta]^T$ due to a unit displacement of one of the supports. The first column corresponds to $u_{ga} = 1$, the second column to $u_{gb} = 1$, the third column to $u_{gc} = 1$ and the fourth column to $u_{gd} = 1$.

5. *Write the equations of motion.*

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\,\boldsymbol{\iota}\,\ddot{\mathbf{u}}_g(t)$$

where

$$\mathbf{u} = \begin{bmatrix} u_x & u_y & u_\theta \end{bmatrix}^T$$

and

$$\ddot{\mathbf{u}}_g(t) = \begin{bmatrix} \ddot{u}_{ga}(t) & \ddot{u}_{gb}(t) & \ddot{u}_{gc}(t) & \ddot{u}_{gd}(t) \end{bmatrix}^T$$

Observe that the equations of motion are uncoupled (matrices **k** and **m** are diagonal). Therefore, because the second row of the influence matrix, which corresponds to displacement u_y, consists of zeros, there is no response in the *y*-direction. However, although the ground motion is only applied in the *x*-direction, torsional motion will occur in general due to the non-zero terms in the third row of the influence matrix. See Eq. (a).

6. *For the case of identical ground motion at all supports.*

$$\ddot{u}_{ga}(t) = \ddot{u}_{gb}(t) = \ddot{u}_{gc}(t) = \ddot{u}_{gd}(t) = \ddot{u}_g($$

$$\text{i.e.,} \quad \ddot{\mathbf{u}}_g(t) = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \ddot{u}_g(t) \qquad \text{(e)}$$

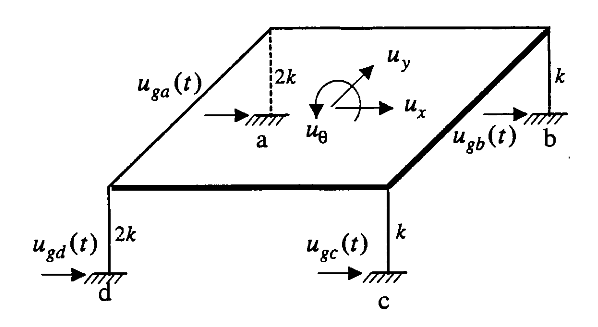
Therefore, the equations of motion become

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\boldsymbol{\iota}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}\ddot{u}_g(t) = -\mathbf{m}\hat{\boldsymbol{\iota}}\ddot{u}_g(t)$$

where $\hat{\boldsymbol{\iota}} = \boldsymbol{\iota}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$

Note that this result implies that, for the case of identical ground motions, the structure responds in the x-direction only.

Problem 9.24



Data (from Problem 9.10).

$m = 0.2331\,\text{kip-sec}^2/\text{in}$

$k = 1.5\,\text{kip/in}$

$b = 25\,\text{ft}$

$\zeta_n = 5\%, \qquad n = 1, 2 \text{ and } 3$

1. *Define DOFs.*

$$\mathbf{u} = \langle u_x \quad u_y \quad u_\theta \rangle^T$$

2. *Formulate the mass matrix.*

From Problem 9.14:

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & \dfrac{b^2}{6} \end{bmatrix} \tag{a}$$

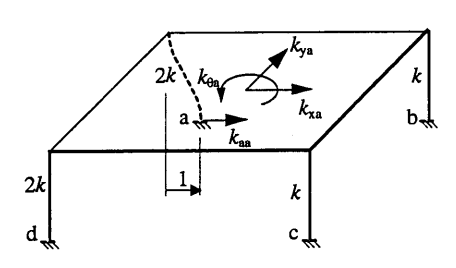
3. *Formulate the stiffness matrix.*

From Problem 9.14:

$$\mathbf{k} = k\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & -b \\ 0 & -b & 3b^2 \end{bmatrix} \tag{b}$$

The stiffness matrix associated with support degrees of freedom, $\mathbf{k}_{gg}$, and the coupling stiffness matrix between the support degrees of freedom and the structural degrees of freedom, $\mathbf{k}_g$, are computed by giving a unit displacement at a support degree of freedom, and computing the reactions at the support and structural degrees of freedom.

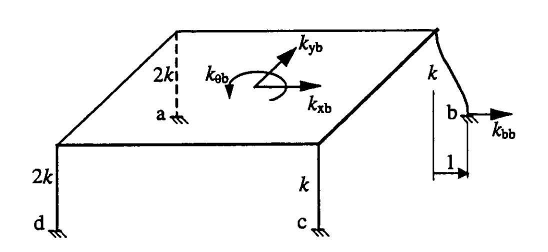
Apply $u_a = 1,\ u_b = 0,\ u_c = 0,\ u_d = 0$



$k_{aa} = 2k \qquad k_{ba} = 0 \qquad k_{ca} = 0 \qquad k_{da} = 0$

$k_{xa} = -2k \qquad k_{ya} = 0 \qquad k_{\theta a} = bk$

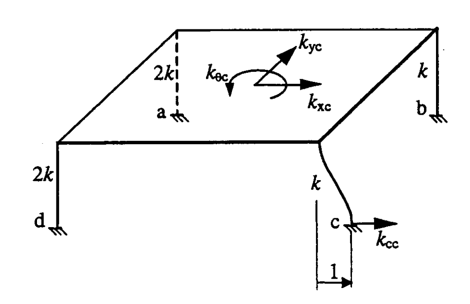
Apply $u_b = 1,\ u_a = 0,\ u_c = 0,\ u_d = 0$



$k_{ab} = 0 \qquad k_{bb} = k \qquad k_{cb} = 0 \qquad k_{db} = 0$

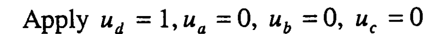
$k_{xb} = -k \qquad k_{yb} = 0 \qquad k_{\theta b} = 0.5bk$

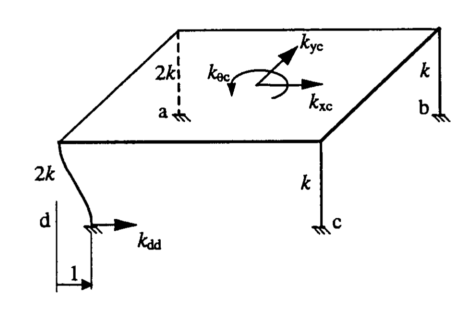
Apply $u_c = 1,\ u_a = 0,\ u_b = 0,\ u_d = 0$



$k_{ac} = 0 \qquad k_{bc} = 0 \qquad k_{cc} = k \qquad k_{dc} = 0$

$k_{xc} = -k \qquad k_{yc} = 0 \qquad k_{\theta c} = -0.5bk$


Apply $u_d = 1, u_a = 0,\ u_b = 0,\ u_c = 0$



$k_{ad} = 0 \quad k_{bd} = 0 \quad k_{cd} = 0 \quad k_{dd} = 2K$

$k_{xd} = -2k \quad k_{yd} = 0 \quad k_{\theta d} = -bk$

Assembling the stiffness influence coefficients gives $\mathbf{k}_{gg}$ and $\mathbf{k}_g$:

$$\mathbf{k}_{gg} = k\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \text{(c)}$$

$$\mathbf{k}_g = k\begin{bmatrix} -2 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ b & 0.5b & -0.5b & -b \end{bmatrix} \qquad \text{(d)}$$

4. *Determine the influence matrix.*

$$\iota = -\mathbf{k}^{-1}\mathbf{k}_g = \begin{bmatrix} 0.3333 & 0.1667 & 0.1667 & 0.3333 \\ -0.0588 & -0.0294 & 0.0294 & 0.0588 \\ -\dfrac{0.3529}{b} & -\dfrac{0.1765}{b} & \dfrac{0.1765}{b} & \dfrac{0.3529}{b} \end{bmatrix}$$

(e)

5. *Formulate the equations of motion.*

The dynamic components of the displacements are governed by

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}_{\text{eff}}(t) \qquad \text{(f)}$$

where $\mathbf{m}$ and $\mathbf{k}$ are defined by Eqs. (a) and (b), and

$$\mathbf{p}_{eff}(t) = -\sum_{\iota} \mathbf{m}\iota\ddot{u}_{g\iota}(t)$$
$$= \left[\mathbf{m}\iota_a\ddot{u}_{ga}(t) + \mathbf{m}\iota_b\ddot{u}_{gb}(t) + \mathbf{m}\iota_c\ddot{u}_{gc}(t) + \mathbf{m}\iota_d\ddot{u}_{gd}(t)\right]$$

(g)

where

$$\iota_a = \left\langle 0.3333 \quad -0.0588 \quad -\frac{0.3529}{b}\right\rangle^T$$

$$\iota_b = \left\langle 0.1667 \quad -0.0294 \quad -\frac{0.1765}{b}\right\rangle^T$$

$$\iota_c = \left\langle 0.1667 \quad 0.0294 \quad \frac{0.1765}{b}\right\rangle^T$$

$$\iota_d = \left\langle 0.3333 \quad -0.0588 \quad -\frac{0.3529}{b}\right\rangle^T$$

6. *Special case: Identical support motions.*

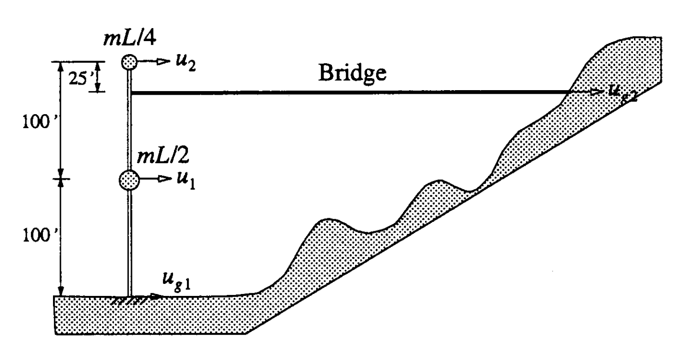
Equation (f) applies with

$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\iota\,\ddot{u}_g(t)$$

where

$$\iota = \langle 1 \ 0 \ 0\rangle^T$$

Problem 9.25



$$A = \frac{\pi}{4}\left[(25)^2 - (22.5)^2\right] = 13,430.31 \text{ in.}^2$$

$$w = 150 \text{ lb/ft}^3 \times \frac{13,430.31}{144} = 13,989.9 \text{ lb/ft}$$

$$m_2 = \frac{13.99 \times 200}{4}\frac{1}{386} = 1.812 \text{ kip}-\text{sec}^2/\text{in.}$$

$$m_1 = 2m_2 = 3.624 \text{ kip}-\text{sec}^2/\text{in.}$$

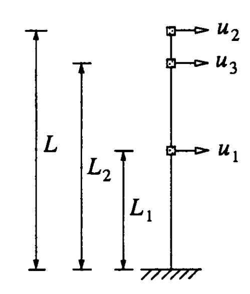
$$I = \frac{\pi(25/2)^4}{4} - \frac{\pi(22.5/2)^4}{4} = 6594.2 \text{ ft}^4$$

$$EI = 4.9225 \times 10^{11} \text{ kip}-\text{in.}^2$$

1. *Formulate the stiffness matrix.*

The stiffness matrix can be formulated by the direct stiffness method and condensing the rotational DOFs. Instead we use a different method.

We use the flexibility approach to determine $\hat{\mathbf{k}}$ for DOFs u_1, u_2 and u_3:



$$\hat{f}_{11} = \frac{L_1^3}{3EI}$$

$$\hat{f}_{21} = \frac{L_1^2}{2EI}\left(L - \frac{L_1}{3}\right)$$

$$\hat{f}_{31} = \frac{L_1^2}{2EI}\left(L_2 - \frac{L_1}{3}\right)$$

$$\hat{f}_{22} = \frac{L^3}{3EI}$$

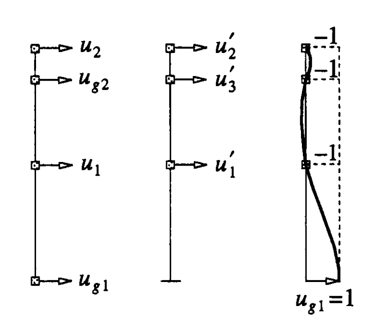
$$\hat{f}_{33} = \frac{L_2^3}{3EI}$$

$$\hat{f}_{23} = \frac{L_2^2}{2EI}\left(L - \frac{L_2}{3}\right)$$

$$\hat{\mathbf{f}} = \begin{bmatrix} \dfrac{L_1^3}{3EI} & \dfrac{L_1^2(L-L_1/3)}{2EI} & \dfrac{L_1^2(L_2-L_1/3)}{2EI} \\ & \dfrac{L^3}{3EI} & \dfrac{L_2^2(L-L_2/3)}{2EI} \\ (sym) & & \dfrac{L_2^3}{3EI} \end{bmatrix}$$

$$\hat{\mathbf{k}} = \hat{\mathbf{f}}^{-1} = \begin{bmatrix} 0.9359 & 0.7701 & -1.3063 \\ & 1.5088 & -2.1375 \\ (sym) & & 3.1294 \end{bmatrix} \times 10^4$$

Define the following transformation:



$$\begin{Bmatrix} u_1' \\ u_2' \\ u_3' \end{Bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}}^{\mathbf{a}} \begin{Bmatrix} u_1 \\ u_2 \\ u_{g1} \\ u_{g2} \end{Bmatrix}$$

The stiffness matrix in DOFs u_1, u_2, u_{g1} and u_{g2} is

$$\bar{\mathbf{k}} = \mathbf{a}^T \hat{\mathbf{k}} \mathbf{a} = \begin{bmatrix} \mathbf{k} & \mathbf{k}_g \\ \mathbf{k}_g^T & \mathbf{k}_{gg} \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} 0.9359 & 0.7701 & -0.3998 & -1.3063 \\ & 1.5088 & -0.1415 & -2.1375 \\ \hline & & 0.2269 & 0.3143 \\ (sym) & & & 3.1294 \end{array}\right] \times 10^4 \text{ kips/in.}$$

2. *Formulate the equations of motion.*

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\boldsymbol{\iota}\ddot{\mathbf{u}}_g(t)$$

where

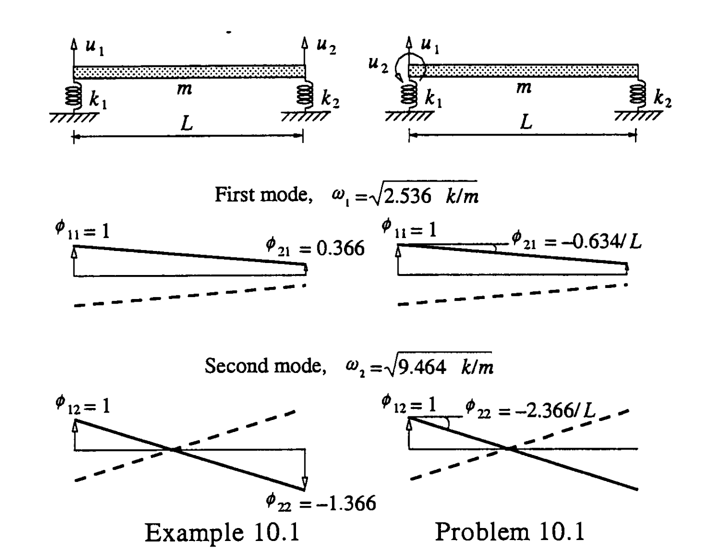
$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \mathbf{u}_g = \begin{Bmatrix} u_{g1} \\ u_{g2} \end{Bmatrix}$$

$$\mathbf{m} = \begin{bmatrix} 3.624 & \\ & 1.812 \end{bmatrix}$$

$$\mathbf{k} = \begin{bmatrix} 0.9359 & 0.7701 \\ 0.7701 & 1.5088 \end{bmatrix} \times 10^4$$

$$\boldsymbol{\iota} = -\mathbf{k}^{-1}\mathbf{k}_g = \begin{bmatrix} 0.6035 & 0.3965 \\ -0.2143 & 1.2143 \end{bmatrix}$$

Problem 10.1



The mass and stiffness matrices were determined in Problem 9.1 with reference to the indicated DOF:

$$\mathbf{m} = \begin{bmatrix} m & mL/2 \\ mL/2 & mL^2/3 \end{bmatrix}$$

$$\mathbf{k} = \begin{bmatrix} k_1 + k_2 & k_2 L \\ k_2 L & k_2 L^2 \end{bmatrix} = \begin{bmatrix} 3k & 2kL \\ 2kL & 2kL^2 \end{bmatrix}$$

Then

$$\mathbf{k} - \omega^2 \mathbf{m} = k \begin{bmatrix} 3-\lambda & 2L - \lambda \dfrac{L}{2} \\ 2L - \lambda \dfrac{L}{2} & 2L^2 - \lambda \dfrac{L^2}{2} \end{bmatrix} \qquad \text{(a)}$$

where

$$\lambda = \omega^2 \frac{m}{k} \qquad \text{(b)}$$

Substituting Eq. (a) in Eq. (10.2.6) gives the frequency equation:

$$\lambda^2 - 12\lambda + 24 = 0$$

which has two solutions $\lambda = (6 - 2\sqrt{3}) = 2.536$ and $\lambda = (6 + 2\sqrt{3}) = 9.464$. The corresponding natural frequencies are

$$\omega_1 = \sqrt{2.536 \frac{k}{m}} \qquad \omega_2 = \sqrt{9.464 \frac{k}{m}} \qquad \text{(c)}$$

The natural modes are determined from Eq. (10.2.5) following the procedure shown in Example 10.1 to obtain

$$\phi_1 = \begin{Bmatrix} 1 \\ -0.634/L \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} 1 \\ -2.366/L \end{Bmatrix} \qquad \text{(d)}$$

The natural frequencies are the same as those obtained in Example 10.1, and the mode shapes are equivalent to those obtained in Example 10.1. The mode shapes in Example 10.1 can be obtained from the mode shapes obtained in this problem, using the transformation matrix

$$\mathbf{a} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}$$

that relates the displacements in the two sets of DOFs:

$$\mathbf{\Phi}_{E10.1} = \mathbf{a}\mathbf{\Phi}_{P10.1}$$

Problem 10.2

***Part a**: Determine natural frequencies and modes.*

The mass and stiffness matrices were determined in Problem 9.2:

$$\mathbf{m} = \frac{mL}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{k} = \frac{162EI}{5L^3}\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix}$$

Then

$$\mathbf{k} - \omega^2\mathbf{m} = \frac{162EI}{5L^3}\begin{bmatrix} 8-\lambda & -7 \\ -7 & 8-\lambda \end{bmatrix} \qquad \text{(a)}$$

where

$$\lambda = \frac{5mL^4}{486EI}\omega^2 \qquad \text{(b)}$$

Substituting Eq. (a) in Eq. (10.2.6) gives the frequency equation:

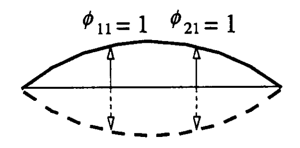
$$\lambda^2 - 16\lambda + 15 = (\lambda - 1)(\lambda - 15) = 0$$

The solution of the frequency equation gives: $\lambda_1 = 1$ and $\lambda_2 = 15$. The corresponding natural frequencies are

$$\omega_1 = 9.859\sqrt{\frac{EI}{mL^4}} \qquad \omega_2 = 38.184\sqrt{\frac{EI}{mL^4}} \qquad \text{(c)}$$

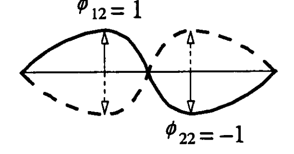
The natural modes are determined from Eq. (10.2.5) following the procedure shown in Example 10.1 to obtain:

$$\phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \qquad \text{(d)}$$



$\omega_1 = 9.859\sqrt{EI/mL^4}$

First mode
(symmetric)



$\omega_2 = 38.184\sqrt{EI/mL^4}$

Second mode
(antisymmetric)

***Part b**: Verify orthogonality properties.*

$$\phi_1^T\mathbf{m}\phi_2 = \frac{mL}{3}\langle 1 \quad 1\rangle\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 0$$

$$\phi_1^T\mathbf{k}\phi_2 = \frac{162EI}{5L^3}\langle 1 \quad 1\rangle\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 0$$

Thus the natural modes satisfy the orthogonality properties.

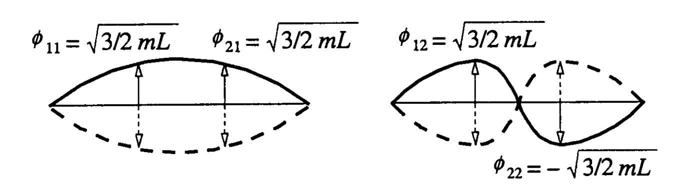
***Part c**: Normalize modes so that $M_n = 1$.*

$$M_1 = \phi_1^T\mathbf{m}\phi_1 = \frac{mL}{3}\langle 1 \quad 1\rangle\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{2mL}{3}$$

$$M_2 = \phi_2^T\mathbf{m}\phi_2 = \frac{mL}{3}\langle 1 \quad -1\rangle\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \frac{2mL}{3}$$

Divide ϕ_1 of Eq. (d) by $\sqrt{2mL/3}$ and ϕ_2 of Eq. (d) by $\sqrt{2mL/3}$ to obtain normalized modes:

$$\bar{\phi}_1 = \sqrt{\frac{3}{2mL}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \bar{\phi}_2 = \sqrt{\frac{3}{2mL}}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \qquad \text{(e)}$$



First mode

Second mode

The modes of Eq. (e) are scalar multiples of the modes in Eq. (d); the shapes of the two sets of modes are the same.

Problem 10.3

The free vibration response of the system without damping is computed using Eq. (10.8.6),

$$\mathbf{u}(t)=\sum_{n=1}^{2}\phi_n\left[q_n(0)\cos\omega_n t+\frac{\dot{q}_n(0)}{\omega_n}\sin\omega_n t\right] \qquad \text{(a)}$$

where according to Eq. (10.8.5)

$$q_n(0)=\frac{\phi_n^T\mathbf{m}\mathbf{u}(0)}{\phi_n^T\mathbf{m}\phi_n} \qquad \dot{q}_n(0)=\frac{\phi_n^T\mathbf{m}\dot{\mathbf{u}}(0)}{\phi_n^T\mathbf{m}\phi_n} \qquad \text{(b)}$$

Using $\mathbf{m}$ and $\mathbf{k}$ from Problem 10.2, the values of $q_1(0)$ and $q_2(0)$ for the initial displacements $\mathbf{u}(0)$ are

$$q_1(0)=\frac{\frac{mL}{3}\langle 1 \;\; 1\rangle\begin{bmatrix}1 & \\ & 1\end{bmatrix}\mathbf{u}(0)}{\frac{mL}{3}\langle 1 \;\; 1\rangle\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}}=\langle 0.5 \;\; 0.5\rangle\mathbf{u}(0) \qquad \text{(c)}$$

$$q_2(0)=\frac{\frac{mL}{3}\langle 1 \;\; -1\rangle\begin{bmatrix}1 & \\ & 1\end{bmatrix}\mathbf{u}(0)}{\frac{mL}{3}\langle 1 \;\; -1\rangle\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{Bmatrix}1\\-1\end{Bmatrix}}=\langle 0.5 \;\; -0.5\rangle\mathbf{u}(0) \qquad \text{(d)}$$

Similarly,

$$\dot{q}_1(0) = \langle 0.5 \quad 0.5\rangle\,\dot{\mathbf{u}}(0)$$
$$\dot{q}_2(0) = \langle 0.5 \quad -0.5\rangle\,\dot{\mathbf{u}}(0)$$

(a) For $u_1(0) = 1$ and $u_2(0) = 0$,

$$\mathbf{u}(0) = \begin{Bmatrix}1\\0\end{Bmatrix} \quad \dot{\mathbf{u}}(0) = \begin{Bmatrix}0\\0\end{Bmatrix}$$

Substituting in Eqs. (c)-(d) gives

$$q_1(0) = 0.5 \qquad q_2(0) = 0.5 \qquad \dot{q}_1(0) = 0 \qquad \dot{q}_2(0) = 0 \qquad \text{(e)}$$

Substituting Eq. (e) in Eq. (a) and using the modes from Problem 10.2 gives

$$\begin{Bmatrix}u_1(t)\\u_2(t)\end{Bmatrix}=\begin{Bmatrix}0.5\cos\left(9.859\sqrt{\frac{EI}{mL^4}}t\right)+0.5\cos\left(38.184\sqrt{\frac{EI}{mL^4}}t\right)\\0.5\cos\left(9.859\sqrt{\frac{EI}{mL^4}}t\right)-0.5\cos\left(38.184\sqrt{\frac{EI}{mL^4}}t\right)\end{Bmatrix} \qquad \text{(f)}$$

(b) For $u_1(0) = 1$ and $u_2(0) = 1$,

$$\mathbf{u}(0)=\begin{Bmatrix}1\\1\end{Bmatrix} \quad \dot{\mathbf{u}}(0)=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Substituting in Eqs. (c)-(d) gives

$$q_1(0) = 1 \quad q_2(0) = 0 \quad \dot{q}_1(0) = 0 \quad \dot{q}_2(0) = 0 \qquad \text{(g)}$$

Substituting Eq. (g) in Eq. (a) and using the modes from Problem 10.2 gives

$$\begin{Bmatrix}u_1(t)\\u_2(t)\end{Bmatrix}=\begin{Bmatrix}\cos\left(9.859\sqrt{\frac{EI}{mL^4}}t\right)\\\cos\left(9.859\sqrt{\frac{EI}{mL^4}}t\right)\end{Bmatrix} \qquad \text{(h)}$$

(c) For $u_1(0) = 1$ and $u_2(0) = -1$,

$$\mathbf{u}(0)=\begin{Bmatrix}1\\-1\end{Bmatrix} \quad \dot{\mathbf{u}}(0)=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Substituting in Eqs. (c)-(d) gives

$$q_1(0) = 0 \qquad q_2(0) = 1 \qquad \dot{q}_1(0) = 0 \qquad \dot{q}_2(0) = 0 \qquad \text{(i)}$$

Substituting Eq. (i) in Eq. (a) and using the modes from Problem 10.2 gives

$$\begin{Bmatrix}u_1(t)\\u_2(t)\end{Bmatrix}=\begin{Bmatrix}\cos\left(38.184\sqrt{\frac{EI}{mL^4}}t\right)\\-\cos\left(38.184\sqrt{\frac{EI}{mL^4}}t\right)\end{Bmatrix} \qquad \text{(j)}$$

In case (a) both modes contribute to the response because the initial displacement condition has components in both modes. In case (b) the initial displacement is proportional to the first mode and therefore only this mode is excited and contributes to the response. In case (c) the initial displacement is proportional to the second mode and therefore only this mode contributes to the response.

Problem 10.4

The free vibration response of the system of Problem 9.2 including damping is computed using Eq. (10.10.4):

$$\mathbf{u}(t)=\sum_{n=1}^{2}\phi_n e^{-\zeta_n\omega_n t}\left[q_n(0)\cos\omega_{nD}t + \frac{\dot{q}_n(0)+\zeta_n\omega_n q_n(0)}{\omega_{nD}}\sin\omega_{nD}t\right] \quad \text{(a)}$$

where $\omega_{nD} = \omega_n\sqrt{1 - \zeta^2}$; ω_n are given by Eq. (c) of Problem 10.2 and $\zeta_n = 0.05$. Therefore,

$$\omega_{1D} = 9.847\sqrt{\frac{EI}{mL^4}} \qquad \omega_{2D} = 38.136\sqrt{\frac{EI}{mL^4}} \quad \text{(b)}$$

For the given initial conditions $u_1(0) = 1$ and $u_2(0) = 0$, Eq. (e) of Problem 10.3 gives

$$q_1(0) = 0.5 \qquad q_2(0) = 0.5 \qquad \dot{q}_1(0) = 0 \qquad \dot{q}_2(0) = 0 \quad \text{(c)}$$

Substituting Eqs. (b) and (c) in Eq. (a) gives

$$\mathbf{u}(t) = \mathbf{u}_1(t) + \mathbf{u}_2(t) \quad \text{(d)}$$

where the modal contributions are

$$\mathbf{u}_1(t)=e^{-0.493t}\left(0.5\cos\omega_{1D}t+0.025\sin\omega_{1D}t\right)\begin{Bmatrix}1\\1\end{Bmatrix} \quad \text{(e)}$$

$$\mathbf{u}_2(t)=e^{-1.909t}\left(0.5\cos\omega_{2D}t+0.025\sin\omega_{2D}t\right)\begin{Bmatrix}1\\-1\end{Bmatrix} \quad \text{(f)}$$

Problem 10.5

Part a

From Problem 9.4, the mass and stiffness matrices are.

$$\mathbf{m}=\frac{m}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$\mathbf{k}=\frac{2EI}{L^3}\begin{bmatrix}14 & -5\\ -5 & 2\end{bmatrix}$$

Thus

$$\mathbf{k}-\omega^2\mathbf{m}=\frac{2EI}{L^3}\begin{bmatrix}2-2\lambda & -5-\lambda\\ -5-\lambda & 14-2\lambda\end{bmatrix} \qquad \text{(a)}$$

where

$$\lambda = \frac{mL^3}{12EI}\omega^2 \qquad \text{(b)}$$

Substituting Eq. (a) in Eq. (10.2.6) gives the frequency equation:

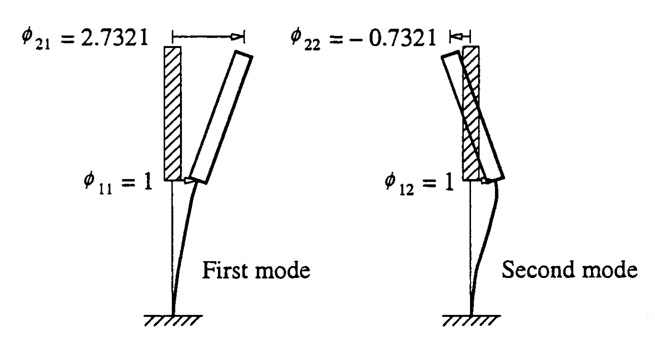
$$\lambda^2 - 14\lambda + 1 = 0$$

The solution of this equation gives: $\lambda_1 = 0.0718$ and $\lambda_2 = 13.928$. The corresponding natural frequencies are

$$\omega_1 = 0.928\sqrt{\frac{EI}{mL^3}} \qquad \omega_2 = 12.928\sqrt{\frac{EI}{mL^3}} \qquad \text{(c)}$$

Following the procedure used in Example 10.1, the natural modes are determined from Eq. (10.2.5):

$$\phi_1=\begin{Bmatrix}1\\ 2.7321\end{Bmatrix} \qquad \phi_2=\begin{Bmatrix}1\\ -0.7321\end{Bmatrix} \qquad \text{(d)}$$



Part b

The free vibration response of the system is computed using Eq. (10.8.6):

$$\mathbf{u}(t)=\sum_{n=1}^{2}\phi_n\left[q_n(0)\cos\omega_n t+\frac{\dot{q}_n(0)}{\omega_n}\sin\omega_n t\right] \qquad \text{(e)}$$

where

$$q_n(0) = \frac{\phi_n^T\mathbf{m}\mathbf{u}(0)}{\phi_n^T\mathbf{m}\phi_n} \qquad \dot{q}_n(0) = \frac{\phi_n^T\mathbf{m}\dot{\mathbf{u}}(0)}{\phi_n^T\mathbf{m}\phi_n} \qquad \text{(f)}$$

Substituting $\dot{\mathbf{u}}(0) = \langle 0 \quad 1\rangle^T$ and $\mathbf{u}(0) = \langle 0 \quad 0\rangle^T$ gives:

$$\dot{q}_1(0)=\frac{\frac{m}{6}\langle 1 \quad 2.7321\rangle\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{Bmatrix}0\\ 1\end{Bmatrix}}{\frac{m}{6}\langle 1 \quad 2.7321\rangle\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{Bmatrix}1\\ 2.7321\end{Bmatrix}}=0.2882 \qquad \text{(g.1)}$$

$$\dot{q}_2(0)=\frac{\frac{m}{6}\langle 1 \quad -0.7321\rangle\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{Bmatrix}0\\ 1\end{Bmatrix}}{\frac{m}{6}\langle 1 \quad -0.7321\rangle\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{Bmatrix}1\\ -0.7321\end{Bmatrix}}=-0.2882 \qquad \text{(g.2)}$$

$$q_1(0) = 0 \qquad q_2(0) = 0 \qquad \text{(h)}$$

Substituting Eqs. (g), (h) and (d) in Eq. (e) gives the displacements:

$$\begin{Bmatrix}u_1(t)\\ u_2(t)\end{Bmatrix}=\sqrt{\frac{mL^3}{EI}}\begin{Bmatrix}0.3110\sin\omega_1 t-0.0223\sin\omega_2 t\\ 0.8947\sin\omega_1 t+0.0164\sin\omega_2 t\end{Bmatrix}$$

Problem 10.6

From Problem 9.5, the mass and stiffness matrices are

$$\mathbf{m} = m\begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix}$$

$$\mathbf{k} = k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

where

$$k \equiv \frac{24EI}{h^3}$$

***Part a*:** Determine the natural frequencies and modes.

$$\det\left[\mathbf{k} - \omega_n^2\mathbf{m}\right] = 0 \Rightarrow$$

$$\left(\frac{m^2}{2}\right)\omega_n^4 - 2k \quad m \quad \omega_n^2 + k^2 = 0 \Rightarrow$$

$$\omega_n^2 = \frac{k}{m}(2 \pm \sqrt{2}) \Rightarrow$$

$$\omega_1 = 0.765\sqrt{\frac{k}{m}} \qquad \omega_2 = 1.848\sqrt{\frac{k}{m}}$$

Substituting for k gives

$$\omega_1 = 3.750\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 9.052\sqrt{\frac{EI}{mh^3}}$$

First mode:

$$\left[\mathbf{k} - \omega_1^2\mathbf{m}\right]\phi_1 = \mathbf{0}$$

$$k\begin{bmatrix} \sqrt{2} & -1 \\ -1 & 1/\sqrt{2} \end{bmatrix}\begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Select $\phi_{11} = 1 \Rightarrow \phi_{21} = \sqrt{2}$

$$\phi_1 = \begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix}$$

Second mode:

$$\left[\mathbf{k} - \omega_2^2\mathbf{m}\right]\phi_2 = \mathbf{0}$$

$$k\begin{bmatrix} -\sqrt{2} & -1 \\ -1 & -1/\sqrt{2} \end{bmatrix}\begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Select $\phi_{12} = 1 \Rightarrow \phi_{22} = -\sqrt{2}$

$$\phi_2 = \begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix}$$

Part b: Verify orthogonality.

$$\phi_1^T\mathbf{m}\phi_2 = \langle 1 \;\; \sqrt{2} \rangle\, m\begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix}\begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix}$$

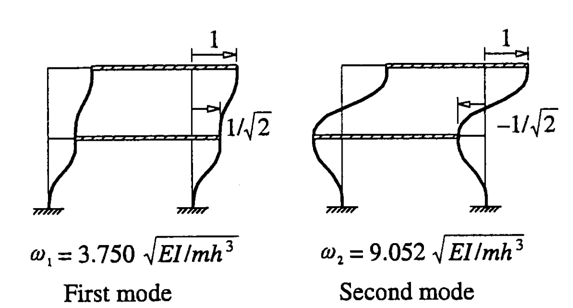
$$= m\langle 1 \;\; \sqrt{2} \rangle\begin{Bmatrix} 1 \\ -1/\sqrt{2} \end{Bmatrix} = 0$$

$$\phi_1^T\mathbf{k}\phi_2 = \langle 1 \;\; \sqrt{2} \rangle\, k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix}$$

$$= k\langle 1 \;\; \sqrt{2} \rangle\begin{Bmatrix} 2+\sqrt{2} \\ -1-\sqrt{2} \end{Bmatrix} = 0$$

***Part c*:** Normalize modes to unit value at roof.

$$\phi_1 = \begin{Bmatrix} 1/\sqrt{2} \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1/\sqrt{2} \\ 1 \end{Bmatrix}$$



$\omega_1 = 3.750\sqrt{EI/mh^3}$ First mode

$\omega_2 = 9.052\sqrt{EI/mh^3}$ Second mode

***Part d*:** Normalize modes so that $M_n = 1$.

$$M_1 = \phi_1^T\mathbf{m}\phi_1 = \langle 1 \;\; \sqrt{2} \rangle\, m\begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix}\begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix} = 2m$$

$$M_2 = \phi_2^T\mathbf{m}\phi_2 = \langle 1 \;\; -\sqrt{2} \rangle\, m\begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix}\begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix} = 2m$$

Divide ϕ_1 from part (a) by $\sqrt{2m}$ and ϕ_2 from part (a) by $\sqrt{2m}$ to obtain the normalized modes:

$$\phi_1 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 1/\sqrt{2} \\ 1 \end{Bmatrix} \quad \phi_2 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 1/\sqrt{2} \\ -1 \end{Bmatrix}$$

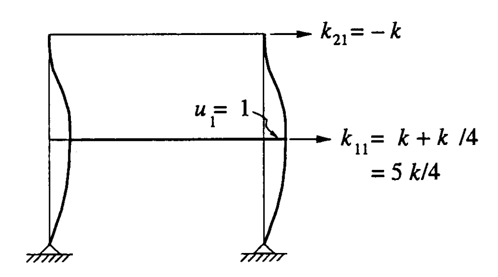
These modes differ from these obtained in Part (c), only by a scale factor; the shapes of the two sets of modes are the same.

Problem 10.7

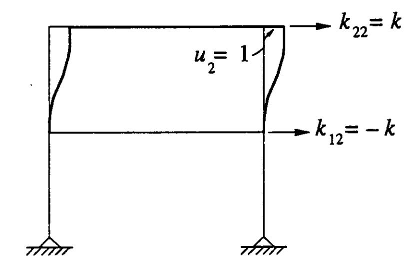
1. *Determine the stiffness matrix.*

Define $k \equiv \dfrac{24EI}{h^3}$

Apply $u_1 = 1$ and $u_2 = 0$ and determine k_{i1}:



Apply $u_2 = 1$ and $u_1 = 0$ and determine k_{i2}:



Thus

$$\mathbf{k} = k\begin{bmatrix} 5/4 & -1 \\ -1 & 1 \end{bmatrix}$$

2. *Determine the mass matrix.*

$$\mathbf{m} = m\begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix}$$

3. *Determine natural frequencies and modes.*

$$\det\left[\mathbf{k} - \omega_n^2\mathbf{m}\right] = 0$$

$$(0.5m^2)\,\omega_n^4 - (1.625km)\,\omega_n^2 + 0.25k^2 = 0$$

$$\omega_1 = 1.971\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 8.609\sqrt{\frac{EI}{mh^3}}$$

Determine first mode:

$$\left[\mathbf{k} - \omega_1^2\mathbf{m}\right]\phi_1 = \mathbf{0}$$

$$k\begin{bmatrix} 1.088 & -1 \\ -1 & 0.919 \end{bmatrix}\begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Select $\phi_{21} = 1 \Rightarrow \phi_{11} = 0.919$

$$\phi_1 = \begin{Bmatrix} 0.919 \\ 1 \end{Bmatrix}$$

Determine second mode:

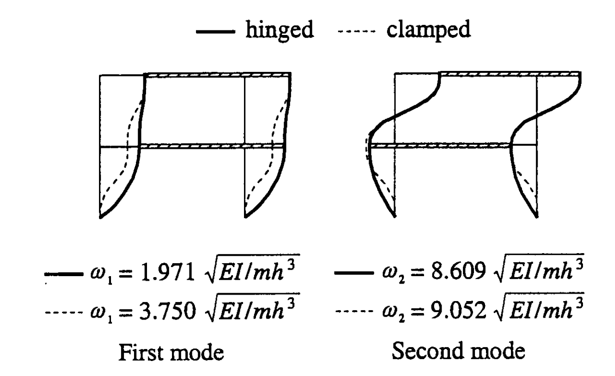
$$\left[\mathbf{k} - \omega_2^2\mathbf{m}\right]\phi_2 = \mathbf{0}$$

$$k\begin{bmatrix} -1.837 & -1 \\ -1 & -0.544 \end{bmatrix}\begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Select $\phi_{22} = 1 \Rightarrow \phi_{12} = -0.544$

$$\phi_2 = \begin{Bmatrix} -0.544 \\ 1 \end{Bmatrix}$$

4. *Compare vibration properties for hinged and clamped cases.*



The system with columns hinged at the base is more flexible, and thus has lower frequencies; the first frequency is reduced by a factor of almost two whereas the second frequency is affected much less.

Problem 10.8

From Problem 10.7,

$$\mathbf{m} = m\begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix} \qquad \mathbf{k} = k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\phi_1 = \begin{Bmatrix} 1/\sqrt{2} \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1/\sqrt{2} \\ 1 \end{Bmatrix}$$

$$\omega_1 = 3.750\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 9.052\sqrt{\frac{EI}{mh^3}}$$

Case a

$$\mathbf{u}(0) = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \qquad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$q_1(0) = \frac{\phi_1^T \mathbf{m} \mathbf{u}(0)}{\phi_1^T \mathbf{m} \phi_1} = 1.707 \qquad \dot{q}_1(0) = 0$$

$$q_2(0) = \frac{\phi_2^T \mathbf{m} \mathbf{u}(0)}{\phi_2^T \mathbf{m} \phi_2} = 0.293 \qquad \dot{q}_2(0) = 0$$

Substituting in Eq. (10.8.8) gives

$$q_1(t) = 1.707 \cos \omega_1 t \qquad q_2(t) = 0.293 \cos \omega_2 t$$

Substituting $q_n(t)$ and ϕ_n in Eq. (10.8.7) gives

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{2} \\ 1 \end{Bmatrix} 1.707 \cos \omega_1 t + \begin{Bmatrix} -1/\sqrt{2} \\ 1 \end{Bmatrix} 0.293 \cos \omega_2 t$$

$$= \begin{Bmatrix} 1.207 \\ 1.707 \end{Bmatrix} \cos \omega_1 t + \begin{Bmatrix} -0.207 \\ 0.293 \end{Bmatrix} \cos \omega_2 t$$

The first mode contributes more to the response than the second mode because the first mode has the more significant component in the specified initial conditions.

Case b

$$\mathbf{u}(0) = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \qquad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Following the procedure of part (a) we obtain

$$q_1(0) = -0.207 \qquad \dot{q}_1(0) = 0$$

$$q_2(0) = 1.207 \qquad \dot{q}_2(0) = 0$$

$$q_1(t) = -0.207 \cos \omega_1 t \quad q_2(t) = 1.207 \cos \omega_2 t$$

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 1/\sqrt{2} \\ 1 \end{Bmatrix} (-0.207) \cos \omega_1 t + \begin{Bmatrix} -1/\sqrt{2} \\ 1 \end{Bmatrix} 1.207 \cos \omega_2 t$$

$$= \begin{Bmatrix} -0.146 \\ -0.207 \end{Bmatrix} \cos \omega_1 t + \begin{Bmatrix} -0.853 \\ 1.207 \end{Bmatrix} \cos \omega_2 t$$

The second mode contributes more to the response than the first mode because the second mode has the more significant component in the specified initial conditions.

Problem 10.9

From Problem 10.8

$$q_1(0) = 1.707 \qquad q_2(0) = 0.293$$

$$\dot{q}_1(0) = 0 \qquad \dot{q}_2(0) = 0$$

Substituting $q_n(0)$ and $\dot{q}_n(0)$ in Eq. (10.10.2) gives

$$q_1(t) = e^{-\zeta_1\omega_1 t}\left(1.707\cos\omega_{1D}t + \frac{1.707\zeta_1}{\sqrt{1-\zeta_1^2}}\sin\omega_{1D}t\right)$$

$$= 1.707\, e^{-0.05\omega_1 t}\left(\cos 0.999\omega_1 t + 0.05\sin 0.999\omega_1 t\right) \qquad \text{(a)}$$

$$q_2(t) = 0.293\, e^{-0.05\omega_2 t}\left(\cos 0.999\omega_2 t + 0.05\sin 0.999\omega_2 t\right) \qquad \text{(b)}$$

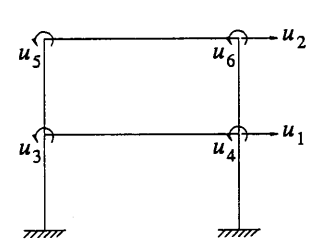
Substituting Eqs. (a) and (b) in Eq. (10.8.7) gives

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 1.207 \\ 1.707 \end{Bmatrix} e^{-0.05\omega_1 t}\left(\cos 0.999\omega_1 t + 0.05\sin 0.999\omega_1 t\right) + \begin{Bmatrix} -0.207 \\ 0.293 \end{Bmatrix} e^{-0.05\omega_2 t}\left(\cos 0.999\omega_2 t + 0.05\sin 0.999\omega_2 t\right)$$

where

$$\omega_1 = 3.750\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 9.052\sqrt{\frac{EI}{mh^3}}$$

Problem 10.10



With reference to the lateral floor displacements u_1 and u_2, the mass matrix and the condensed stiffness matrix (from Problem 9.6) are

$$\mathbf{m}_{tt} = m\begin{bmatrix} 1 & \\ & 0.5 \end{bmatrix} \qquad \hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 37.15 & -15.12 \\ -15.12 & 10.19 \end{bmatrix}$$

1. *Determine natural frequencies.*

$$\det\left[\hat{\mathbf{k}}_{tt} - \omega_n^2 \mathbf{m}_{tt}\right] = 0$$

$$\omega_1 = 2.407\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 7.193\sqrt{\frac{EI}{mh^3}}$$

2. *Determine first mode.*

$$\left[\hat{\mathbf{k}}_{tt} - \omega_1^2 \mathbf{m}_{tt}\right]\boldsymbol{\phi}_1 = \mathbf{0}$$

$$\frac{EI}{h^3}\begin{bmatrix} 31.36 & -15.12 \\ -15.12 & 7.295 \end{bmatrix}\begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} 0.482 \\ 1 \end{Bmatrix}$$

3. *Determine second mode.*

$$\left[\hat{\mathbf{k}}_{tt} - \omega_2^2 \mathbf{m}_{tt}\right]\boldsymbol{\phi}_2 = \mathbf{0}$$

$$\frac{EI}{h^3}\begin{bmatrix} -14.59 & -15.12 \\ -15.12 & -15.68 \end{bmatrix}\begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\boldsymbol{\phi}_2 = \begin{Bmatrix} -1.037 \\ 1 \end{Bmatrix}$$

4. *Determine joint rotations.*

The joint rotations corresponding to the lateral displacements $\boldsymbol{\phi}_n$ are computed using Eq. (9.3.3):

$$\mathbf{u}_0 = \mathbf{T}\,\mathbf{u}_t \qquad \text{(a)}$$

where

$$\mathbf{T} = -\mathbf{k}_{00}^{-1}\,\mathbf{k}_{0t} \qquad \text{(b)}$$

Substituting $\mathbf{k}_{00}$ and $\mathbf{k}_{0t}$ from Problem 9.6 in Eq. (b) gives

$$\mathbf{T} = \frac{1}{h}\begin{bmatrix} -0.164 & -0.411 \\ -0.164 & -0.411 \\ 0.904 & -0.740 \\ 0.904 & -0.740 \end{bmatrix}$$

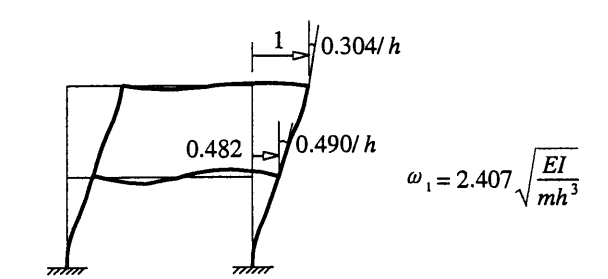
The joint rotations associated with the first mode are obtained by substituting $\mathbf{u}_t = \boldsymbol{\phi}_1$ in Eq. (a):

$$\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{h}\begin{bmatrix} -0.164 & -0.411 \\ -0.164 & -0.411 \\ 0.904 & -0.740 \\ 0.904 & -0.740 \end{bmatrix}\begin{Bmatrix} 0.482 \\ 1 \end{Bmatrix} = \frac{1}{h}\begin{Bmatrix} -0.490 \\ -0.490 \\ -0.304 \\ -0.304 \end{Bmatrix}$$

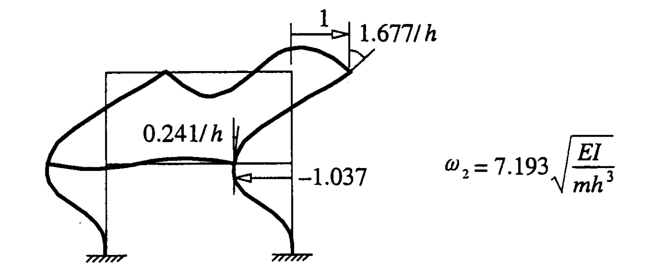
Similarly, $\mathbf{u}_t = \boldsymbol{\phi}_2$ in Eq. (a) gives the joint rotations associated with the second mode:

$$\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \frac{1}{h}\begin{bmatrix} -0.164 & -0.411 \\ -0.164 & -0.411 \\ 0.904 & -0.740 \\ 0.904 & -0.740 \end{bmatrix}\begin{Bmatrix} -1.037 \\ 1 \end{Bmatrix} = \frac{1}{h}\begin{Bmatrix} -0.241 \\ -0.241 \\ -1.677 \\ -1.677 \end{Bmatrix}$$

5. *Summary.*



First mode



Second mode

Problem 10.11

From Problem 9.7,

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}$$

$$\mathbf{k} = \frac{24EI}{h^3}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

***Part a**: Natural frequencies and modes.*

$$\mathbf{k} - \omega^2\mathbf{m} = \frac{24EI}{h^3}\begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-0.5\lambda \end{bmatrix} \quad \text{(a)}$$

where

$$\lambda = \frac{mh^3}{24EI}\omega^2$$

Substituting Eq. (a) in Eq. (10.2.6) gives the frequency equation:

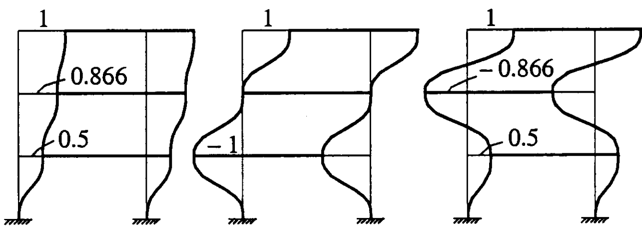
$$\lambda^3 - 6\lambda^2 + 9\lambda - 2 = 0$$

The solution gives $\lambda_1 = 2 - \sqrt{3} = 0.2679$, $\lambda_2 = 2$ and $\lambda_3 = 2 + \sqrt{3} = 3.7321$. The corresponding natural frequencies are

$$\omega_1 = 2.5359\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 6.9282\sqrt{\frac{EI}{mh^3}} \qquad \omega_3 = 9.4641\sqrt{\frac{EI}{mh^3}} \quad \text{(b)}$$

Following the procedure used in Example 10.1, the mode shapes are determined from Eq. (10.2.5):

$$\phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} \quad \text{(c)}$$



***Part b**: Verify modal orthogonality.*

$$\phi_1^T\mathbf{m}\phi_2 = m\langle 0.5 \;\; 0.866 \;\; 1\rangle\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} = 0$$

$$\phi_1^T\mathbf{m}\phi_3 = m\langle 0.5 \;\; 0.866 \;\; 1\rangle\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}\begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} = 0$$

$$\phi_2^T\mathbf{m}\phi_3 = m\langle -1 \;\; 0 \;\; 1\rangle\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}\begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} = 0$$

(e)

$$\phi_1^T\mathbf{k}\phi_2 = \frac{24EI}{h^3}\langle 0.5 \;\; 0.866 \;\; 1\rangle\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} = 0$$

$$\phi_1^T\mathbf{k}\phi_3 = \frac{24EI}{h^3}\langle 0.5 \;\; 0.866 \;\; 1\rangle\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} = 0$$

$$\phi_2^T\mathbf{k}\phi_3 = \frac{24EI}{h^3}\langle -1 \;\; 0 \;\; 1\rangle\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} = 0$$

(f)

Thus the computed modes satisfy the orthogonality properties.

***Part c**: Normalize modes so that $M_n = 1$.*

$$M_1 = \phi_1^T\mathbf{m}\phi_1 = m\langle 0.5 \;\; 0.866 \;\; 1\rangle\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}\begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} = 1.5m$$

$$M_2 = \phi_2^T\mathbf{m}\phi_2 = 1.5m$$

$$M_3 = \phi_3^T\mathbf{m}\phi_3 = 1.5m$$

Divide ϕ_1 by $\sqrt{1.5m}$, ϕ_2 by $\sqrt{1.5m}$ and ϕ_3 by $\sqrt{1.5m}$ to obtain the normalized modes:

$$\phi_1 = \sqrt{\frac{2}{3m}}\begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \quad \phi_2 = \sqrt{\frac{2}{3m}}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \quad \phi_3 = \sqrt{\frac{2}{3m}}\begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} \quad \text{(g)}$$

These modes are scalar multiples of the modes in (a); the shapes of the two sets of modes are the same.

Problem 10.12

From Problem 9.8,

$$\mathbf{m} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\mathbf{k} = k\begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

where $k=8EI/h^3$

***Part a**: Natural frequencies and modes.*

$$\mathbf{k} - \omega^2\mathbf{m} = k\begin{bmatrix} 5-\lambda & -2 & 0 \\ -2 & 3-\lambda & -1 \\ 0 & -1 & 1-\dfrac{\lambda}{2} \end{bmatrix} \qquad \text{(a)}$$

where $\lambda=m\omega^2/k$

$$\det(\mathbf{k} - \omega^2\mathbf{m}) = (5-\lambda)\left[(3-\lambda)(1-\frac{\lambda}{2})-1\right]+2\left[-2(1-\frac{\lambda}{2})\right]$$

$$= -\frac{\lambda^3}{2} + 5\lambda^2 - \frac{25\lambda}{2} + 6$$

Substituting Eq. (a) in Eq. (10.2.6) gives the frequency equation:

$$\lambda^3 - 10\lambda^2 + 25\lambda - 12 = 0 \qquad \text{(b)}$$

The solution gives:

$$\lambda_1 = 0.6277$$

$$\lambda_2 = 3$$

$$\lambda_3 = 6.372$$

The corresponding natural frequencies are:

$$\omega_1 = 2.241\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 4.899\sqrt{\frac{EI}{mh^3}} \quad \omega_3 = 7.14\sqrt{\frac{EI}{mh^3}} \qquad \text{(c)}$$

Substituting λ_1=0.6277 in Eq. (a) gives

$$k\begin{bmatrix} 4.3723 & -2 & 0 \\ -2 & 2.3723 & -1 \\ 0 & -1 & 0.686 \end{bmatrix}\begin{bmatrix} \phi_{11} \\ \phi_{12} \\ \phi_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $\phi_{13} = 1$, then the third and first equations give:

$$-\phi_{12} + (0.686)1 = 0 \Rightarrow \phi_{12} = 0.686$$

$$4.3723\phi_{11} - 2(0.6862) = 0 \Rightarrow \phi_{11} = 0.314$$

$$\phi_1 = \begin{bmatrix} 0.314 \\ 0.686 \\ 1 \end{bmatrix}$$

Substituting λ_2=3 in Eq. (a) gives

$$k\begin{bmatrix} 2 & -2 & 0 \\ -2 & 0 & -1 \\ 0 & -1 & -\dfrac{1}{2} \end{bmatrix}\begin{bmatrix} \phi_{21} \\ \phi_{22} \\ \phi_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let ϕ_{23}=1, then the third and first equations give:

$$-\phi_{22} - 0.5 = 0 \Rightarrow \phi_{22} = -0.5$$

$$-2\phi_{21} - 1 = 0 \Rightarrow \phi_{21} = -0.5$$

$$\phi_2 = \begin{bmatrix} -0.5 \\ -0.5 \\ 1 \end{bmatrix}$$

Substituting λ_3=6.372 in Eq. (a) gives

$$k\begin{bmatrix} -1.372 & -2 & 0 \\ -2 & -3.372 & -1 \\ 0 & -1 & -2.186 \end{bmatrix}\begin{bmatrix} \phi_{31} \\ \phi_{32} \\ \phi_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let ϕ_{33}=1, then the third and first equations give:

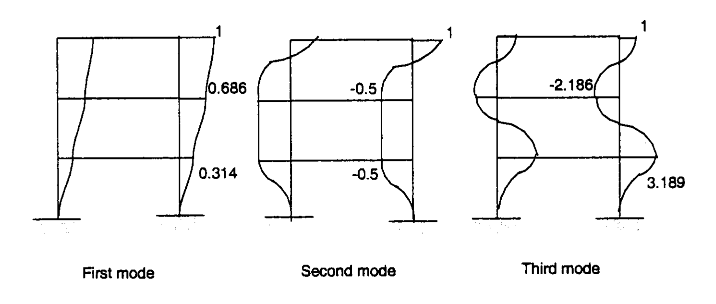
$$-\phi_{32} - (2.186)1 = 0 \Rightarrow \phi_{32} = -2.186$$

$$-1.3723\phi_{31} - 2(-2.186) = 0 \Rightarrow \phi_{31} = 3.186$$

$$\phi_3 = \begin{bmatrix} 3.186 \\ -2.186 \\ 1 \end{bmatrix}$$

In summary, the modes are:

$$\phi_1 = \begin{bmatrix} 0.314 \\ 0.686 \\ 1 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -0.5 \\ -0.5 \\ 1 \end{bmatrix} \quad \phi_3 = \begin{bmatrix} 3.186 \\ -2.186 \\ 1 \end{bmatrix} \qquad \text{(d)}$$



***Part b**: Verify modal orthogonality.*

$$\phi_1^T \mathbf{m} \phi_2 = m\begin{bmatrix}0.314 & 0.686 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}-0.5\\ -0.5\\ 1\end{bmatrix} = 0$$

$$\phi_1^T \mathbf{m} \phi_3 = m\begin{bmatrix}0.314 & 0.686 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}3.186\\ -2.186\\ 1\end{bmatrix} = 0$$

$$\phi_2^T \mathbf{m} \phi_3 = m\begin{bmatrix}-0.5 & -0.5 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}3.186\\ -2.186\\ 1\end{bmatrix} = 0 \tag{e}$$

$$\phi_1^T \mathbf{k} \phi_2 = k\begin{bmatrix}0.314 & 0.686 & 1\end{bmatrix}\begin{bmatrix}5 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix}\cdot\begin{bmatrix}-0.5\\ -0.5\\ 1\end{bmatrix} = 0$$

$$\phi_1^T \mathbf{k} \phi_3 = k\begin{bmatrix}0.314 & 0.686 & 1\end{bmatrix}\begin{bmatrix}5 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}3.186\\ -2.186\\ 1\end{bmatrix} = 0$$

$$\phi_2^T \mathbf{k} \phi_3 = k\begin{bmatrix}-0.5 & -0.5 & 1\end{bmatrix}\begin{bmatrix}5 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}3.186\\ -2.186\\ 1\end{bmatrix} = 0 \tag{f}$$

Thus the computed modes satisfy the orthogonality properties.

***Part c**: Normalize modes so that M_n=1.*

$$M_1 = \phi_1^T \mathbf{m} \phi_1$$

$$= m\,\langle 0.314 \quad 0.686 \quad 1\rangle\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0.314\\ 0.686\\ 1\end{bmatrix}$$

$$=1.069m$$

$$M_2 = \phi_2^T \mathbf{m} \phi_2$$

$$= m\begin{bmatrix}-0.5 & -0.5 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}-0.5\\ -0.5\\ 1\end{bmatrix}$$

$$= m$$

$$M_2 = \phi_3^T \mathbf{m} \phi_3$$

$$= m\begin{bmatrix}3.186 & -2.186 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}3.186\\ -2.186\\ 1\end{bmatrix}$$

$$= 15.46m$$

Divide ϕ_1 by $\sqrt{1.069m}$, ϕ_2 by $\sqrt{m}$ and ϕ_3 by $\sqrt{15.46m}$ to obtain the normalized modes:

$$\phi_1 = \frac{1}{\sqrt{1.069m}}\begin{bmatrix}0.314\\ 0.686\\ 1\end{bmatrix} \quad \phi_2 = \frac{1}{\sqrt{m}}\begin{bmatrix}-0.5\\ -0.5\\ 1\end{bmatrix} \quad \phi_3 = \frac{1}{\sqrt{15.46m}}\begin{bmatrix}3.189\\ -2.186\\ 1\end{bmatrix}$$

These modes are multiples of the modes in (a); the shapes of the two sets of modes are the same.

Problem 10.13

When the columns are hinged at the base, the stiffness of the first story is

$$k_1 = 2\left(\frac{3EI}{h^3}\right) = \frac{6EI}{h^3}$$

The stiffness of the second and third stories does not change. Following the procedure in Problem 9.7 gives

$$\mathbf{k} = \frac{24EI}{h^3}\begin{bmatrix} 1.25 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

The mass matrix is the same as in Problem 10.12:

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}$$

Then

$$\mathbf{k} - \omega^2\mathbf{m} = \frac{24EI}{h^3}\begin{bmatrix} 1.25-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-0.5\lambda \end{bmatrix} \qquad \text{(a)}$$

where

$$\lambda = \frac{mh^3}{24EI}\omega^2$$

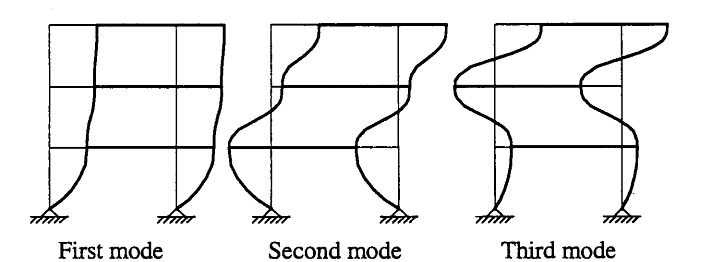
Substituting Eq. (a) in Eq. (10.2.6) gives the frequency equation:

$$4\lambda^3 - 21\lambda^2 + 24\lambda - 2 = 0$$

Following the procedure of Problem 10.12 we obtain

$$\omega_1 = 1.4726\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 6.0413\sqrt{\frac{EI}{mh^3}} \quad \omega_3 = 9.3453\sqrt{\frac{EI}{mh^3}} \qquad \text{(b)}$$

$$\phi_1 = \begin{Bmatrix} 0.8234 \\ 0.9548 \\ 1 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} -0.8851 \\ 0.2396 \\ 1 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0.3430 \\ -0.8195 \\ 1 \end{Bmatrix} \qquad \text{(c)}$$



The structure with columns hinged at the base is more flexible than the structure with clamped columns, and thus has lower natural frequencies. The fundamental frequency is less than half, whereas the higher frequencies are affected less.

The modes are also affected by the column fixity. Notice that the fundamental mode of the structure with hinged columns indicates a flexible first story relative to the other stories.

Problem 10.14

When the columns are hinged at the base, the stiffness of the first story is

$$k_1 = 2\left(\frac{3EI}{h^3}\right) = \frac{6EI}{h^3}$$

The stiffness of the second and third stories does not change. Following the procedure in Problem 9.8 gives

$$\mathbf{k} = \frac{EI}{h^3}\begin{bmatrix} 22 & -16 & 0 \\ -16 & 24 & -8 \\ 0 & -8 & 8 \end{bmatrix} = k\begin{bmatrix} \frac{11}{4} & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

where $k = 8EI/h^3$.

The mass matrix is the same as in Problem 10.12:

$$\mathbf{m} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Then

$$\mathbf{k} - \omega^2\mathbf{m} = k\begin{bmatrix} \frac{11}{4} - \lambda & -2 & 0 \\ -2 & 3-\lambda & -1 \\ 0 & -1 & 1 - \frac{1}{2}\lambda \end{bmatrix} \qquad \text{(a)}$$

where $\lambda = \frac{m}{k}\omega^2$

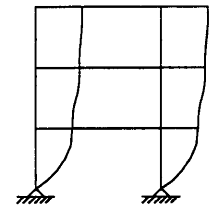
Substituting Eq. (a) in Eq. (10.2.6) gives:

$$-\frac{\lambda^3}{2} + \frac{31\lambda^2}{8} - \frac{55\lambda}{8} + \frac{3}{2} = 0$$

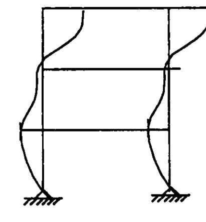
Following the procedure of Problem 10.12 gives:

$$\omega_1 = 1.423\sqrt{\frac{EI}{mh^3}}, \quad \omega_2 = 4.257\sqrt{\frac{EI}{mh^3}}, \quad \omega_3 = 6.469\sqrt{\frac{EI}{mh^3}}$$

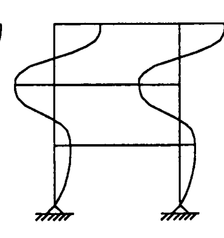
$$\varphi_1 = \begin{bmatrix} 0.7 \\ 0.873 \\ 1 \end{bmatrix} \varphi_2 = \begin{bmatrix} -0.549 \\ -0.133 \\ 1 \end{bmatrix} \varphi_3 = \begin{bmatrix} 1.301 \\ -1.614 \\ 1 \end{bmatrix}$$



First mode



Second mode



Third mode

The structure with columns hinged at the base is more flexible than the structure with clamped columns, and thus has lower natural frequencies. The fundamental frequency is less than half, whereas the higher frequencies are affected less.

The modes are also affected by column fixity. Notice that the fundamental mode of the structure with hinged columns indicates a flexible first story relative to the other stories.

Problem 10.15

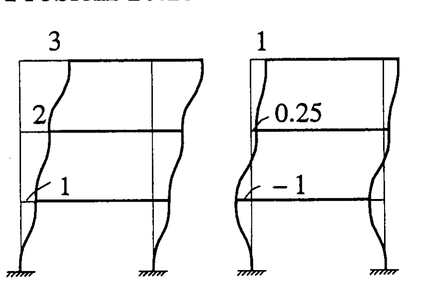


Fig. P10.15a Fig. P10.15b Fig. P10.15c

From Problem 10.11,

$$\omega_1 = 2.5359\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 6.9282\sqrt{\frac{EI}{mh^3}} \quad \omega_3 = 9.4641\sqrt{\frac{EI}{mh^3}} \tag{a}$$

and

$$\phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} \tag{b}$$

The response of the system to initial displacements is obtained from Eqs. (10.8.6) and (10.8.5):

$$\mathbf{u}(t) = \sum_{n=1}^{3} \phi_n \left[q_n(0)\cos\omega_n t + \frac{\dot{q}_n(0)}{\omega_n}\sin\omega_n t \right] \tag{c}$$

where

$$q_n(0) = \frac{\phi_n^T \mathbf{m}\mathbf{u}(0)}{\phi_n^T \mathbf{m}\phi_n} \qquad \dot{q}_n(0) = \frac{\phi_n^T \mathbf{m}\dot{\mathbf{u}}(0)}{\phi_n^T \mathbf{m}\phi_n} \tag{d}$$

Case a

The initial conditions (from Fig. P10.15a) are

$$\mathbf{u}(0) = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} \quad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Then from Eq. (d):

$$\begin{aligned} q_1(0) &= 2.4800 & \dot{q}_1(0) &= 0 \\ q_2(0) &= 0.3333 & \dot{q}_2(0) &= 0 \\ q_3(0) &= 0.1786 & \dot{q}_3(0) &= 0 \end{aligned} \tag{e}$$

Substituting Eqs. (b) and (e) in Eq. (c) gives $\mathbf{u}(t)$ in inches:

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} 1.2440 \\ 2.1547 \\ 2.4880 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} -0.3333 \\ 0 \\ 0.3333 \end{Bmatrix}\cos\omega_2 t + \begin{Bmatrix} 0.0893 \\ -0.1547 \\ 0.1786 \end{Bmatrix}\cos\omega_3 t \tag{f}$$

Case b

The initial conditions (from Fig. P10.15b) are

$$\mathbf{u}(0) = \begin{Bmatrix} -1 \\ 0.25 \\ 1 \end{Bmatrix} \quad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Then from Eq. (d),

$$\begin{aligned} q_1(0) &= 0.1433 & \dot{q}_1(0) &= 0 \\ q_2(0) &= 1 & \dot{q}_2(0) &= 0 \\ q_3(0) &= -0.1433 & \dot{q}_3(0) &= 0 \end{aligned} \tag{g}$$

The displacement response is

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} 0.0717 \\ 0.1241 \\ 0.1433 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}\cos\omega_2 t + \begin{Bmatrix} -0.0717 \\ 0.1241 \\ -0.1433 \end{Bmatrix}\cos\omega_3 t \tag{h}$$

Case c

The initial conditions (from Fig. P10.15c) are

$$\mathbf{u}(0) = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} \quad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Then from Eq. (d),

$$\begin{aligned} q_1(0) &= 0.0893 & \dot{q}_1(0) &= 0 \\ q_2(0) &= -0.3333 & \dot{q}_2(0) &= 0 \\ q_3(0) &= 1.2440 & \dot{q}_3(0) &= 0 \end{aligned} \tag{i}$$

The displacement response is

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} 0.0447 \\ 0.0774 \\ 0.0893 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} 0.3333 \\ 0 \\ -0.3333 \end{Bmatrix}\cos\omega_2 t + \begin{Bmatrix} 0.6220 \\ -1.0774 \\ 1.2440 \end{Bmatrix}\cos\omega_3 t \tag{j}$$

Although all three modes contribute to the response in each case, in the three cases the dominant response is due to the first, second, and third modes, respectively.

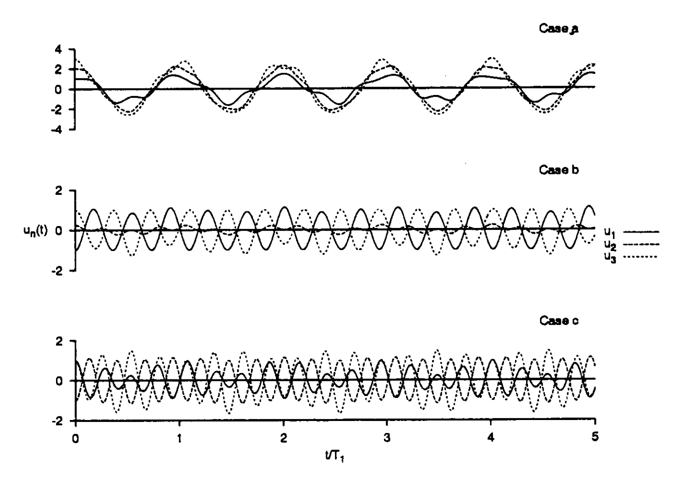


Fig. P10.15d

Problem 10.16

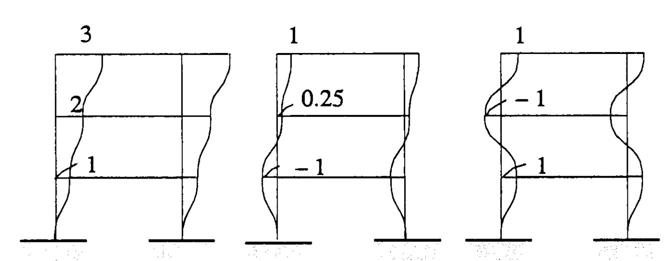


Fig. P10.16a Fig. P10.16b Fig. P10.16a

From Problem 10.12,

$$\omega_1 = 2.241\sqrt{\frac{EI}{mh^3}} \quad \omega_2=4.899\sqrt{\frac{EI}{mh^3}} \quad \omega_3=7.14\sqrt{\frac{EI}{mh^3}} \quad \text{(a)}$$

$$\phi_1 = \begin{bmatrix} 0.314 \\ 0.686 \\ 1 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -0.5 \\ -0.5 \\ 1 \end{bmatrix} \quad \phi_3 = \begin{bmatrix} 3.186 \\ -2.186 \\ 1 \end{bmatrix} \quad \text{(b)}$$

The response of the system to initial displacements is obtained from Eqs. (10.8.6) and (10.8.5):

$$\mathbf{u}(t) = \sum_{n=1}^{3}\phi_n\left[q_n(0)\cos\omega_n t + \frac{\dot{q}_n(0)}{\omega_n}\sin\omega_n t\right] \quad \text{(c)}$$

where

$$q_n(0) = \frac{\phi_n^T \mathbf{m}\mathbf{u}(0)}{\phi_n^T \mathbf{m}\phi_n} \quad \dot{q}_n(0) = \frac{\phi_n^T \mathbf{m}\dot{\mathbf{u}}(0)}{\phi_n^T \mathbf{m}\phi_n} \quad \text{(d)}$$

Case a

The initial conditions from (Fig. P10.16a) are

$$\mathbf{u}(0) = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}, \quad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Then from Eq. (d):

$$q_1(0) = 2.98 \quad \dot{q}_1(0) = 0$$
$$q_2(0) = 0 \quad \dot{q}_2(0) = 0 \quad \text{(e)}$$
$$q_3(0) = 0.0205 \quad \dot{q}_3(0) = 0$$

Substituting Eqs. (b) and (e) in Eq. (c) gives $\mathbf{u}(t)$ in inches:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} 0.935 \\ 2.04 \\ 2.98 \end{bmatrix}\cos\omega_1 t + \begin{bmatrix} 0.065 \\ -0.0668 \\ 0.0205 \end{bmatrix}\cos\omega_3 t$$

Case b

The initial conditions (from Fig. P10.16b) are

$$u(0) = \begin{Bmatrix} -1 \\ 0.25 \\ 1 \end{Bmatrix}, \quad \dot{\mathbf{u}}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then from Eq. (d):

$$q_1(0) = 0.334 \quad \dot{q}_1(0) = 0$$
$$q_2(0) = 0.875 \quad \dot{q}_2(0) = 0$$
$$q_3(0) = -0.209 \quad \dot{q}_3(0) = 0$$

The displacement response is

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} 0.1048 \\ 0.229 \\ 0.334 \end{bmatrix}\cos\omega_1 t + \begin{bmatrix} -0.4375 \\ -0.4375 \\ 0.875 \end{bmatrix}\cos\omega_2 t + \begin{bmatrix} -0.666 \\ 0.456 \\ -0.2089 \end{bmatrix}\cos\omega_3 t$$

Case c

The initial conditions (Fig. P10.16c) are:

$$\mathbf{u}(0) = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}, \quad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Then from Eq. (d):

$$q_1(0)=0.12 \quad \dot{q}_1(0)=0$$
$$q_2(0)=0.5 \quad \dot{q}_2(0)=0$$
$$q_3(0)=0.38 \quad \dot{q}_3(0)=0$$

The displacement response is

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} 0.038 \\ 0.082 \\ 0.12 \end{bmatrix}\cos\omega_1 t + \begin{bmatrix} -0.25 \\ -0.25 \\ 0.5 \end{bmatrix}\cos\omega_2 t + \begin{bmatrix} 1.21 \\ -0.83 \\ 0.38 \end{bmatrix}\cos\omega_3 t$$

Although all three modes contribute to the response in each case, in the three cases the dominant response is due to the first, second, and third modes, respectively.

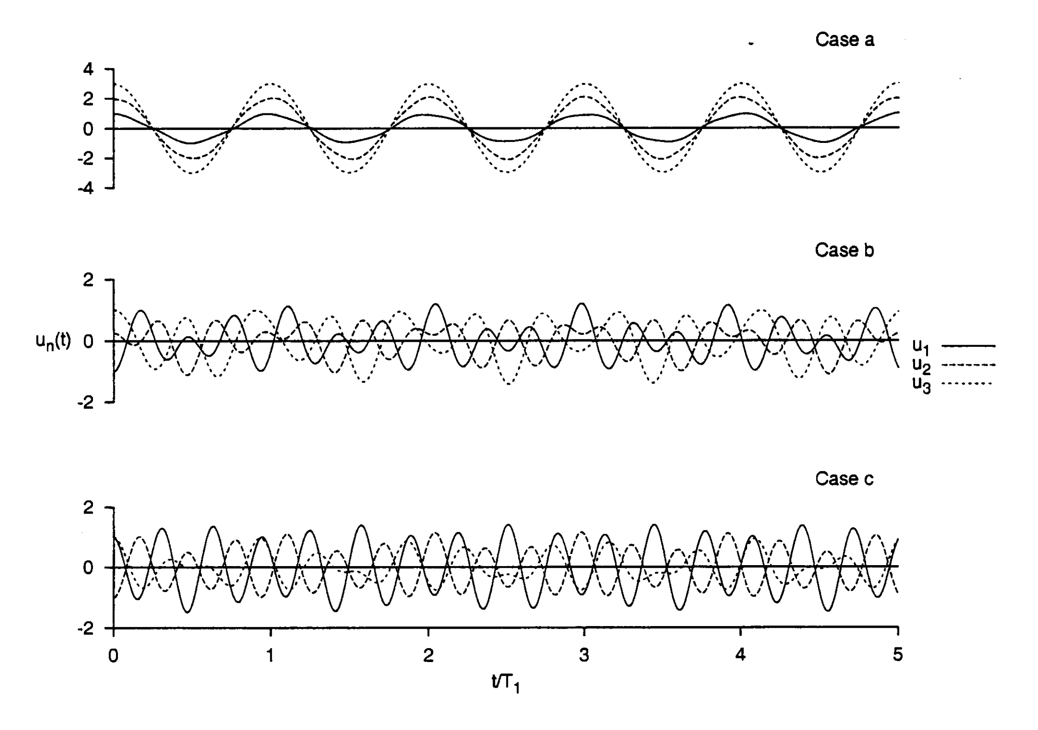


Fig. P10.16d

Problem 10.17

From Problem 10.11,

$$\omega_1 = 2.5359\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 6.9282\sqrt{\frac{EI}{mh^3}} \qquad \omega_3 = 9.4641\sqrt{\frac{EI}{mh^3}} \tag{a}$$

The response of the system is given by Eqs. (10.8.7) and (10.10.2):

$$\mathbf{u}(t) = \sum_{n=1}^{2} \phi_n e^{-\zeta_n \omega_n t}\left[q_n(0)\cos\omega_{nD}t + \frac{\dot{q}_n(0) + \zeta_n\omega_n q_n(0)}{\omega_{nD}}\sin\omega_{nD}t\right] \tag{b}$$

where

$$\omega_{nD} = \omega_n\sqrt{1 - \zeta_n^2}$$

For $\zeta_n = 5\%$,

$$\omega_{1D} = 2.5327\sqrt{\frac{EI}{mh^3}} \qquad \zeta_1\omega_1 = 0.1268\sqrt{\frac{EI}{mh^3}}$$
$$\omega_{2D} = 6.9195\sqrt{\frac{EI}{mh^3}} \qquad \zeta_2\omega_2 = 0.3461\sqrt{\frac{EI}{mh^3}} \tag{c}$$
$$\omega_{3D} = 9.4523\sqrt{\frac{EI}{mh^3}} \qquad \zeta_3\omega_3 = 0.4732\sqrt{\frac{EI}{mh^3}}$$

Substituting Eq. (c) and values of $q_n(0)$ from Problem 10.15 in Eq. (b) gives the following response for the first set of initial conditions:

$$\mathbf{u}(t) = \begin{Bmatrix} 1.2440 \\ 2.1547 \\ 2.4880 \end{Bmatrix} e^{-\zeta_1\omega_1 t}\cos\omega_{1D}t + \begin{Bmatrix} 0.0623 \\ 0.1079 \\ 0.1249 \end{Bmatrix} e^{-\zeta_1\omega_1 t}\sin\omega_{1D}t$$
$$+ \begin{Bmatrix} -0.3333 \\ 0 \\ 0.3333 \end{Bmatrix} e^{-\zeta_2\omega_2 t}\cos\omega_{2D}t + \begin{Bmatrix} -0.0167 \\ 0 \\ 0.0167 \end{Bmatrix} e^{-\zeta_2\omega_2 t}\sin\omega_{2D}t$$
$$+ \begin{Bmatrix} 0.0893 \\ -0.1547 \\ 0.1786 \end{Bmatrix} e^{-\zeta_3\omega_3 t}\cos\omega_{3D}t + \begin{Bmatrix} 0.0045 \\ -0.0077 \\ 0.0089 \end{Bmatrix} e^{-\zeta_3\omega_3 t}\sin\omega_{3D}t$$

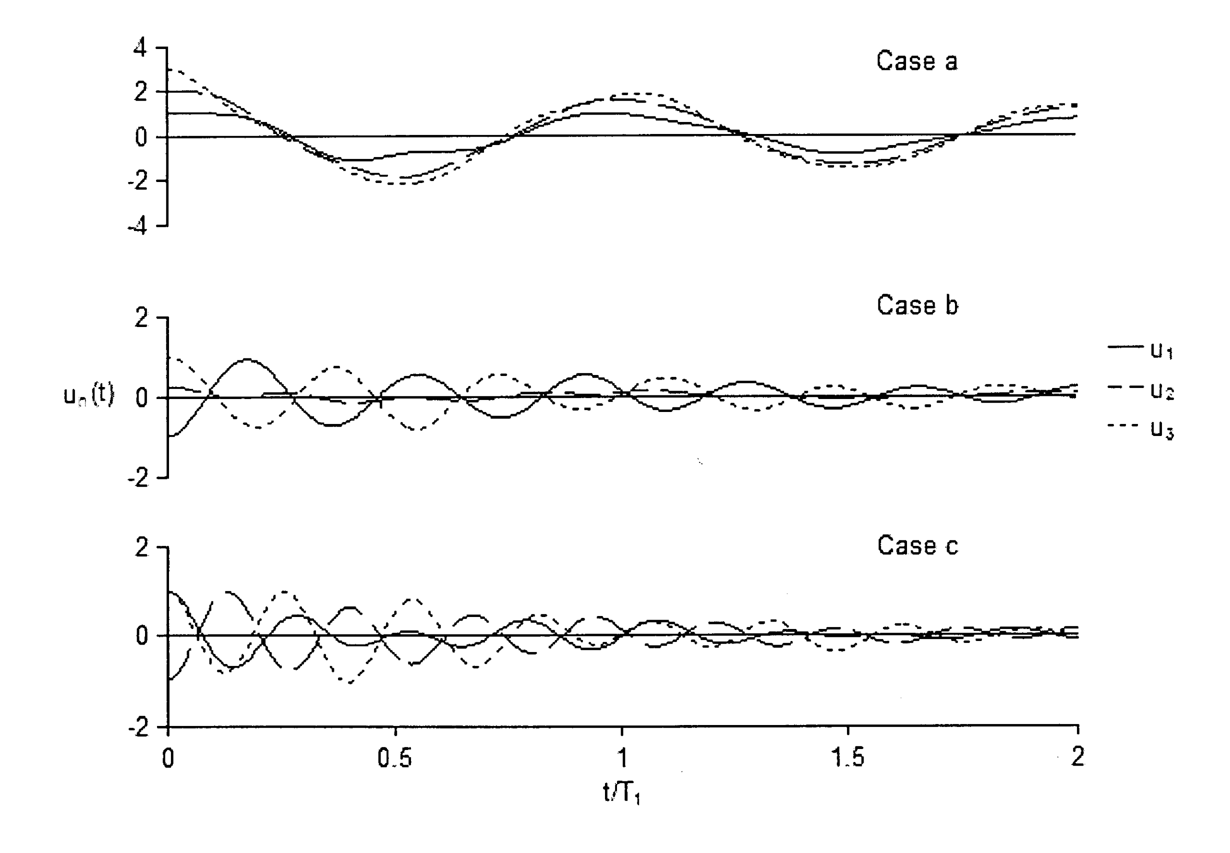


Fig. P10.17

Problem 10.18

From Problem 10.12,

$$\omega_1 = 2.241\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 4.899\sqrt{\frac{EI}{mh^3}} \quad \omega_3 = 7.14\sqrt{\frac{EI}{mh^3}} \tag{a}$$

The response of the system is given by Eqs. (10.8.7) and (10.10.2):

$$\mathbf{u}(t) = \sum_{n=1}^{3} \phi_n e^{-\varsigma_n \omega_n t}\left[q_n(0)\cos\omega_{nD}t + \frac{\dot{q}_n(0) + \varsigma_n\omega_n q_n(0)}{\omega_{nD}}\sin\omega_{nD}t\right] \tag{b}$$

where

$$\omega_{nD} = \omega_n\sqrt{1-\varsigma_n^2}$$

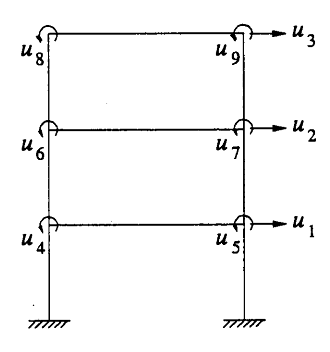
For $\varsigma_n = 5\%$,

$$\begin{aligned}
\omega_{1D} &= 2.2382\sqrt{\frac{EI}{mh^3}} & \varsigma_1\omega_1 &= 0.1121\sqrt{\frac{EI}{mh^3}} \\
\omega_{2D} &= 4.8929\sqrt{\frac{EI}{mh^3}} & \varsigma_2\omega_2 &= 0.2450\sqrt{\frac{EI}{mh^3}} \\
\omega_{3D} &= 7.1311\sqrt{\frac{EI}{mh^3}} & \varsigma_3\omega_3 &= 0.3570\sqrt{\frac{EI}{mh^3}}
\end{aligned} \tag{c}$$

Substituting Eq. (c) and values of $q_n(0)$ from Problem 10.16 gives the following response for the first set of initial conditions:

$$\mathbf{u}(t) = \begin{Bmatrix} 0.9358 \\ 2.0443 \\ 2.98 \end{Bmatrix} e^{-\varsigma_1\omega_1 t}\cos\omega_{1D}t + \begin{Bmatrix} 0.0469 \\ 0.1023 \\ 0.1492 \end{Bmatrix} e^{-\varsigma_1\omega_1 t}\sin\omega_{1D}t$$
$$+ \begin{Bmatrix} 0.0653 \\ 0.0448 \\ 0.0205 \end{Bmatrix} e^{-\varsigma_3\omega_3 t}\cos\omega_{3D}t + \begin{Bmatrix} 0.0033 \\ -0.0022 \\ 0.0010 \end{Bmatrix} e^{-\varsigma_3\omega_3 t}\sin\omega_{3D}t$$

Problem 10.19



With reference to the lateral floor displacements u_1, u_2 and u_3, the mass matrix and the condensed stiffness matrix (from Problem 9.9) are

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}$$

$$\hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 40.85 & -23.26 & 5.11 \\ -23.26 & 31.09 & -14.25 \\ 5.11 & -14.25 & 10.06 \end{bmatrix}$$

Obtained by using the same procedure as in Problem 10.11, the natural frequencies and modes are

$$\omega_1 = 1.4576\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 4.7682\sqrt{\frac{EI}{mh^3}} \quad \omega_3 = 8.1980\sqrt{\frac{EI}{mh^3}} \tag{a}$$

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} 0.3156 \\ 0.7451 \\ 1 \end{Bmatrix} \quad \boldsymbol{\phi}_2 = \begin{Bmatrix} -0.7409 \\ -0.3572 \\ 1 \end{Bmatrix} \quad \boldsymbol{\phi}_3 = \begin{Bmatrix} 1.2546 \\ -1.2024 \\ 1 \end{Bmatrix} \tag{b}$$

The joint rotations corresponding to the modes of Eq. (b) are computed using Eq. (9.3.3):

$$\mathbf{u}_0 = \mathbf{T}\,\mathbf{u}_t \tag{c}$$

where

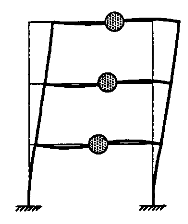
$$\mathbf{T} = -\mathbf{k}_{00}^{-1}\,\mathbf{k}_{0t} \tag{d}$$

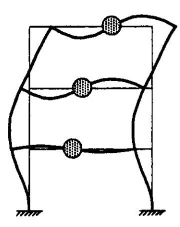
Substituting $\mathbf{k}_{00}$ and $\mathbf{k}_{0t}$ from Problem 9.9 in Eq. (d) gives

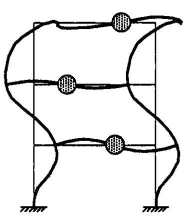
$$\mathbf{T} = \frac{1}{h}\begin{bmatrix} -0.1084 & -0.5342 & 0.0744 \\ -0.1084 & -0.5342 & 0.0744 \\ 0.5961 & -0.0619 & -0.4258 \\ 0.5961 & -0.0619 & -0.4258 \\ -0.1703 & 0.8748 & -0.7355 \\ -0.1703 & 0.8748 & -0.7355 \end{bmatrix}$$

The joint rotations associated with each mode are obtained by substituting $\mathbf{u}_t = \boldsymbol{\phi}_n$, $n = 1$, 2 and 3, in Eq. (c); the results are

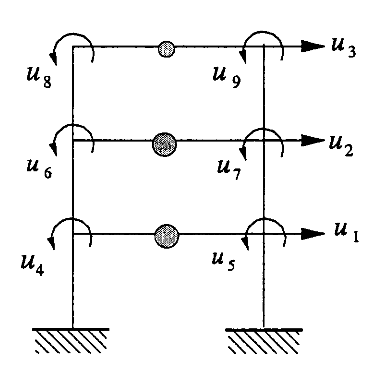
$$(\boldsymbol{\phi}_1)_0 = \frac{1}{h}\begin{Bmatrix} -0.3548 \\ -0.3548 \\ -0.2838 \\ -0.2838 \\ -0.1374 \\ -0.1374 \end{Bmatrix} \quad (\boldsymbol{\phi}_2)_0 = \frac{1}{h}\begin{Bmatrix} 0.3486 \\ 0.3486 \\ -0.8453 \\ -0.8453 \\ -0.9218 \\ -0.9218 \end{Bmatrix} \quad (\boldsymbol{\phi}_3)_0 = \frac{1}{h}\begin{Bmatrix} 0.5837 \\ 0.5837 \\ 0.3966 \\ 0.3966 \\ -2.0011 \\ -2.0011 \end{Bmatrix}$$


First mode


Second mode


Third mode

Problem 10.20



With reference to the lateral floor displacements u_1, u_2 and u_3, the mass matrix and the condensed stiffness matrix (from Problem 9.10) are

$$\mathbf{m} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$\hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 39.38 & -22.68 & 5.486 \\ & 27.13 & -11.75 \\ Symm & & 7.418 \end{bmatrix}$$

Obtained by using the same procedure as in Problem 10.11, the natural frequencies and modes are:

$$\omega_1 = 1.197\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 4.178\sqrt{\frac{EI}{mh^3}} \quad \omega_3 = 7.903\sqrt{\frac{EI}{mh^3}} \tag{a}$$

$$\phi_1 = \begin{bmatrix} 0.273 \\ 0.698 \\ 1 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -0.706 \\ -0.441 \\ 1 \end{bmatrix} \quad \phi_3 = \begin{bmatrix} 1.529 \\ -1.315 \\ 1 \end{bmatrix} \tag{b}$$

The joint rotations corresponding to the modes of Eq. (b) are computed using Eq. (9.3.3):

$$\mathbf{u}_0 = \mathbf{T}\,\mathbf{u}_t \tag{c}$$

where

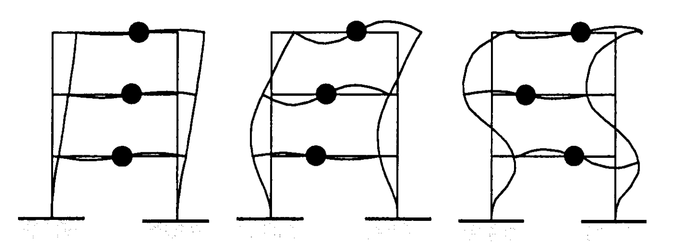
$$\mathbf{T} = -\mathbf{k}_{oo}^{-1} \cdot \mathbf{k}_{oT} \tag{d}$$

Substituting $\mathbf{k}_{00}$ and $\mathbf{k}_{0t}$ from Problem 9.10 in Eq. (d) gives

$$\mathbf{T} = \frac{1}{h}\begin{bmatrix} -0.1512 & -0.6084 & 0.0962 \\ -0.1512 & -0.6084 & 0.0962 \\ 0.7184 & -0.11 & -0.457 \\ 0.7184 & -0.11 & -0.457 \\ -0.2612 & 1.131 & -0.925 \\ -0.2612 & 1.131 & -0.925 \end{bmatrix}$$

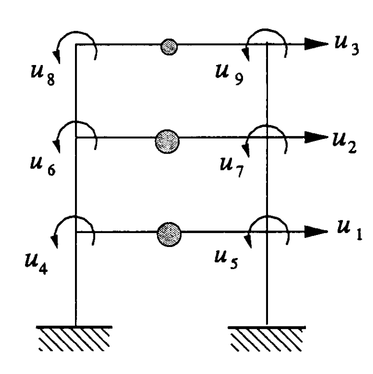
The joint rotations associated with each mode are obtained by substituting $\mathbf{u}_t = \phi_n$, n = 1, 2, 3, in Eq. (c): The results are:

$$(\phi_1)_0 = \frac{1}{h}\begin{bmatrix} -0.369 \\ -0.369 \\ -0.338 \\ -0.338 \\ -0.208 \\ -0.208 \end{bmatrix} (\phi_2)_0 = \frac{1}{h}\begin{bmatrix} 0.472 \\ 0.472 \\ -0.917 \\ -0.917 \\ -1.238 \\ -1.238 \end{bmatrix} (\phi_3)_0 = \frac{1}{h}\begin{bmatrix} 0.667 \\ 0.667 \\ 0.782 \\ 0.782 \\ -2.808 \\ -2.808 \end{bmatrix}$$



First mode   Second mode   Third mode

Problem 10.21



With reference to the lateral floor displacements u_1, u_2 and u_3, the mass matrix and the condensed stiffness matrix (from Problem 9.11) are

$$\mathbf{m} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$\hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 33.36 & -14.91 & 1.942 \\ & 15.96 & -5.489 \\ Symm & & 3.923 \end{bmatrix}$$

Obtained by using the same procedure as in Problem 10.12, the natural frequencies and modes are:

$$\omega_1 = 1.329\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 3.514\sqrt{\frac{EI}{mh^3}} \quad \omega_3 = 6.562\sqrt{\frac{EI}{mh^3}} \tag{a}$$

$$\phi_1 = \begin{bmatrix} 0.234 \\ 0.639 \\ 1 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -0.512 \\ -0.591 \\ 1 \end{bmatrix} \quad \phi_3 = \begin{bmatrix} 3.324 \\ -2.032 \\ 1 \end{bmatrix} \tag{b}$$

The joint rotations corresponding to the modes of Eq. (b) are computed using Eq. (9.3.3):

$$\mathbf{u}_0 = \mathbf{T}\,\mathbf{u}_t \tag{c}$$

where

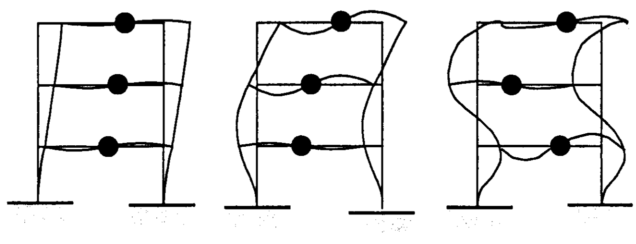
$$\mathbf{T} = -\mathbf{k}_{oo}^{-1} \cdot \mathbf{k}_{oT} \tag{d}$$

Substituting $\mathbf{k}_{00}$ and $\mathbf{k}_{0t}$ from Problem 9.11 in Eq. (d) gives

$$\mathbf{T} = \frac{1}{h}\begin{bmatrix} -0.3006 & -0.3695 & 0.0313 \\ -0.3006 & -0.3695 & 0.0313 \\ 0.6795 & -0.321 & -0.227 \\ 0.6795 & -0.321 & -0.227 \\ -0.1942 & 0.9489 & -0.7923 \\ -0.1942 & 0.9489 & -0.7923 \end{bmatrix}$$

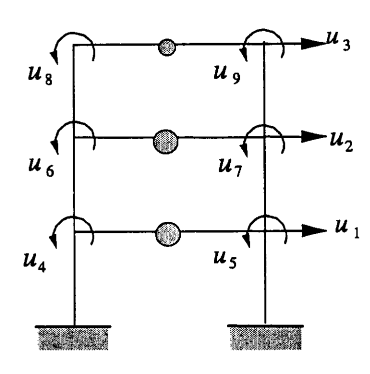
The joint rotations associated with each mode are obtained by substituting $\mathbf{u}_t = \phi_n$, n = 1, 2, 3, in Eq. (c): The results are:

$$(\phi_1)_0 = \frac{1}{h}\begin{bmatrix} -0.275 \\ -0.275 \\ -0.273 \\ -0.273 \\ -0.231 \\ -0.231 \end{bmatrix} (\phi_2)_0 = \frac{1}{h}\begin{bmatrix} 0.404 \\ 0.404 \\ -0.385 \\ -0.385 \\ -1.254 \\ -1.254 \end{bmatrix} (\phi_3)_0 = \frac{1}{h}\begin{bmatrix} -0.217 \\ -0.217 \\ 2.684 \\ 2.684 \\ -3.366 \\ -3.366 \end{bmatrix}$$



First mode Second mode Third mode

Problem 10.22



With reference to the lateral floor displacements u_1, u_2 and u_3, the mass matrix and the condensed stiffness matrix (from Problem 9.12) are

$$\mathbf{m} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$\hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 30.77 & -14.01 & 2.43 \\ & 13.82 & -4.80 \\ Symm & & 2.92 \end{bmatrix}$$

Obtained by using the same procedure as in Problem 10.12, the natural frequencies and modes are:

$$\omega_1 = 1.043\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 3.081\sqrt{\frac{EI}{mh^3}} \quad \omega_3 = 6.314\sqrt{\frac{EI}{mh^3}} \tag{a}$$

$$\phi_1 = \begin{bmatrix} 0.1997 \\ 0.5966 \\ 1 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -0.5454 \\ -0.6555 \\ 1 \end{bmatrix} \quad \phi_3 = \begin{bmatrix} 3.2201 \\ -1.9163 \\ 1 \end{bmatrix} \tag{b}$$

The joint rotations corresponding to the modes of Eq. (b) are computed using Eq. (9.3.3):

$$\mathbf{u}_0 = \mathbf{T}\mathbf{u}_t \tag{c}$$

where

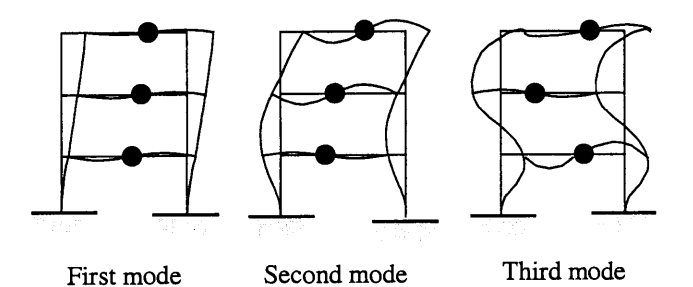
$$\mathbf{T} = -\mathbf{k}_{oo}^{-1}\ \mathbf{k}_{oT} \tag{d}$$

Substituting $\mathbf{k}_{00}$ and $\mathbf{k}_{0t}$ from Problem 9.12 in Eq. (d) gives

$$\mathbf{T} = \frac{1}{h}\begin{bmatrix} -0.4005 & -0.4152 & 0.0458 \\ -0.4005 & -0.4152 & 0.0458 \\ 0.9530 & -0.4569 & -0.2803 \\ 0.9530 & -0.4569 & -0.2803 \\ -0.3465 & 1.2570 & -0.9890 \\ -0.3465 & 1.2570 & -0.9890 \end{bmatrix}$$

The joint rotations associated with each mode are obtained by substituting $\mathbf{u}_t = \phi_n$, n = 1, 2, 3, in Eq. (c): The results are:

$$(\phi_1)_0 = \frac{1}{h}\begin{bmatrix} -0.2819 \\ -0.2819 \\ -0.3626 \\ -0.3626 \\ -0.3083 \\ -0.3083 \end{bmatrix} \quad (\phi_2)_0 = \frac{1}{h}\begin{bmatrix} 0.5364 \\ 0.5364 \\ -0.5006 \\ -0.5006 \\ -1.6240 \\ -1.6240 \end{bmatrix} \quad (\phi_3)_0 = \frac{1}{h}\begin{bmatrix} -0.4482 \\ -0.4482 \\ 3.6639 \\ 3.6639 \\ -4.5137 \\ -4.5137 \end{bmatrix}$$



First mode Second mode Third mode

Problem 10.23

Part a

From Problem 9.13, the mass and stiffness matrices of the system are

$$\mathbf{m} = m\begin{bmatrix} 5 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

$$\mathbf{k} = \frac{3EI}{10L^3}\begin{bmatrix} 28 & 6 & -6 \\ 6 & 7 & 3 \\ -6 & 3 & 7 \end{bmatrix}$$

Then

$$\mathbf{k}-\omega^2\mathbf{m} = \frac{3EI}{10L^3}\begin{bmatrix} 28-5\lambda & 6 & -6 \\ 6 & 7-\lambda & 3 \\ -6 & 3 & 7-\lambda \end{bmatrix} \quad \text{(a)}$$

where

$$\lambda = \frac{10mL^3}{3EI}\omega^2 \quad \text{(b)}$$

Substituting Eq. (a) in Eq. (10.2.6) gives the frequency equation:

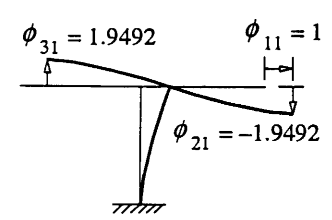
$$5\lambda^3 - 98\lambda^2 + 520\lambda - 400 = 0$$

The roots of this equation are $\lambda_1 = 0.9219$, $\lambda_2 = 8.6780$ and $\lambda_3 = 10$. The corresponding natural frequencies are

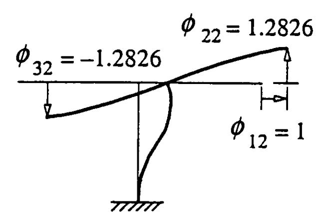
$$\omega_1 = 0.5259\sqrt{\frac{EI}{mL^3}} \quad \omega_2 = 1.6135\sqrt{\frac{EI}{mL^3}} \quad \omega_3 = 1.7321\sqrt{\frac{EI}{mL^3}} \quad \text{(c)}$$

Following the procedure used in Example 10.1, the natural modes are determined from Eq. (10.2.5):

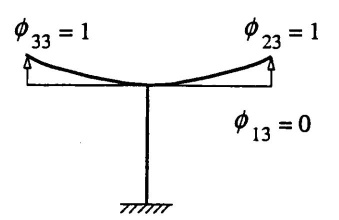
$$\phi_1 = \begin{Bmatrix} 1 \\ -1.9492 \\ 1.9492 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} 1 \\ 1.2826 \\ -1.2826 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \quad \text{(d)}$$



First mode



Second mode



Third mode

Part b

The vectors of initial displacements and velocities are

$$\mathbf{u}(0) = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{(e)}$$

Substituting **m**, **k** and Eqs. (d)-(e) in Eq. (10.8.5) gives

$$\left.\begin{aligned} q_1(0) &= \frac{\phi_1^T\mathbf{m}\mathbf{u}(0)}{\phi_1^T\mathbf{m}\phi_1} = 0.3969 \quad \dot{q}_1(0) = 0 \\ q_2(0) &= \frac{\phi_2^T\mathbf{m}\mathbf{u}(0)}{\phi_2^T\mathbf{m}\phi_2} = 0.6031 \quad \dot{q}_2(0) = 0 \\ q_3(0) &= \frac{\phi_3^T\mathbf{m}\mathbf{u}(0)}{\phi_3^T\mathbf{m}\phi_3} = 0 \quad \dot{q}_3(0) = 0 \end{aligned}\right\} \quad \text{(f)}$$

The free vibration response is given by Eq. (10.8.6) as

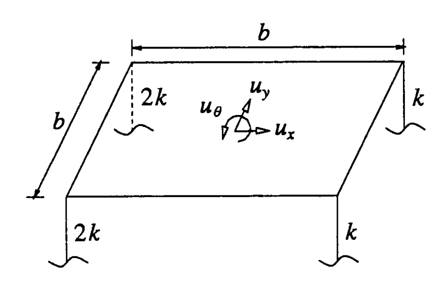
$$\mathbf{u}(t) = \sum_{n=1}^{3}\phi_n\left[q_n(0)\cos\omega_n t + \frac{\dot{q}_n(0)}{\omega_n}\sin\omega_n t\right] \quad \text{(g)}$$

Substituting for ϕ_n, $q_n(0)$ and $\dot{q}_n(0)$ gives

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} 0.3969 \\ -0.7736 \\ 0.7736 \end{Bmatrix}\cos\omega_1 t + \begin{Bmatrix} 0.6031 \\ 0.7736 \\ -0.7736 \end{Bmatrix}\cos\omega_2 t \quad \text{(h)}$$

The third mode does not contribute to the free-vibration response because the initial conditions do not contain a component in this mode.

Problem 10.24



1. *Data.*

$$m = 90 \text{ kips/g} \quad k = 1.5 \text{ kips/in.} \quad b = 25 \text{ ft}$$

2. *Determine the mass and stiffness matrices.*

From Problem 9.140,

$$\mathbf{m} = \frac{90}{386}\begin{bmatrix} 1 & & \\ & 1 & \\ & & (25 \times 12)^2/6 \end{bmatrix} = 0.2331\begin{bmatrix} 1 & & \\ & 1 & \\ & & 15{,}000 \end{bmatrix}$$

$$\mathbf{k} = 1.5\begin{bmatrix} 6 & 0 & 0 \\ & 6 & -(25 \times 12) \\ (sym) & & 3 \times (25 \times 12)^2 \end{bmatrix}$$

$$= 1.5\begin{bmatrix} 6 & 0 & 0 \\ & 6 & -300 \\ (sym) & & 270{,}000 \end{bmatrix}$$

3. *Determine natural frequencies.*

$$\det\left[\mathbf{k} - \omega^2\mathbf{m}\right] = 0$$

$$\begin{vmatrix} 9 - 0.2331\omega^2 & 0 & 0 \\ 0 & 9 - 0.2331\omega^2 & -450 \\ 0 & -450 & 405{,}000 - 3495\omega^2 \end{vmatrix} = 0$$

$$\left(9 - 0.2331\omega^2\right)\left(9 - 0.2331\omega^2\right) \times$$
$$\left(405{,}000 - 3495\omega^2 - 450^2\right) = 0$$

The roots of the above equation are

$$\omega_1^2 = 35.55 \quad \omega_2^2 = 38.63 \quad \omega_3^2 = 118.85$$

Thus the natural frequencies are

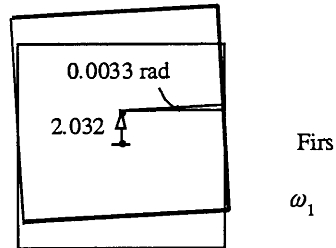
$$\omega_1 = 5.96 \quad \omega_2 = 6.21 \quad \omega_3 = 10.90$$

4. *Determine natural modes.*

$$\left(\mathbf{k} - \omega_n^2\mathbf{m}\right)\phi_n = \mathbf{0} \Rightarrow \phi_n$$

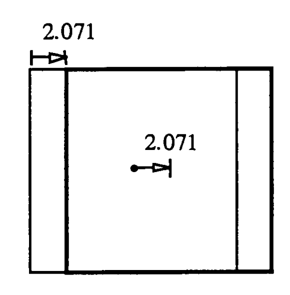
$$\Phi = \begin{bmatrix} 0 & 2.071 & 0 \\ 2.0322 & 0 & -0.3988 \\ 0.0033 & 0 & 0.0166 \end{bmatrix}$$

The natural modes are sketched next.



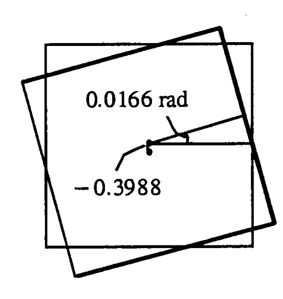
First mode

$\omega_1 = 5.96$ rads/sec



Second mode

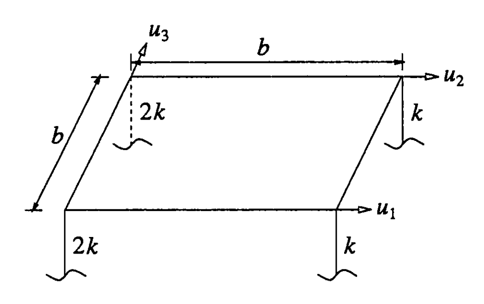
$\omega_2 = 6.21$ rads/sec



Third mode

$\omega_3 = 10.90$ rads/sec

Problem 10.25



1. *Determine the mass and stiffness matrices.*

From Problem 9.15,

$$\mathbf{m} = 0.2331\begin{bmatrix} 2/3 & -1/6 & 1/2 \\ -1/6 & 2/3 & -1/2 \\ 1/2 & -1/2 & 1 \end{bmatrix}$$

$$\mathbf{k} = 15\begin{bmatrix} 5 & -2 & 2 \\ -2 & 5 & -2 \\ 2 & -2 & 6 \end{bmatrix}$$

2. *Determine natural frequencies.*

$$\det\left[\mathbf{k} - \omega^2\mathbf{m}\right] = 0 \qquad \text{(a)}$$

The roots of Eq. (a) are

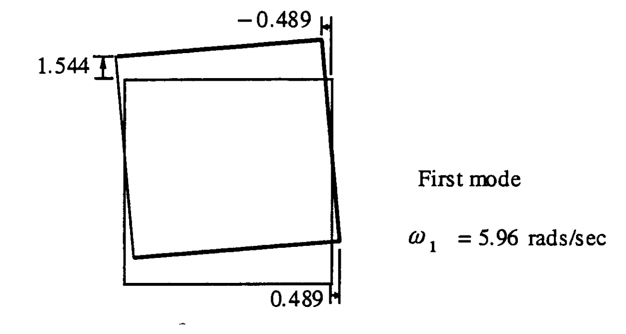
$$\omega_1 = 5.96 \qquad \omega_2 = 6.21 \qquad \omega_3 = 10.90$$

which are the same as in Problem 10.24.

3. *Determine natural modes.*

$$\overline{\Phi} = \begin{bmatrix} 0.4885 & 2.071 & 2.4889 \\ -0.4885 & 2.071 & -2.4889 \\ 1.5437 & 0 & -2.8878 \end{bmatrix} \qquad \text{(b)}$$

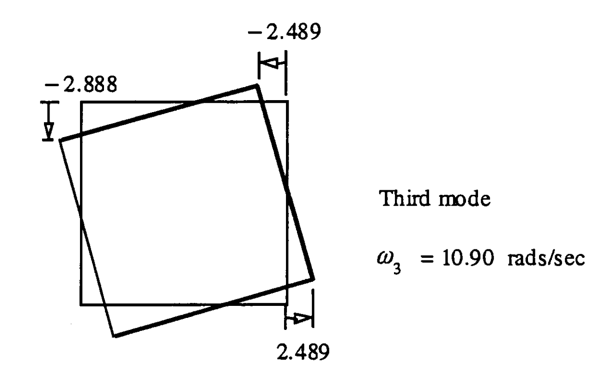
The natural modes are sketched next.



Second mode

ω_2 = 6.21 rads/sec

2.071



4. *Compare these modes with Problem 10.24.*

The two sets of DOF are related by

$$\begin{Bmatrix} u_x \\ u_y \\ u_\theta \end{Bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 1 \\ 1/b & -1/b & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{(c)}$$

or

$$\mathbf{u} = \mathbf{a}\,\overline{\mathbf{u}} \qquad \text{(d)}$$

where

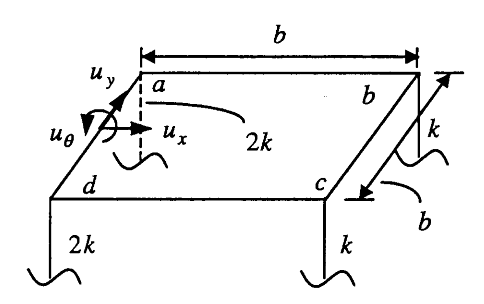
$$b = 25 \times 12 = 300 \text{ in.}$$

Substituting $\overline{\mathbf{u}} = \overline{\phi}_n$, $n = 1$, 2 and 3, from Eq. (b) in Eq. (d) gives

$$\Phi = \mathbf{a}\,\overline{\Phi} = \begin{bmatrix} 0 & 2.071 & 0 \\ 2.0332 & 0 & -0.3988 \\ 0.0033 & 0 & 0.0166 \end{bmatrix}$$

These modes are the same as in Problem 10.24.

Problem 10.26



1. *Data.*

$$m = 90 \text{ kips/g} \quad k = 1.5 \text{ kips/in.} \quad b = 25 \text{ ft}$$

2. *Determine mass and stiffness matrices.*

From Problem 9.16,

$$\mathbf{m} = \frac{90}{386}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & (25\times12)/2 \\ 0 & (25\times12)/2 & 5(25\times12)^2/12 \end{bmatrix}$$

$$= 0.2331\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 150 \\ 0 & 150 & 37{,}500 \end{bmatrix}$$

$$\mathbf{k} = 1.5\begin{bmatrix} 6 & 0 & 0 \\ & 6 & 2(25\times12) \\ (\text{sym}) & & 7(25\times12)^2/2 \end{bmatrix}$$

$$= 1.5\begin{bmatrix} 6 & 0 & 0 \\ & 6 & 600 \\ (\text{sym}) & & 315{,}000 \end{bmatrix}$$

3. *Determine natural frequencies.*

$$\det\left[\mathbf{k} - \omega^2\mathbf{m}\right] = 0 \qquad \text{(a)}$$

The roots of Eq. (a) are

$$\omega_1^2 = 35.55 \quad \omega_2^2 = 38.63 \quad \omega_3^2 = 118.85$$

Thus the natural frequencies are

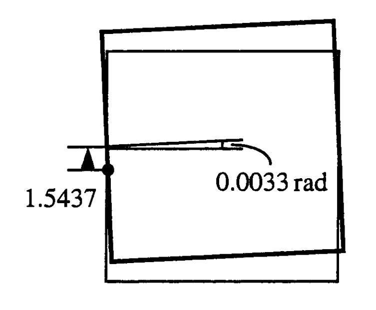
$$\omega_1 = 5.96 \quad \omega_2 = 6.21 \quad \omega_3 = 10.90$$

which are the same as in Problem 10.24.

4. *Determine natural modes.*

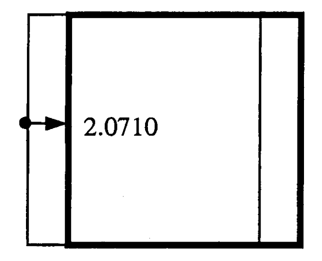
$$\overline{\mathbf{\Phi}} = \begin{bmatrix} 0 & 2.0710 & 0 \\ 1.5437 & 0 & -2.8878 \\ 0.0033 & 0 & 0.0166 \end{bmatrix} \qquad \text{(b)}$$

The natural modes are sketched next.



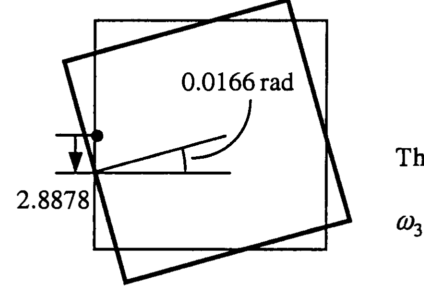
First mode

$\omega_1 = 5.96$ rad/sec



Second mode

$\omega_2 = 6.21$ rad/sec



Third mode

$\omega_3 = 10.90$ rad/sec

5. *Compare these modes with Problem 10.24.*

The two sets of DOF are related by

$$\begin{Bmatrix} u_x \\ u_y \\ u_\theta \end{Bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & b/2 \\ & & 1 \end{bmatrix}\begin{Bmatrix} \bar{u}_x \\ \bar{u}_y \\ \bar{u}_\theta \end{Bmatrix} \qquad \text{(c)}$$

or

$$\mathbf{u} = \mathbf{a}\bar{\mathbf{u}} \qquad \text{(d)}$$

where

$$b = 25\times12 = 300 \text{ in.}$$

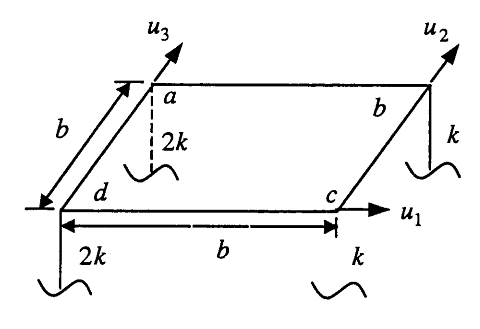
In Eq. (d), $\mathbf{u}$ represents the degrees of freedom defined in Problem 9.14, while $\bar{\mathbf{u}}$ represents the degrees of freedom defined in Problem 9.16.

Substituting $\bar{\mathbf{u}} = \bar{\phi}_n$, $n = 1, 2$ and 3, from Eq. (b) in Eq. (d) gives

$$\mathbf{\Phi} = \mathbf{a}\bar{\mathbf{\Phi}} = \begin{bmatrix} 0 & 2.071 & 0 \\ 2.0322 & 0 & -0.3988 \\ 0.0033 & 0 & 0.0166 \end{bmatrix}$$

These modes are the same as in Problem 10.24.

Problem 10.27



1. *Determine mass and stiffness matrices.*

From Problem 9.17,

$$\mathbf{m} = 0.2331\begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 2/3 & -1/6 \\ 1/2 & -1/6 & 2/3 \end{bmatrix}$$

$$\mathbf{k} = 1.5\begin{bmatrix} 6 & -3 & 3 \\ -3 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

2. *Determine natural frequencies.*

$$\det\left[\mathbf{k} - \omega^2\mathbf{m}\right] = 0 \qquad \text{(a)}$$

The roots of Eq. (a) are

$$\omega_1^2 = 35.55 \qquad \omega_2^2 = 38.63 \qquad \omega_3^2 = 118.85$$

Thus the natural frequencies are

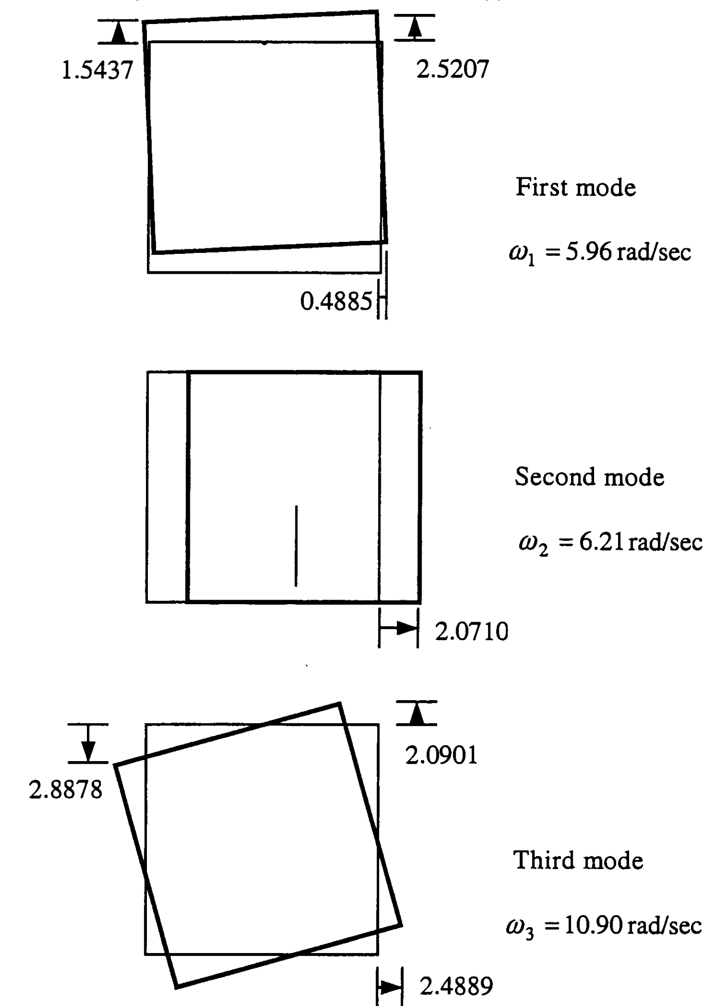
$$\omega_1 = 5.96 \qquad \omega_2 = 6.21 \qquad \omega_3 = 10.90$$

which are the same as in Problem 10.24.

3. *Determine natural modes.*

$$\overline{\mathbf{\Phi}} = \begin{bmatrix} 0.4885 & 2.0710 & 2.4889 \\ 2.5207 & 0 & 2.0901 \\ 1.5437 & 0 & -2.8878 \end{bmatrix} \qquad \text{(b)}$$

The natural modes are sketched next.



4. *Compare these modes with Problem 10.24.*

The two sets of DOF are related by

$$\begin{Bmatrix} u_x \\ u_y \\ u_\theta \end{Bmatrix} = \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/b & -1/b \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{(c)}$$

or

$$\mathbf{u} = \mathbf{a}\overline{\mathbf{u}} \qquad \text{(d)}$$

where

$$b = 25 \times 12 = 300 \text{ in.}$$

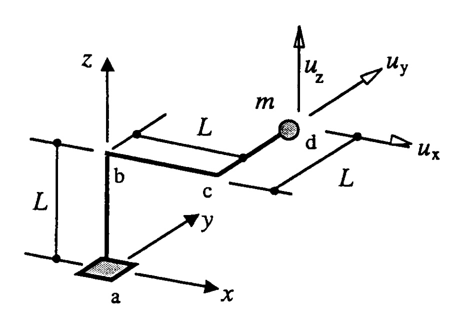
In Eq. (d), $\mathbf{u}$ represents the degrees of freedom defined in Problem 9.14, while $\overline{\mathbf{u}}$ represents the degrees of freedom defined in Problem 9.17.

Substituting $\overline{\mathbf{u}} = \overline{\phi}_n$, $n = 1, 2$ and 3, from Eq. (b) in Eq. (d) gives

$$\mathbf{\Phi} = \mathbf{a}\overline{\mathbf{\Phi}} = \begin{bmatrix} 0 & 2.071 & 0 \\ 2.0322 & 0 & -0.3988 \\ 0.0033 & 0 & 0.0166 \end{bmatrix}$$

These modes are the same as in Problem 10.24.

Problem 10.28



1. *Determine mass and stiffness matrices.*

From solution to Problem 9.18:

$$\mathbf{m} = \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix}$$

and

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 0.9283 & 0.9088 & 0.2345 \\ 0.9088 & 1.4294 & 0.2985 \\ 0.2345 & 0.2985 & 0.3234 \end{bmatrix}$$

assuming $GJ = \frac{4}{5}EI$

2. *Determine the natural frequencies.*

$$\det[\mathbf{k} - \omega^2\mathbf{m}] = 0$$

$$\frac{EI}{L^3}\begin{vmatrix} 0.9283-\omega^2\left(\frac{mL^3}{EI}\right) & 0.9088 & 0.2345 \\ 0.9088 & 1.4294-\omega^2\left(\frac{mL^3}{EI}\right) & 0.2985 \\ 0.2345 & 0.2985 & 0.3234-\omega^2\left(\frac{mL^3}{EI}\right) \end{vmatrix} = 0$$

Solving for the roots of the characteristic equation yields the natural frequencies of the system.

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}}$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}}$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}}$$

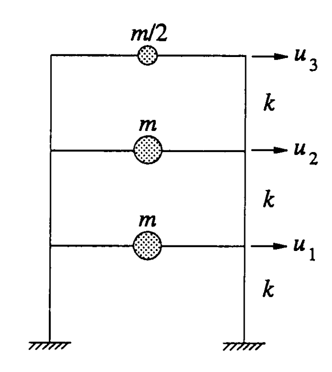
3. *Determine the natural modes.*

$$(\mathbf{k} - \omega_n^2\mathbf{m})\phi_n = 0 \quad \Rightarrow \quad \phi_n$$

$$\Phi = \begin{bmatrix} 0.7767 & -0.2084 & 0.5943 \\ -0.4923 & 0.3875 & 0.7794 \\ -0.3928 & -0.8980 & 0.1984 \end{bmatrix}$$

Note: $\phi_n^T\phi_n = 1$.

Problem 10.29



1. *Data:*

$$w = 100 \text{ kips} \Rightarrow m = 0.2588 \text{ kip-sec}^2/\text{in.}$$

$$k = 326.32 \text{ kips/in.}$$

$$\mathbf{m} = 0.2588\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}$$

$$\mathbf{k} = 326.32\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Inverse iteration equations:

$$\mathbf{k}\,\overline{\mathbf{x}}_{j+1} = \mathbf{m}\,\mathbf{x}_j \qquad \text{(a)}$$

$$\lambda^{(j+1)} = \frac{\overline{\mathbf{x}}_{j+1}^T \mathbf{k}\,\overline{\mathbf{x}}_{j+1}}{\overline{\mathbf{x}}_{j+1}^T \mathbf{m}\,\overline{\mathbf{x}}_{j+1}} \qquad \text{(b)}$$

Implement Eqs. (a) and (b) to obtain Table P10.29.

Table P10.29

Iteration	$\mathbf{x}_j$	$\overline{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$	$\begin{Bmatrix} 0.0036 \\ 0.0063 \\ 0.0075 \end{Bmatrix}$	338.68	$\begin{Bmatrix} 0.7777 \\ 1.3826 \\ 1.6418 \end{Bmatrix}$
2	$\begin{Bmatrix} 0.7777 \\ 1.3826 \\ 1.6418 \end{Bmatrix}$	$\begin{Bmatrix} 0.0024 \\ 0.0041 \\ 0.0048 \end{Bmatrix}$	337.87	$\begin{Bmatrix} 0.7989 \\ 1.3894 \\ 1.6094 \end{Bmatrix}$
3	$\begin{Bmatrix} 0.7989 \\ 1.3894 \\ 1.6094 \end{Bmatrix}$	$\begin{Bmatrix} 0.0024 \\ 0.0041 \\ 0.0048 \end{Bmatrix}$	337.86	$\begin{Bmatrix} 0.8020 \\ 1.3899 \\ 1.6055 \end{Bmatrix}$

The fundamental natural frequency and mode are

$$\omega_1 = \sqrt{337.86} = 18.3809$$

$$\phi_1 = \langle 0.8020 \quad 1.3899 \quad 1.6055 \rangle^T$$

Problem 10.30

Excitation frequency:

$$f = 430 \text{ rpm} = \frac{430}{60} = 7.17 \text{ Hz}$$

$$\omega = 2\pi f = 45.03 \text{ rads/sec}$$

Shifted eigenvalue problem:

$$\breve{\mathbf{k}}\,\phi = \breve{\lambda}\,\mathbf{m}\,\phi$$

$$\breve{\mathbf{k}} = \mathbf{k} - \mu\mathbf{m} \qquad \breve{\lambda} = \lambda - \mu$$

and **m** and **k** are given in Problem 10.29.

Inverse iteration equations:

$$\breve{\mathbf{k}}\,\overline{\mathbf{x}}_{j+1} = \mathbf{m}\,\mathbf{x}_j$$

$$\breve{\lambda}^{(j+1)} = \frac{\overline{\mathbf{x}}_{j+1}^T\,\breve{\mathbf{k}}\,\overline{\mathbf{x}}_{j+1}}{\overline{\mathbf{x}}_{j+1}^T\,\mathbf{m}\,\overline{\mathbf{x}}_{j+1}}$$

Use $\mu = \omega^2 = (45.03)^2 = 2027.7$ as a shift and implement the inverse iteration equations to obtain Table P10.30.

Table P10.30

Iteration	$\mathbf{x}_j$	μ	$\overline{\mathbf{x}}_{j+1}$	$\breve{\lambda}^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{Bmatrix} 1 \\ 0 \\ -0.5 \end{Bmatrix}$	2027.7	$\begin{Bmatrix} 0.0017 \\ -0.0001 \\ -0.0017 \end{Bmatrix}$	2517.3	$\begin{Bmatrix} 1.5839 \\ -0.1323 \\ -1.6357 \end{Bmatrix}$
2	$\begin{Bmatrix} 1.5839 \\ -0.1323 \\ -1.6357 \end{Bmatrix}$	2027.7	$\begin{Bmatrix} 0.0033 \\ 0.0000 \\ -0.0032 \end{Bmatrix}$	2521.3	$\begin{Bmatrix} 1.6239 \\ 0.0143 \\ -1.5663 \end{Bmatrix}$
3	$\begin{Bmatrix} 1.6239 \\ 0.0143 \\ -1.5663 \end{Bmatrix}$	2027.7	$\begin{Bmatrix} 0.0032 \\ 0.0000 \\ -0.0033 \end{Bmatrix}$	2521.7	$\begin{Bmatrix} 1.6020 \\ -0.0087 \\ -1.6110 \end{Bmatrix}$
4	$\begin{Bmatrix} 1.6020 \\ -0.0087 \\ -1.6110 \end{Bmatrix}$	2027.7	$\begin{Bmatrix} 0.0033 \\ 0.0000 \\ -0.0032 \end{Bmatrix}$	2521.8	$\begin{Bmatrix} 1.6063 \\ 0.0017 \\ -1.6023 \end{Bmatrix}$

The natural frequency of the structure closest to the machine frequency is

$$\omega_n = \sqrt{2521.8} = 50.2174$$

Problem 10.31

Shifted eigenvalue problem:

$$\breve{\mathbf{k}}\,\phi = \breve{\lambda}\,\mathbf{m}\,\phi$$

$$\breve{\mathbf{k}} = \mathbf{k} - \mu\mathbf{m} \qquad \breve{\lambda} = \lambda - \mu$$

and **m** and **k** are given in Problem 10.30.

Inverse iteration equations:

$$\breve{\mathbf{k}}\,\overline{\mathbf{x}}_{j+1} = \mathbf{m}\,\mathbf{x}_j$$

$$\lambda^{(j+1)} = \frac{\overline{\mathbf{x}}_{j+1}^T\breve{\mathbf{k}}\,\overline{\mathbf{x}}_{j+1}}{\overline{\mathbf{x}}_{j+1}^T\mathbf{m}\,\overline{\mathbf{x}}_{j+1}}$$

1. *First mode.*

Select $\mu = 300$ and implement the inverse iteration equations to obtain Table P10.31a.

Table P10.31a: First Mode

Iteration	$\mathbf{x}_j$	μ	$\overline{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{Bmatrix}1\\2\\3\end{Bmatrix}$	300	$\begin{Bmatrix}0.0327\\0.0569\\0.0659\end{Bmatrix}$	337.869	$\begin{Bmatrix}0.7993\\1.3891\\1.6096\end{Bmatrix}$
2	$\begin{Bmatrix}0.7993\\1.3891\\1.6096\end{Bmatrix}$	300	$\begin{Bmatrix}0.0212\\0.0367\\0.0424\end{Bmatrix}$	337.856	$\begin{Bmatrix}0.8024\\1.3900\\1.6051\end{Bmatrix}$

The fundamental natural frequency and mode are

$$\omega_1 = \sqrt{337.856} = 18.3809$$

$$\phi_1 = \langle 0.8024 \quad 1.3900 \quad 1.6051\rangle^T$$

2. *Second mode.*

Select $\mu = 2000$ and implement the inverse iteration equations to obtain Table P10.31b.

Table P10.31b: Second Mode

Iteration	$\mathbf{x}_j$	μ	$\overline{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{Bmatrix}1\\0\\-0.5\end{Bmatrix}$	2000	$\begin{Bmatrix}0.0016\\-0.0001\\-0.0016\end{Bmatrix}$	2516.5	$\begin{Bmatrix}1.5813\\-0.1405\\-1.6394\end{Bmatrix}$
2	$\begin{Bmatrix}1.5813\\-0.1405\\-1.6394\end{Bmatrix}$	2000	$\begin{Bmatrix}0.0031\\0.0000\\-0.0030\end{Bmatrix}$	2521.1	$\begin{Bmatrix}1.6264\\0.0171\\-1.5611\end{Bmatrix}$
3	$\begin{Bmatrix}1.6264\\0.0171\\-1.5611\end{Bmatrix}$	2000	$\begin{Bmatrix}0.0031\\0.0000\\-0.0031\end{Bmatrix}$	2521.7	$\begin{Bmatrix}1.6011\\-0.0106\\-1.6126\end{Bmatrix}$

The second natural frequency and mode are

$$\omega_2 = \sqrt{2521.7} = 50.2167$$

$$\phi_2 = \langle 1.6011 \quad -0.0106 \quad -1.6126\rangle^T$$

3. *Third mode.*

Select $\mu = 4000$ and implement the inverse iteration equations to obtain Table P10.31c.

Table P10.31c: Third Mode

Iteration	$\mathbf{x}_j$	μ	$\overline{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{Bmatrix}1\\-1\\1\end{Bmatrix}$	4000	$\begin{Bmatrix}0.0006\\-0.0015\\0.0020\end{Bmatrix}$	4669.7	$\begin{Bmatrix}0.5813\\-1.3977\\1.7735\end{Bmatrix}$
2	$\begin{Bmatrix}0.5813\\-1.3977\\1.7735\end{Bmatrix}$	4000	$\begin{Bmatrix}0.0013\\-0.0019\\0.0021\end{Bmatrix}$	4697.6	$\begin{Bmatrix}0.9010\\-1.3837\\1.5084\end{Bmatrix}$
3	$\begin{Bmatrix}0.9010\\-1.3837\\1.5084\end{Bmatrix}$	4000	$\begin{Bmatrix}0.0011\\-0.0020\\0.0023\end{Bmatrix}$	4703.9	$\begin{Bmatrix}0.7550\\-1.3901\\1.6503\end{Bmatrix}$
4	$\begin{Bmatrix}0.7550\\-1.3901\\1.6503\end{Bmatrix}$	4000	$\begin{Bmatrix}0.0012\\-0.0020\\0.0022\end{Bmatrix}$	4705.3	$\begin{Bmatrix}0.8248\\-1.3897\\1.5827\end{Bmatrix}$

The third natural frequency and mode are

$$\omega_3 = \sqrt{4705.3} = 68.5952$$

$$\phi_3 = \langle 0.8248 \quad -1.3897 \quad 1.5827\rangle^T$$

4. *Compare with exact results.*

$$\omega_1 = 18.4883 \quad \phi_1 = \langle 0.8024 \quad 1.3898 \quad 1.6048 \rangle^T$$

$$\omega_2 = 50.5111 \quad \phi_2 = \langle 1.6011 \quad 0 \quad -1.6011 \rangle^T$$

$$\omega_3 = 68.9994 \quad \phi_3 = \langle 0.8248 \quad -1.4286 \quad 1.6496 \rangle^T$$

Problem 10.32

Implement the iteration procedure of Eqs. (10.14.3) to (10.14.5) with the starting shift computed from Eq. (10.14.4) using the starting vector. The results are summarized below.

1. *First mode.*

Table P10.32a: First Mode

Iteration	$\mathbf{x}_j$	μ	$\overline{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$	398.18	$\begin{Bmatrix} -0.0208 \\ -0.0358 \\ -0.0410 \end{Bmatrix}$	337.89	$\begin{Bmatrix} -0.8078 \\ -1.3913 \\ -1.5973 \end{Bmatrix}$
2	$\begin{Bmatrix} -0.8078 \\ -1.3913 \\ -1.5973 \end{Bmatrix}$	337.89	$\begin{Bmatrix} 22.2645 \\ 38.5633 \\ 44.5291 \end{Bmatrix}$	337.86	$\begin{Bmatrix} 0.8025 \\ 1.3900 \\ 1.6050 \end{Bmatrix}$

The fundamental natural frequency and mode are

$$\omega_1 = \sqrt{337.86} = 18.3809$$

$$\phi_1 = \langle 0.8025 \quad 1.3900 \quad 1.6050 \rangle^T$$

2. *Second mode.*

Table P10.32b: Second Mode

Iteration	$\mathbf{x}_j$	μ	$\overline{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{Bmatrix} 1 \\ 0.5 \\ -0.5 \end{Bmatrix}$	2063.3	$\begin{Bmatrix} 0.0017 \\ -0.0022 \\ -0.0021 \end{Bmatrix}$	2478.1	$\begin{Bmatrix} 1.4524 \\ -0.1647 \\ -1.8587 \end{Bmatrix}$
2	$\begin{Bmatrix} 1.4524 \\ -0.1647 \\ -1.8587 \end{Bmatrix}$	2478.1	$\begin{Bmatrix} 0.0364 \\ 0.0001 \\ -0.0363 \end{Bmatrix}$	2521.8	$\begin{Bmatrix} 1.6070 \\ 0.0048 \\ -1.6010 \end{Bmatrix}$

The second natural frequency and mode are

$$\omega_2 = \sqrt{2521.8} = 50.2175$$

$$\phi_2 = \langle 1.6070 \quad 0.0048 \quad -1.6010 \rangle^T$$

3. *Third mode.*

Table P10.32c: Third Mode

Iteration	$\mathbf{x}_j$	μ	$\overline{\mathbf{x}}_{j+1}$	$\lambda^{(j+1)}$	$\mathbf{x}_{j+1}$
1	$\begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$	4539.2	$\begin{Bmatrix} 0.0036 \\ -0.0065 \\ 0.0076 \end{Bmatrix}$	4704.6	$\begin{Bmatrix} 0.7645 \\ -1.3936 \\ 1.6355 \end{Bmatrix}$
2	$\begin{Bmatrix} 0.7645 \\ -1.3936 \\ 1.6355 \end{Bmatrix}$	4704.6	$\begin{Bmatrix} 0.7275 \\ -1.2600 \\ 1.4549 \end{Bmatrix}$	4705.7	$\begin{Bmatrix} 0.8025 \\ -1.3900 \\ 1.6050 \end{Bmatrix}$

The third natural frequency and mode are

$$\omega_3 = \sqrt{4705.7} = 68.5981$$

$$\phi_3 = \langle 0.8025 \quad -1.3900 \quad 1.6050 \rangle^T$$

Problem 11.1

1. *Set up mass and stiffness matrices.*

$$\mathbf{m} = \frac{1}{386}\begin{bmatrix} 100 & & \\ & 100 & \\ & & 50 \end{bmatrix}$$

$$\mathbf{k} = \frac{168}{9}\begin{bmatrix} 16 & -7 & 0 \\ -7 & 10 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

2. *Determine a_0 and a_1 from Eq. (11.4.9).*

$$\begin{bmatrix} \dfrac{1}{12.01} & 12.01 \\ \dfrac{1}{38.90} & 38.90 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = 2\begin{Bmatrix} 0.05 \\ 0.05 \end{Bmatrix}$$

$$a_0 = 0.9177 \qquad a_1 = 1.964 \times 10^{-3}$$

3. *Evaluate the damping matrix.*

$$\mathbf{c} = a_0\mathbf{m} + a_1\mathbf{k}$$

$$= \begin{bmatrix} 0.824 & -0.257 & 0 \\ & 0.604 & -0.110 \\ (sym) & & 0.229 \end{bmatrix}$$

4. *Compute ζ_2 from Eq. (11.4.8)*

$$\zeta_2 = \frac{a_0}{2}\frac{1}{\omega_2} + \frac{a_1}{2}\omega_2 = 0.0430$$

Problem 11.2

1. *Caughey series for a 3-DOF system:*

$$\mathbf{c} = a_0\mathbf{m} + a_1\mathbf{k} + a_2\mathbf{k}\,\mathbf{m}^{-1}\,\mathbf{k} \qquad \text{(a)}$$

(b) *Determine a_0, a_1 and a_2 from Eq. (11.4.14).*

$$\begin{bmatrix} \frac{1}{12.01} & 12.01 & (12.01)^3 \\ \frac{1}{25.47} & 25.47 & (25.47)^3 \\ \frac{1}{38.90} & 38.90 & (38.90)^3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = 2 \begin{Bmatrix} 0.05 \\ 0.05 \\ 0.05 \end{Bmatrix}$$

$a_0 = 0.7400 \quad a_1 = 3.3137 \times 10^{-3} \quad a_2 = -8.1417 \times 10^{-7}$

(c) *Evaluate* **c**.

Substituting a_0, a_1 and a_2 in Eq. (a) gives

$$\mathbf{c} = \begin{bmatrix} 0.848 & -0.234 & -0.023 \\ & 0.628 & -0.133 \\ (sym) & & 0.252 \end{bmatrix}$$

Problem 11.3

$$\mathbf{c} = \mathbf{m}\left[\sum_{n=1}^{N} \frac{2\zeta_n\omega_n}{M_n}\phi_n\,\phi_n^T\right]\mathbf{m} \qquad \text{(a)}$$

1. *Determine the individual terms in Eq. (a).*

$$\mathbf{c}_1 = \frac{2\,(0.05)\,(12.01)}{1}\,\mathbf{m}\,\phi_1\,\phi_1^T\,\mathbf{m}$$

$$= \begin{bmatrix} 0.0328 & 0.0655 & 0.0491 \\ & 0.1310 & 0.0983 \\ (sym) & & 0.0737 \end{bmatrix}$$

$$\mathbf{c}_3 = \frac{2\,(0.05)\,(38.90)}{1}\,\mathbf{m}\phi_3\phi_3^T\mathbf{m}$$

$$= \begin{bmatrix} 0.6500 & -0.4643 & 0.0929 \\ & 0.3316 & -0.0663 \\ (sym) & & 0.0133 \end{bmatrix}$$

2. *Determine* **c**.

$$\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_3$$

$$= \begin{bmatrix} 0.683 & -0.399 & 0.142 \\ & 0.463 & 0.032 \\ (sym) & & 0.087 \end{bmatrix}$$

3. *Check damping in the second mode.*

$$C_2 = \phi_2^T\,\mathbf{c}\,\phi_2 = 0$$

Therefore the second mode is undamped.

Problem 11.4

1. *Determine individual terms in Eq. (a) of Problem 11.3.*

The first and third terms $\mathbf{c}_1$ and $\mathbf{c}_3$ are already computed in Problem 11.3 and

$$\mathbf{c}_2 = \frac{2\,(0.05)\,(25.47)}{1}\,\mathbf{m}\,\phi_2\,\phi_2^T\,\mathbf{m}$$

$$= \begin{bmatrix} 0.1651 & 0.1651 & -0.1650 \\ & 0.1652 & -0.1650 \\ (sym) & & 0.1650 \end{bmatrix}$$

2. *Determine* $\mathbf{c}$.

$$\mathbf{c} = \sum_{n=1}^{3} \mathbf{c}_n = \begin{bmatrix} 0.848 & -0.234 & -0.023 \\ & 0.628 & -0.133 \\ (sym) & & 0.252 \end{bmatrix}$$

Problem 12.1

Part a

The equations of motion are

$$\begin{bmatrix} m & \\ & m/2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_o \end{Bmatrix} \sin \omega t \quad \text{(a)}$$

(i) *Direct Solution*

The steady-state solution is assumed as

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} \sin \omega t \quad \text{(b)}$$

Substituting Eq. (b) into Eq. (a) gives

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2/2 \end{bmatrix} \begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_o \end{Bmatrix}$$

$$\begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2/2 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ p_o \end{Bmatrix}$$

$$= \frac{p_o}{(m^2/2)(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \begin{Bmatrix} k \\ 2k - m\omega^2 \end{Bmatrix}$$

where, from Problem 10.6, $\omega_1 = 0.765\sqrt{k/m}$ and $\omega_2 = 1.848\sqrt{k/m}$, or

$$u_{1o} = \frac{p_o/k}{\left[1 - (\omega/\omega_1)^2\right]\left[1 - (\omega/\omega_2)^2\right]}$$

$$u_{2o} = \frac{2p_o(2k - m\omega^2)}{m^2\omega_1^2\omega_2^2\left[1 - (\omega/\omega_1)^2\right]\left[1 - (\omega/\omega_2)^2\right]}$$

(ii) *Modal Analysis*

From Problem 10.6, the following data are available:

$$\omega_1 = 0.765\sqrt{k/m} \qquad \omega_2 = 1.848\sqrt{k/m}$$

$$\phi_1 = \langle 0.707 \quad 1 \rangle^T \qquad \phi_2 = \langle -0.707 \quad 1 \rangle^T$$

From Eqs. (12.3.4) and (10.4.7), M_n, K_n and $P_n(t)$ are obtained:

$$M_1 = m \qquad M_2 = m$$

$$K_1 = 0.586k \qquad K_2 = 3.414k$$

$$P_1(t) = p_o \sin \omega t \qquad P_2(t) = p_o \sin \omega t$$

1. *Set up modal equations.*

$$M_n\ddot{q}_n + K_n q_n = P_{no} \sin \omega t; \quad P_{no} = p_o \quad \text{(c)}$$

2. *Solve modal equations.*

$$q_1(t) = \frac{p_o}{0.586k}\mathcal{C}_1 \sin \omega t; \; \mathcal{C}_1 = \frac{1}{1 - (\omega/\omega_1)^2} \quad \text{(d)}$$

$$q_2(t) = \frac{p_o}{3.414k}\mathcal{C}_2 \sin \omega t; \; \mathcal{C}_2 = \frac{1}{1 - (\omega/\omega_2)^2} \quad \text{(e)}$$

3. *Determine modal responses.*

$$\mathbf{u}_1(t) = \phi_1 q_1(t) \qquad \mathbf{u}_2(t) = \phi_2 q_2(t) \quad \text{(f)}$$

4. *Combine modal responses.*

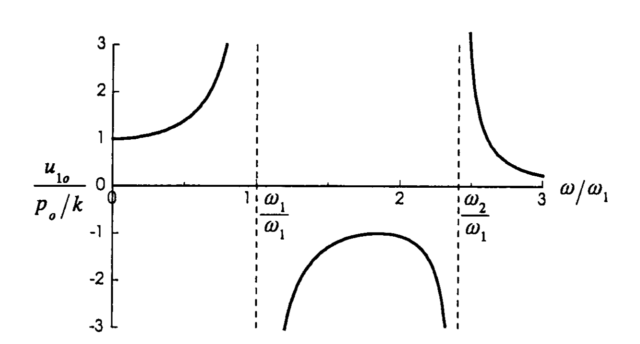
$$u_1(t) = \frac{p_o}{k}(1.207\mathcal{C}_1 - 0.207\mathcal{C}_2) \sin \omega t$$

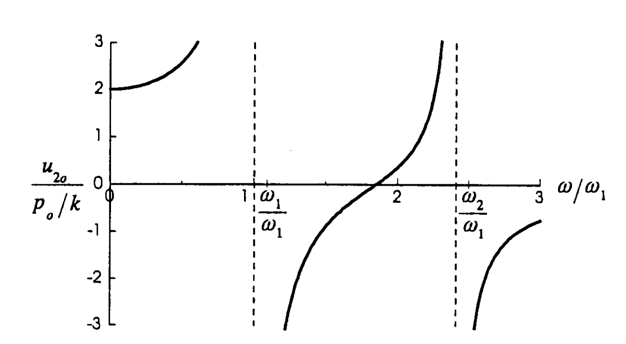
$$u_2(t) = \frac{p_o}{k}(1.707\mathcal{C}_1 + 0.293\mathcal{C}_2) \sin \omega t$$

Part b

By algebraic manipulation it can be shown that these results are equivalent to those obtained by solving the coupled equations.

***Part c**: Frequency Response Curves*





Problem 12.2

***Part a**: Story shears from displacements.*

$$V_{1n}(t) = ku_{1n}(t) = k\,\phi_{1n}q_n(t)$$

$$V_{2n}(t) = k\left[u_{2n}(t) - u_{1n}(t)\right] = k\left[\phi_{2n} - \phi_{1n}\right]q_n(t)$$

From Problem 12.1,

$$\omega_1 = 0.765\sqrt{k/m} \qquad \omega_2 = 1.848\sqrt{k/m}$$

$$\phi_1 = \langle 0.707 \quad 1\rangle^T \qquad \phi_2 = \langle -0.707 \quad 1\rangle^T$$

$$q_1(t) = 1.707\frac{p_o}{k}\mathcal{C}_1 \sin \omega t$$

$$q_2(t) = 0.293\frac{p_o}{k}\mathcal{C}_2 \sin \omega t$$

First mode responses:

$$V_{11}(t) = 1.207 p_o \mathcal{C}_1 \sin \omega t$$

$$V_{21}(t) = 0.5 p_o \mathcal{C}_1 \sin \omega t$$

Second mode responses:

$$V_{12}(t) = -0.207 p_o \mathcal{C}_2 \sin \omega t$$

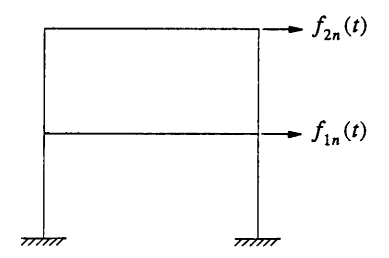
$$V_{22}(t) = 0.5 p_o \mathcal{C}_2 \sin \omega t$$

Total responses:

$$V_1(t) = V_{11}(t) + V_{12}(t) = (1.207\mathcal{C}_1 - 0.207\mathcal{C}_2)\, p_o \sin \omega t$$

$$V_2(t) = V_{21}(t) + V_{22}(t) = 0.5(\mathcal{C}_1 + \mathcal{C}_2)\, p_o \sin \omega t$$

***Part b**: Story shears from equivalent static forces.*



The equivalent static forces are

$$f_{jn}(t) = \omega_n^2 m_j\, \phi_{jn} q_n(t)$$

For first mode ($n = 1$),

$$f_{11}(t) = \omega_1^2 m\,(0.707)\, q_1(t) = 0.707 p_o\, \mathcal{C}_1 \sin \omega t$$

$$f_{21}(t) = \omega_1^2\left(\frac{m}{2}\right)(1)\, q_1(t) = 0.5 p_o\, \mathcal{C}_1 \sin \omega t$$

For second mode ($n = 2$),

$$f_{12}(t) = \omega_2^2 m\,(-0.707)\, q_2(t) = -0.707 p_o\, \mathcal{C}_2 \sin \omega t$$

$$f_{22}(t) = \omega_2^2\left(\frac{m}{2}\right)(1)\, q_2(t) = 0.5 p_o\, \mathcal{C}_2 \sin \omega t$$

Static analysis of the structure shown in the figure gives the story shears:

$$V_{1n}(t) = f_{1n}(t) + f_{2n}(t) \qquad V_{2n}(t) = f_{2n}(t)$$

Substituting for $f_{jn}(t)$ and combining modal responses, we get the story shears:

$$V_1(t) = V_{11}(t) + V_{12}(t) = (1.207\mathcal{C}_1 - 0.207\mathcal{C}_2)\, p_o \sin \omega t$$

$$V_2(t) = V_{21}(t) + V_{22}(t) = 0.5(\mathcal{C}_1 + \mathcal{C}_2)\, p_o \sin \omega t$$

These are identical to those obtained in part (a) directly from displacements.

Problem 12.3

1. *Set up modal equations.*

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = P_{no} \sin \omega t \qquad \text{(a)}$$

where M_n, K_n and P_{no} are available (Problem 12.1) and C_n is known in terms of ζ_n.

2. *Solve modal equations.*

$$q_n(t) = \frac{P_{no}}{K_n}\left[\mathcal{C}_n \sin \omega t + \mathcal{D}_n \cos \omega t\right] \qquad \text{(b)}$$

where

$$\begin{aligned} \mathcal{C}_n &= \frac{1 - (\omega/\omega_n)^2}{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta_n \omega/\omega_n\right]^2} \\ \mathcal{D}_n &= \frac{-2\zeta_n \omega/\omega_n}{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta_n \omega/\omega_n\right]^2} \end{aligned} \qquad \text{(c)}$$

Substituting for P_{no} and K_n into Eq. (b) gives

$$\begin{aligned} q_1(t) &= \frac{p_o}{0.586k}\left[\mathcal{C}_1 \sin \omega t + \mathcal{D}_1 \cos \omega t\right] \\ &= 1.707\frac{p_o}{k}\left[\mathcal{C}_1 \sin \omega t + \mathcal{D}_1 \cos \omega t\right] \end{aligned}$$

$$\begin{aligned} q_2(t) &= \frac{p_o}{3.414k}\left[\mathcal{C}_2 \sin \omega t + \mathcal{D}_2 \cos \omega t\right] \\ &= 0.293\frac{p_o}{k}\left[\mathcal{C}_2 \sin \omega t + \mathcal{D}_2 \cos \omega t\right] \end{aligned}$$

3. *Combine modal responses.*

$$\begin{aligned} u_1(t) &= \phi_{11} q_1(t) + \phi_{12} q_2(t) \\ &= \frac{p_o}{k}\left[(1.207\mathcal{C}_1 - 0.207\mathcal{C}_2) \sin \omega t \right. \\ &\quad \left. + (1.207\mathcal{D}_1 - 0.207\mathcal{D}_2) \cos \omega t\right] \end{aligned}$$

$$\begin{aligned} u_2(t) &= \phi_{21} q_1(t) + \phi_{22} q_2(t) \\ &= \frac{p_o}{k}\left[(1.707\mathcal{C}_1 + 0.293\mathcal{C}_2) \sin \omega t \right. \\ &\quad \left. + (1.707\mathcal{D}_1 + 0.293\mathcal{D}_2) \cos \omega t\right] \end{aligned}$$

4. *Determine displacement amplitudes.*

$$u_{1o} = \frac{p_o}{k}\sqrt{(1.207\mathcal{C}_1 - 0.207\mathcal{C}_2)^2 + (1.207\mathcal{D}_1 - 0.207\mathcal{D}_2)^2}$$

$$u_{2o} = \frac{p_o}{k}\sqrt{(1.707\mathcal{C}_1 + 0.293\mathcal{C}_2)^2 + (1.707\mathcal{D}_1 + 0.293\mathcal{D}_2)^2}$$

Problem 12.4

1. *Set up equations of motion.*

$$\begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_o\delta(t) \\ 0 \end{Bmatrix} \qquad \text{(a)}$$

2. *Set up modal equations.*

From Problem 12.1,

$$\omega_1 = 0.765\sqrt{k/m} \qquad \omega_2 = 1.848\sqrt{k/m}$$

$$\phi_1 = \langle 0.707 \quad 1\rangle^T \qquad \phi_2 = \langle -0.707 \quad 1\rangle^T$$

$$M_1 = m \qquad M_2 = m$$

$$K_1 = 0.586k \qquad K_2 = 3.414k$$

The generalized modal forces are

$$P_1(t) = \phi_1^T\mathbf{p}(t) = 0.707\,p_o\delta(t)$$

$$P_2(t) = \phi_2^T\mathbf{p}(t) = -0.707\,p_o\delta(t)$$

Substituting these in Eq. (12.3.3) gives the modal equations:

$$M_1\ddot{q}_1 + K_1q_1 = 0.707\,p_o\delta(t) \qquad \text{(b.1)}$$

$$M_2\ddot{q}_1 + K_2q_1 = -0.707\,p_o\delta(t) \qquad \text{(b.2)}$$

3. *Solve modal equations.*

For an SDF system subjected to an impulse force, the governing equation is

$$m\ddot{u} + ku = p_o\delta(t) \qquad \text{(c)}$$

and its solution is

$$u(t) = \frac{p_o}{m\omega_n}\sin\omega_n t \qquad \text{(d)}$$

Adapting this result, the solutions of Eqs. (b.1) and (b.2) are

$$q_1(t) = \frac{0.707\,p_o}{M_1\omega_1}\sin\omega_1 t;\; q_2(t) = \frac{-0.707\,p_o}{M_2\omega_2}\sin\omega_2 t$$

Substituting for M_n gives

$$q_1(t) = \frac{0.707\,p_o}{m\omega_1}\sin\omega_1 t;\; q_2(t) = \frac{-0.707\,p_o}{m\omega_2}\sin\omega_2 t \qquad \text{(e)}$$

4. *Combine modal responses.*

$$\mathbf{u}(t) = \sum_{n=1}^{2}\phi_n q_n(t)$$

$$u_1(t) = \phi_{11}q_1(t) + \phi_{12}q_2(t) = \frac{p_o}{2m}\left(\frac{\sin\omega_1 t}{\omega_1} + \frac{\sin\omega_2 t}{\omega_2}\right) \qquad \text{(f)}$$

$$u_2(t) = \phi_{21}q_1(t) + \phi_{22}q_2(t) = \frac{0.707\,p_o}{m}\left(\frac{\sin\omega_1 t}{\omega_1} - \frac{\sin\omega_2 t}{\omega_2}\right) \qquad \text{(g)}$$

Alternative method

1. *Determine initial velocities.*

Ignoring the stiffness elements and using the impulse-momentum relationship of Eq. (4.1.3) for each mass gives

$$\dot{u}_1(0) = \frac{p_o}{m} \qquad \dot{u}_2(0) = 0 \qquad \text{(h)}$$

2. *Determine initial velocities in modal coordinates.*

$$\dot{q}_n(0) = \frac{\phi_n^T\mathbf{m}\dot{\mathbf{u}}(0)}{M_n}$$

$$\dot{q}_1(0) = \frac{\langle 0.707 \quad 1\rangle\, m \begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix} \begin{Bmatrix} p_o/m \\ 0 \end{Bmatrix}}{m} = 0.707\frac{p_o}{m}$$

Similarly,

$$\dot{q}_2(0) = -0.707\frac{p_o}{m}$$

3. *Determine free vibration response.*

$$q_n(t) = \frac{\dot{q}_n(0)}{\omega_n}\sin\omega_n t$$

$$q_1(t) = \frac{0.707\,p_o}{m\omega_1}\sin\omega_1 t;\; q_2(t) = \frac{-0.707\,p_o}{m\omega_2}\sin\omega_2 t \qquad \text{(i)}$$

Note that these results are the same as Eq. (e).

Problem 12.5

1. *Set up equations of motion.*

$$\begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_o(t) \\ 0 \end{Bmatrix} \qquad \text{(a)}$$

2. *Set up modal equations.*

From Problem 12.1,

$$\omega_1 = 0.765\sqrt{k/m} \qquad \omega_2 = 1.848\sqrt{k/m}$$

$$\phi_1 = \langle 0.707 \quad 1 \rangle^T \qquad \phi_2 = \langle -0.707 \quad 1 \rangle^T$$

$$M_1 = m \qquad M_2 = m$$

$$K_1 = 0.586k \qquad K_2 = 3.414k$$

The generalized modal forces are

$$P_1(t) = \phi_1^T \mathbf{p}(t) = 0.707 p_o$$

$$P_2(t) = \phi_2^T \mathbf{p}(t) = -0.707 p_o$$

Substituting these in Eq. (12.3.3) gives the modal equations:

$$M_1 \ddot{q}_1 + K_1 q_1 = 0.707 p_o \qquad \text{(b.1)}$$

$$M_2 \ddot{q}_2 + K_2 q_2 = -0.707 p_o \qquad \text{(b.2)}$$

3. *Solve modal equations.*

For an SDF system subjected to a suddenly applied force, the governing equation is

$$m\ddot{u} + ku = p_o \qquad \text{(c)}$$

and its solution is

$$u(t) = \frac{p_o}{k}(1 - \cos \omega_n t) \qquad \text{(d)}$$

Adapting this result, the solutions of Eqs. (b.1) and (b.2) are

$$q_1(t) = \frac{0.707 p_o}{K_1}(1 - \cos \omega_1 t)$$

$$= \frac{1.207 p_o}{k}(1 - \cos \omega_1 t)$$

$$q_2(t) = \frac{-0.707 p_o}{K_2}(1 - \cos \omega_2 t)$$

$$= \frac{-0.207 p_o}{k}(1 - \cos \omega_2 t)$$

4. *Combine modal responses.*

$$\mathbf{u}(t) = \phi_1 q_1(t) + \phi_2 q_2(t)$$

$$u_1(t) = \frac{0.853 p_o}{k}(1 - \cos \omega_1 t) + \frac{0.147 p_o}{k}(1 - \cos \omega_2 t)$$

$$= \frac{p_o}{k}(1 - 0.853 \cos \omega_1 t - 0.147 \cos \omega_2 t)$$

$$u_2(t) = \frac{1.207 p_o}{k}(1 - \cos \omega_1 t) - \frac{0.207 p_o}{k}(1 - \cos \omega_2 t)$$

$$= \frac{p_o}{k}(1 - 1.207 \cos \omega_1 t + 0.207 \cos \omega_2 t)$$

5. *Determine second story drift.*

$$\Delta_2(t) = u_2(t) - u_1(t)$$

$$= \frac{0.354 p_o}{k}(-\cos \omega_1 t + \cos \omega_2 t)$$

Problem 12.6

1. *Set up equations of motion.*

$$\begin{bmatrix} m & 0 \\ 0 & m/2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_o(t) \\ 0 \end{Bmatrix} \qquad \text{(a)}$$

where

$$p(t) = \begin{cases} p_o & t \le t_d \\ 0 & t \ge t_d \end{cases} \; ; \qquad t_d = \frac{T_1}{2}$$

2. *Set up modal equations.*

From Problem 12.1,

$$\omega_1 = 0.765\sqrt{k/m} \qquad \omega_2 = 1.848\sqrt{k/m}$$

$$\phi_1 = \langle 0.707 \quad 1 \rangle^T \qquad \phi_2 = \langle -0.707 \quad 1 \rangle^T$$

$$M_1 = m \qquad M_2 = m$$

$$K_1 = 0.586k \qquad K_2 = 3.414k$$

The generalized modal forces are

$$P_1(t) = \phi_1^T \mathbf{p}(t) = \begin{cases} 0.707 p_o & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases}$$

$$P_2(t) = \phi_2^T \mathbf{p}(t) = \begin{cases} -0.707 p_o & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases}$$

Substituting these in Eq. (12.3.3) gives the modal equations:

$$M_1 \ddot{q}_1 + K_1 q_1 = \begin{cases} 0.707 p_o & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(b)}$$

$$M_2 \ddot{q}_2 + K_2 q_2 = \begin{cases} -0.707 p_o & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(c)}$$

3. *Solve modal equations.*

For an undamped SDF system subjected to a rectangular pulse force of amplitude p_o and duration t_d, the response is

$$u(t) = \begin{cases} \dfrac{p_o}{k}\left(1 - \cos\dfrac{2\pi t}{T_n}\right) & 0 \le t \le t_d \\ \dfrac{p_o}{k}\left(2\sin\dfrac{\pi t_d}{T_n}\right)\sin\left[2\pi\left(\dfrac{t}{T_n} - \dfrac{t_d}{2T_n}\right)\right] & t \ge t_d \end{cases} \qquad \text{(d)}$$

where

$$T_n = \frac{2\pi}{\omega_n}$$

Adapting this result, the solution of Eq. (b) is

$$q_1(t) = \begin{cases} \dfrac{0.707 p_o}{K_1}\left(1 - \cos\dfrac{2\pi t}{T_1}\right) & 0 \le t \le t_d \\ \dfrac{0.707 p_o}{K_1}\left(2\sin\dfrac{\pi t_d}{T_1}\right)\sin\left[2\pi\left(\dfrac{t}{T_1} - \dfrac{t_d}{2T_1}\right)\right] & t \ge t_d \end{cases} \qquad \text{(e)}$$

Substituting $K_1 = 0.586k$ and $t_d = T_1/2$ gives

$$q_1(t) = \begin{cases} 1.207\dfrac{p_o}{k}\left(1 - \cos\dfrac{2\pi t}{T_1}\right) & 0 \le t \le \dfrac{T_1}{2} \\ 2.414\dfrac{p_o}{k}\sin\left[2\pi\left(\dfrac{t}{T_1} - \dfrac{1}{4}\right)\right] & t \ge \dfrac{T_1}{2} \end{cases} \qquad \text{(f)}$$

Similarly the solution of Eq. (c) is

$$q_2(t) = \begin{cases} -0.207\dfrac{p_o}{k}\left(1 - \cos\dfrac{2\pi t}{T_2}\right) & 0 \le t \le \dfrac{T_1}{2} \\ 0.252\dfrac{p_o}{k}\sin\left[2\pi\left(\dfrac{t}{T_2} - \dfrac{T_1}{4T_2}\right)\right] & t \ge \dfrac{T_1}{2} \end{cases} \qquad \text{(g)}$$

4. *Combine modal responses.*

$$\mathbf{u}(t) = \phi_1 q_1(t) + \phi_2 q_2(t) \qquad \text{(h)}$$

Problem 12.7

Part a

From Problem 9.8, the mass and stiffness matrices are

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix} \qquad \mathbf{k} = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

where $k = 24EI/h^3$.

From Problem 10.11, the natural frequencies ω_n and modes ϕ_n are given by

$$\omega_1^2 = (2 - \sqrt{3})\frac{k}{m};\ \omega_2^2 = 2\frac{k}{m};\ \omega_3^2 = (2 + \sqrt{3})\frac{k}{m}$$

$$\phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix}$$

(i) *Direct Solution*

The equations of motion are

$$m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ p_o \end{Bmatrix}\sin \omega t \tag{a}$$

The steady-state response is assumed as

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} u_{1o} \\ u_{2o} \\ u_{3o} \end{Bmatrix}\sin \omega t \tag{b}$$

Substituting Eq. (b) in Eq. (a) gives

$$\begin{Bmatrix} u_{1o} \\ u_{2o} \\ u_{3o} \end{Bmatrix} = \left[\mathbf{k} - \omega^2\mathbf{m}\right]^{-1}\begin{Bmatrix} 0 \\ 0 \\ p_o \end{Bmatrix} = \frac{1}{\det\left[\mathbf{k} - \omega^2\mathbf{m}\right]}\operatorname{adj}\left[\mathbf{k} - \omega^2\mathbf{m}\right]\begin{Bmatrix} 0 \\ 0 \\ p_o \end{Bmatrix} \tag{c}$$

The determinant can be expressed in terms of the natural frequencies:

$$\begin{aligned}\det\left[\mathbf{k} - \omega^2\mathbf{m}\right] &= m_1 m_2 m_3 (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)(\omega_3^2 - \omega^2) \\ &= \frac{1}{2}m^3\left(1 - \frac{\omega^2}{\omega_1^2}\right)\left(1 - \frac{\omega^2}{\omega_2^2}\right)\left(1 - \frac{\omega^2}{\omega_3^2}\right)\omega_1^2\omega_2^2\omega_3^2 \\ &= k^3\left(1 - \frac{\omega^2}{\omega_1^2}\right)\left(1 - \frac{\omega^2}{\omega_2^2}\right)\left(1 - \frac{\omega^2}{\omega_3^2}\right)\end{aligned} \tag{d}$$

Similarly,

$$\operatorname{adj}\left[\mathbf{k} - \omega^2\mathbf{m}\right]\begin{Bmatrix} 0 \\ 0 \\ p_o \end{Bmatrix} = k^2 p_o\begin{Bmatrix} 1 \\ 2\left(1 - \omega^2/\omega_2^2\right) \\ 4\left(1 - \omega^2/\omega_2^2\right)^2 - 1 \end{Bmatrix} \tag{e}$$

Substituting Eqs. (d) and (e) in Eq. (c) gives

$$\begin{Bmatrix} u_{1o} \\ u_{2o} \\ u_{3o} \end{Bmatrix} = \frac{p_o}{k}\frac{1}{\left(1 - \frac{\omega^2}{\omega_1^2}\right)\left(1 - \frac{\omega^2}{\omega_2^2}\right)\left(1 - \frac{\omega^2}{\omega_3^2}\right)}\begin{Bmatrix} 1 \\ 2\left(1 - \omega^2/\omega_2^2\right) \\ 4\left(1 - \omega^2/\omega_2^2\right)^2 - 1 \end{Bmatrix} \tag{f}$$

Substituting Eq. (f) in Eq. (b) gives $u_j(t)$.

(ii) *Modal Analysis*

Using Eq. (12.3.4), the generalized modal mass, stiffness, and force are

$$\begin{aligned} M_1 &= 1.5m & K_1 &= 1.5(2 - \sqrt{3})k & P_1(t) &= p_o \sin \omega t \\ M_2 &= 1.5m & K_2 &= 3k & P_2(t) &= p_o \sin \omega t \\ M_3 &= 1.5m & K_3 &= 1.5(2 + \sqrt{3})k & P_3(t) &= p_o \sin \omega t \end{aligned} \tag{g}$$

The modal equations and their steady-state solution are

$$M_n\ddot{q}_n + K_n q_n = P_{no}\sin \omega t$$

$$q_n(t) = \frac{P_{no}}{K_n}\mathcal{C}_n \sin \omega t \tag{h}$$

where

$$\mathcal{C}_n = \frac{1}{1 - (\omega/\omega_n)^2} \tag{i}$$

Substituting Eq. (i) in Eq. (h) gives

$$q_1(t) = \frac{p_o}{k}\frac{2(2+\sqrt{3})}{3}\mathcal{C}_1 \sin \omega t$$

$$q_2(t) = \frac{p_o}{k}\frac{1}{3}\mathcal{C}_2 \sin \omega t \qquad \text{(j)}$$

$$q_3(t) = \frac{p_o}{k}\frac{2(2-\sqrt{3})}{3}\mathcal{C}_3 \sin \omega t$$

Substituting Eq. (j) and ϕ_n in Eq. (12.3.2) gives the floor displacements:

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \frac{p_o}{3k}\begin{Bmatrix} (2+\sqrt{3})\mathcal{C}_1 - \mathcal{C}_2 + (2-\sqrt{3})\mathcal{C}_3 \\ (3+2\sqrt{3})\mathcal{C}_1 + 0 + (3-2\sqrt{3})\mathcal{C}_3 \\ (4+2\sqrt{3})\mathcal{C}_1 + \mathcal{C}_2 + (4-2\sqrt{3})\mathcal{C}_3 \end{Bmatrix} \sin \omega t \qquad \text{(k)}$$

Part b

The direct solution Eq. (f) can be written in terms of $\mathcal{C}_n$:

$$\frac{u_{1o}}{p_o/k} = \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3$$

$$\frac{u_{2o}}{p_o/k} = 2\mathcal{C}_1\mathcal{C}_3 \qquad \text{(l)}$$

$$\frac{u_{3o}}{p_o/k} = \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3\left(\frac{4}{\mathcal{C}_2^2} - 1\right)$$

This result is equivalent to Eq. (k) from modal analysis. This equivalence can be proven for u_2, for example, as follows: The direct solution gives

$$\frac{u_{2o}}{p_o/k} = 2\mathcal{C}_1\mathcal{C}_3 = 2\frac{\omega_1^2}{\left(\omega_1^2 - \omega^2\right)}\frac{\omega_3^2}{\left(\omega_3^2 - \omega^2\right)}$$

From modal analysis,

$$\frac{u_{2o}}{p_o/k} = \frac{1}{3}\left[(3+2\sqrt{3})\mathcal{C}_1 + (3-2\sqrt{3})\mathcal{C}_3\right]$$

$$= \frac{1}{3}\left[(3+2\sqrt{3})\frac{\omega_1^2}{\omega_1^2 - \omega^2} + (3-2\sqrt{3})\frac{\omega_3^2}{\omega_3^2 - \omega^2}\right]$$

$$= \frac{1}{3}\left[\frac{6\omega_1^2\omega_3^2 - \sqrt{3}\omega^2\left[\omega_1^2(\sqrt{3}+2) - \omega_3^2(\sqrt{3}-2)\right]}{(\omega_1^2 - \omega^2)(\omega_3^2 - \omega^2)}\right]$$

$$= 2\frac{\omega_1^2}{\left(\omega_1^2 - \omega^2\right)}\frac{\omega_3^2}{\left(\omega_3^2 - \omega^2\right)} \qquad \text{(m)}$$

Hence, both methods give the same results. The same procedure can be applied to show such equivalence for displacements u_1 and u_3.

Part c

The amplitude of displacements is given by Eq. (l) where $\mathcal{C}_n$ are defined by Eq. (i). These amplitudes are plotted against ω/ω_1 in Fig. P12.7.

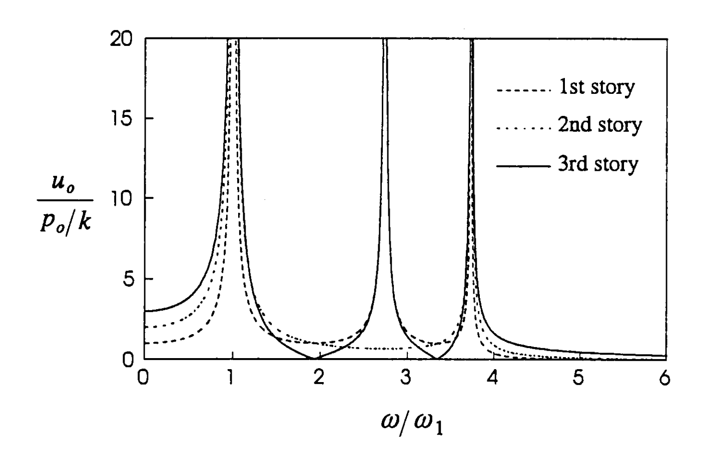


Fig. P12.7

Problem 12.8

Part a*: Story shears from displacements.*

$$
\begin{aligned}
V_{1n}(t) &= k_1 u_{1n}(t) = k_1 \phi_{1n} q_n(t) \\
V_{2n}(t) &= k_2 \left[u_{2n}(t) - u_{1n}(t) \right] \\
&= k_2 \left(\phi_{2n} - \phi_{1n} \right) q_n(t) \\
V_{3n}(t) &= k_3 \left[u_{3n}(t) - u_{2n}(t) \right] \\
&= k_3 \left(\phi_{3n} - \phi_{2n} \right) q_n(t)
\end{aligned}
\qquad \text{(a)}
$$

Substituting $k_1 = k_2 = k_3 = k$, and the values of ϕ_{jn}, and q_n from Eq. (j) of Problem 12.7 in Eq. (a) gives the story shears for each mode:

Mode 1

$$
\begin{aligned}
V_{11}(t) &= \frac{p_o}{3}\left(2 + \sqrt{3}\right) \mathcal{C}_1 \sin \omega t \\
V_{21}(t) &= \frac{p_o}{3}\left(1 + \sqrt{3}\right) \mathcal{C}_1 \sin \omega t \\
V_{31}(t) &= \frac{p_o}{3}(1) \mathcal{C}_1 \sin \omega t
\end{aligned}
\qquad \text{(b.1)}
$$

Mode 2

$$
\begin{aligned}
V_{12}(t) &= \frac{p_o}{3}(-1) \mathcal{C}_2 \sin \omega t \\
V_{22}(t) &= \frac{p_o}{3}(1) \mathcal{C}_2 \sin \omega t \\
V_{32}(t) &= \frac{p_o}{3}(1) \mathcal{C}_2 \sin \omega t
\end{aligned}
\qquad \text{(b.2)}
$$

Mode 3

$$
\begin{aligned}
V_{13}(t) &= \frac{p_o}{3}\left(2 - \sqrt{3}\right) \mathcal{C}_3 \sin \omega t \\
V_{23}(t) &= \frac{p_o}{3}\left(1 - \sqrt{3}\right) \mathcal{C}_3 \sin \omega t \\
V_{33}(t) &= \frac{p_o}{3}(1) \mathcal{C}_3 \sin \omega t
\end{aligned}
\qquad \text{(b.3)}
$$

The total story shears are

$$
V_j(t) = V_{j1}(t) + V_{j2}(t) + V_{j3}(t) \qquad \text{(c)}
$$

Substituting Eqs. (b) in Eq. (c) gives

$$
\begin{aligned}
V_1(t) &= \frac{p_o}{3}\left[\left(2 + \sqrt{3}\right)\mathcal{C}_1 - \mathcal{C}_2 + \left(2 - \sqrt{3}\right)\mathcal{C}_3\right] \sin \omega t \\
V_2(t) &= \frac{p_o}{3}\left[\left(1 + \sqrt{3}\right)\mathcal{C}_1 + \mathcal{C}_2 + \left(1 - \sqrt{3}\right)\mathcal{C}_3\right] \sin \omega t \\
V_3(t) &= \frac{p_o}{3}\left[\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3\right] \sin \omega t
\end{aligned}
\qquad \text{(d)}
$$

Part b*: Story shears from equivalent static forces.*

Substituting $q_n(t)$ from Problem 12.7 in Eq. (12.6.2) gives the equivalent static forces:

$$
f_{jn}(t) = \omega_n^2 m_j \phi_{jn} q_n(t) = \frac{\omega_n^2 p_o \mathcal{C}_n}{K_n} m_j \phi_{jn} \sin \omega t \qquad \text{(e)}
$$

Substituting ω_n^2, K_n, m_j and ϕ_{jn} (from Problems 10.11 and 12.7) in Eq. (e) gives f_{jn} for each mode:

Mode 1

$$
\begin{aligned}
f_{11}(t) &= \frac{p_o}{3} \mathcal{C}_1 \sin \omega t \\
f_{21}(t) &= \frac{\sqrt{3} p_o}{3} \mathcal{C}_1 \sin \omega t \\
f_{31}(t) &= \frac{p_o}{3} \mathcal{C}_1 \sin \omega t
\end{aligned}
\qquad \text{(f.1)}
$$

Mode 2

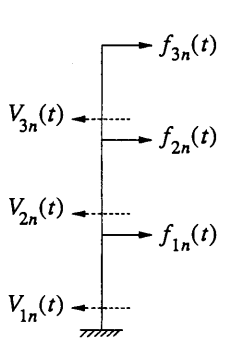
$$
\begin{aligned}
f_{12}(t) &= -\frac{2 p_o}{3} \mathcal{C}_2 \sin \omega t \\
f_{22}(t) &= 0 \\
f_{32}(t) &= \frac{p_o}{3} \mathcal{C}_2 \sin \omega t
\end{aligned}
\qquad \text{(f.2)}
$$

Mode 3

$$
\begin{aligned}
f_{13}(t) &= \frac{p_o}{3} \mathcal{C}_3 \sin \omega t \\
f_{23}(t) &= -\frac{\sqrt{3} p_o}{3} \mathcal{C}_3 \sin \omega t \\
f_{33}(t) &= \frac{p_o}{3} \mathcal{C}_3 \sin \omega t
\end{aligned}
\qquad \text{(f.3)}
$$

Static analysis of the system gives the story shears due to the nth mode:

$$
\left.
\begin{aligned}
V_{3n}(t) &= f_{3n}(t) \\
V_{2n}(t) &= f_{2n}(t) + f_{3n}(t) \\
V_{1n}(t) &= f_{1n}(t) + f_{2n}(t) + f_{3n}(t)
\end{aligned}
\right\} \qquad \text{(g)}
$$



Problem 12.9

Using the results from Problem 10.11 and substituting $m = 100$ kips/g and $k = 24EI/h^3 = 326.32$ kips/in. gives ω_n in rads/sec and ϕ_n:

$$\omega_1 = 18.38 \qquad \omega_2 = 50.22 \qquad \omega_3 = 68.60 \tag{a}$$

$$\phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} \tag{b}$$

The excitation force due to the shaker is

$$p_3(t) = 2\left(\frac{m_e}{2}\right) e\,\omega^2 \sin\omega t$$
$$= 2\left(\frac{0.02}{386.4}\right)(12)\,\omega^2 \sin\omega t$$
$$= 1.242 \times 10^{-3}\,\omega^2 \sin\omega t$$

Thus

$$\mathbf{p}(t) = \begin{Bmatrix} 0 \\ 0 \\ 1.242 \times 10^{-3}\,\omega^2 \end{Bmatrix} \sin\omega t \tag{c}$$

The generalized modal masses and stiffnesses have been computed in Eq. (g) of Problem 12.7. Substituting $m = 400/386.4 = 1.0352$ kip$-$sec^2/in. and $k = 326.32$ kips/in. gives

$$\left.\begin{aligned} M_1 &= 0.3882 & K_1 &= 131.16 & P_{1o} &= 1.242 \times 10^{-3}\,\omega^2 \\ M_2 &= 0.3882 & K_2 &= 978.97 & P_{2o} &= 1.242 \times 10^{-3}\,\omega^2 \\ M_3 &= 0.3882 & K_3 &= 1826.80 & P_{3o} &= 1.242 \times 10^{-3}\,\omega^2 \end{aligned}\right\} \tag{d}$$

The modal equations are

$$M_n\ddot{q}_n + C_n\dot{q}_n + K_n q_n = P_{no}\sin\omega t \tag{e}$$

For $\zeta_n = 0.05$ and the values of ω_n, ϕ_{jn}, K_n, M_n and P_{no} defined in Eqs. (a), (b) and (d), the solution is

$$q_n(t) = \frac{p_o}{K_n}\omega^2\left[\mathcal{C}_n \sin\omega t + \mathcal{D}_n \cos\omega t\right] \tag{f}$$

where

$$p_o = 1.242 \times 10^{-3}$$

$$\left.\begin{aligned} \mathcal{C}_n &= \frac{1 - (\omega/\omega_n)^2}{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta_n\omega/\omega_n\right]^2} \\ \mathcal{D}_n &= \frac{-2\zeta_n\omega/\omega_n}{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta_n\omega/\omega_n\right]^2} \end{aligned}\right\} \tag{g}$$

Substituting $q_n(t)$ and ϕ_n in Eq. (12.5.2) gives the lateral displacements. In particular, the roof displacement, $u_3(t)$ is

$$u_3(t) = \phi_{31}q_1(t) + \phi_{32}q_2(t) + \phi_{33}q_3(t)$$
$$= \omega^2 p_o\left[\left(\frac{\mathcal{C}_1}{K_1} + \frac{\mathcal{C}_2}{K_2} + \frac{\mathcal{C}_3}{K_3}\right)\sin\omega t + \left(\frac{\mathcal{D}_1}{K_1} + \frac{\mathcal{D}_2}{K_2} + \frac{\mathcal{D}_3}{K_3}\right)\cos\omega t\right] \tag{h}$$

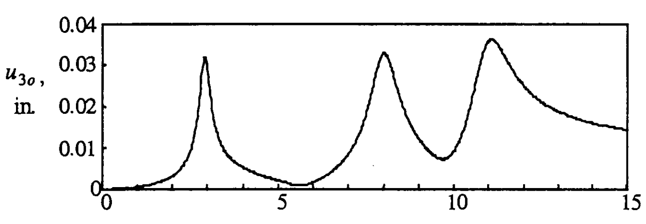
The amplitude of the displacement at the roof is

$$u_{3o} = \omega^2 p_o\sqrt{\left(\frac{\mathcal{C}_1}{K_1} + \frac{\mathcal{C}_2}{K_2} + \frac{\mathcal{C}_3}{K_3}\right)^2 + \left(\frac{\mathcal{D}_1}{K_1} + \frac{\mathcal{D}_2}{K_2} + \frac{\mathcal{D}_3}{K_3}\right)^2} \tag{i}$$

The amplitude of the roof acceleration is

$$\ddot{u}_{3o} = \omega^2 u_{3o} \tag{j}$$

The frequency response curves for roof displacement and roof acceleration are shown in Figs. P12.9a-b.



Exciting frequency, Hz
Fig. P12.9a

0.5, 0.4, 0.3, 0.2, 0.1, 0
$\ddot{u}_{3o}$, g
0, 5, 10, 15

Exciting frequency, Hz
Fig. P12.9b

Problem 12.10

Using the results from Problem 10.11 and substituting $m = 100$ kips/g and $k = 24EI/h^3 =$ 326.32 kips/in. gives ω_n in rads/sec and ϕ_n:

$$\omega_1 = 18.38 \qquad \omega_2 = 50.22 \qquad \omega_3 = 68.60$$

$$\phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} \tag{a}$$

The generalized modal masses and stiffnesses have been computed in Eq. (g) of Problem 12.7. Substituting $m = 400/386.4 = 1.0352 \text{ kip}-\text{sec}^2/\text{in.}$ and $k = 326.32$ kips/in. gives

$$\begin{aligned} M_1 &= 0.3882 \qquad K_1 = 131.16 \\ M_2 &= 0.3882 \qquad K_2 = 978.97 \\ M_3 &= 0.3882 \qquad K_3 = 1826.80 \end{aligned} \tag{b}$$

The generalized modal forces due to the impulsive force at the second floor are

$$P_1(t) = \phi_1^T \mathbf{p}(t) = \langle 0.5 \quad 0.886 \quad 1 \rangle \begin{Bmatrix} 0 \\ 20\,\delta(t) \\ 0 \end{Bmatrix} \tag{c.1}$$

$$= 17.32\,\delta(t)$$

$$P_2(t) = \phi_2^T \mathbf{p}(t) = 0 \tag{c.2}$$

$$P_3(t) = \phi_3^T \mathbf{p}(t) = -17.32\,\delta(t) \tag{c.3}$$

The modal equations and their solutions are

$$M_n \ddot{q}_n + K_n q_n = P_{no}\,\delta(t) \tag{d}$$

$$q_n(t) = \frac{P_{no}}{M_n \omega_n} \sin \omega_n t \tag{e}$$

Substituting P_{no}, M_n and ω_n in Eq. (e) gives

$$\left.\begin{aligned} q_1(t) &= 2.427 \sin \omega_1 t \\ q_2(t) &= 0 \\ q_3(t) &= -0.650 \sin \omega_3 t \end{aligned}\right\} \tag{f}$$

Substituting ϕ_n and $q_n(t)$ in Eq. (12.5.2) gives the lateral displacements:

$$\mathbf{u}(t) = \begin{Bmatrix} 1.214 \\ 2.102 \\ 2.427 \end{Bmatrix} \sin \omega_1 t + \begin{Bmatrix} -0.325 \\ 0.563 \\ -0.650 \end{Bmatrix} \sin \omega_3 t \tag{g}$$

Substituting Eq. (f) in Eq. (g) gives $V_{jn}(t)$ for each mode:

Mode 1

$$\begin{aligned} V_{31}(t) &= \frac{p_o}{3}(1)\,\mathcal{C}_1 \sin \omega t \\ V_{21}(t) &= \frac{p_o}{3}\left(1 + \sqrt{3}\right)\mathcal{C}_1 \sin \omega t \\ V_{11}(t) &= \frac{p_o}{3}\left(2 + \sqrt{3}\right)\mathcal{C}_1 \sin \omega t \end{aligned} \tag{h.1}$$

Mode 2

$$\begin{aligned} V_{32}(t) &= \frac{p_o}{3}(1)\,\mathcal{C}_2 \sin \omega t \\ V_{22}(t) &= \frac{p_o}{3}(1)\,\mathcal{C}_2 \sin \omega t \\ V_{12}(t) &= \frac{p_o}{3}(-1)\,\mathcal{C}_2 \sin \omega t \end{aligned} \tag{h.2}$$

Mode 3

$$\begin{aligned} V_{33}(t) &= \frac{p_o}{3}(1)\,\mathcal{C}_3 \sin \omega t \\ V_{23}(t) &= \frac{p_o}{3}\left(1 - \sqrt{3}\right)\mathcal{C}_3 \sin \omega t \\ V_{13}(t) &= \frac{p_o}{3}\left(2 - \sqrt{3}\right)\mathcal{C}_3 \sin \omega t \end{aligned} \tag{h.3}$$

These modal contributions to the story shears are the same as in Eq. (b) determined from displacements.

Problem 12.11

Part a

From Problem 12.10,

$$\left.\begin{array}{lll} \omega_1 = 13.38 & \omega_2 = 50.22 & \omega_3 = 68.60 \\ \phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} & \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} & \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} \\ M_1 = 0.3882 & M_2 = 0.3882 & M_3 = 0.3882 \\ K_1 = 131.16 & K_2 = 978.97 & K_3 = 1826.80 \end{array}\right\} \quad \text{(a)}$$

The generalized modal forces are

$$P_1(t) = \phi_1^T \mathbf{p}(t) = \langle 0.5 \quad 0.886 \quad 1 \rangle \begin{Bmatrix} 200 \\ 0 \\ 0 \end{Bmatrix} = 100$$

$$P_2(t) = \phi_2^T \mathbf{p}(t) = -200 \qquad \text{(b)}$$

$$P_3(t) = \phi_3^T \mathbf{p}(t) = 100$$

The modal equations and their solutions are

$$M_n \ddot{q}_n + K_n q_n = P_{no} \qquad \text{(c)}$$

$$q_n(t) = \frac{P_{no}}{K_n}(1 - \cos \omega_n t) \qquad \text{(d)}$$

Substituting P_{no}, M_n and ω_n in Eq. (d) gives

$$\left.\begin{array}{l} q_1(t) = 0.7624\,(1 - \cos \omega_1 t) \\ q_2(t) = -0.2043\,(1 - \cos \omega_2 t) \\ q_3(t) = 0.0547\,(1 - \cos \omega_3 t) \end{array}\right\} \qquad \text{(e)}$$

Substituting ϕ_n and $q_n(t)$ in Eq. (12.5.2) gives the lateral displacements:

$$\mathbf{u}(t) = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} 0.7624\,(1 - \cos \omega_1 t) + \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} (-0.2043)\,(1 - \cos \omega_2 t) + \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} 0.0547\,(1 - \cos \omega_3 t)$$

or,

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} 0.6129 \\ 0.6129 \\ 0.6129 \end{Bmatrix} - \begin{Bmatrix} 0.3812 \\ 0.6603 \\ 0.7624 \end{Bmatrix} \cos \omega_1 t - \begin{Bmatrix} 0.2043 \\ 0 \\ -0.2043 \end{Bmatrix} \cos \omega_2 t - \begin{Bmatrix} 0.0274 \\ -0.0474 \\ 0.0547 \end{Bmatrix} \cos \omega_3 t$$

Part b

The second-story drift is

$$\begin{aligned} \Delta_2(t) &= u_2(t) - u_1(t) \\ &= -0.2791 \cos \omega_1 t + 0.2043 \cos \omega_2 t \\ &\quad + 0.0748 \cos \omega_3 t \end{aligned}$$

Problem 12.12

From Problem 12.10,

$$\omega_1 = 18.38 \quad \omega_2 = 50.22 \quad \omega_3 = 68.60$$

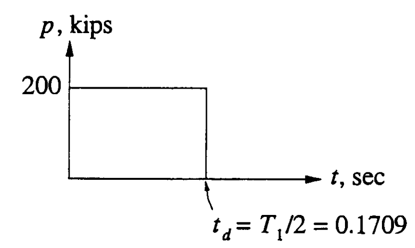
$$T_1 = 0.3418 \quad T_2 = 0.1244 \quad T_3 = 0.0916$$

$$\phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix}$$

$$M_1 = 0.3882 \quad M_2 = 0.3882 \quad M_3 = 0.3882$$

$$K_1 = 131.16 \quad K_2 = 978.97 \quad K_3 = 1826.80 \qquad \text{(a)}$$

The rectangular pulse force at the third floor is shown in the accompanying figure:



The generalized modal forces are

$$\begin{Bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.866 & 1 \\ -1 & 0 & 1 \\ 0.5 & -0.866 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 200 \end{Bmatrix} = \begin{Bmatrix} 200 \\ 200 \\ 200 \end{Bmatrix} \qquad \text{(b)}$$

The modal equations are

$$M_n \ddot{q}_n + K_n q_n = \begin{cases} P_{no} & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(c)}$$

The solution of Eq. (c) is

$$q_n(t) = \begin{cases} \dfrac{P_{no}}{K_n}\left(1 - \cos\dfrac{2\pi t}{T_n}\right) & 0 \le t \le t_d \\ \dfrac{P_{no}}{K_n}\left(2\sin\dfrac{\pi t_d}{T_n}\right)\sin 2\pi\left(\dfrac{t}{T_n} - \dfrac{1}{2}\dfrac{t_d}{T_n}\right) & t \ge t_d \end{cases} \qquad \text{(d)}$$

Substituting P_{no}, K_n and T_n in Eq. (d) gives

$$\left.\begin{aligned} q_1(t) &= 1.5249\left(1 - \cos 2\pi\frac{t}{T_1}\right) \\ q_2(t) &= 0.2043\left(1 - \cos 2\pi\frac{t}{T_2}\right) \\ q_3(t) &= 0.1095\left(1 - \cos 2\pi\frac{t}{T_3}\right) \end{aligned}\right\} \quad 0 \le t \le t_d \qquad \text{(e)}$$

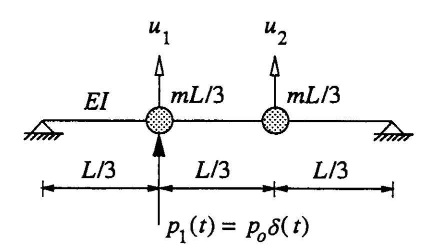
$$\left.\begin{aligned} q_1(t) &= 3.0498\sin 2\pi\left(\frac{t}{T_1} - 0.250\right) \\ q_2(t) &= -0.3729\sin 2\pi\left(\frac{t}{T_2} - 0.687\right) \\ q_3(t) &= -0.0895\sin 2\pi\left(\frac{t}{T_3} - 0.933\right) \end{aligned}\right\} \quad t \ge t_d \qquad \text{(f)}$$

The story displacements are

$$\mathbf{u}(t) = \sum_{n=1}^{3} \phi_n q_n(t) \qquad \text{(g)}$$

where ϕ_n are known and $q_n(t)$ is given by Eq. (e) if $0 \le t \le t_d$ and by Eq. (f) if $t \ge t_d$.

Problem 12.13



$E = 30{,}000$ ksi $\quad I = 100$ in.^{4} $\quad L = 150$ in.

$mL = 0.864$ kip$-$sec^2/in. $\quad p_o = 10$ kips

1. *Determine stiffness and mass matrices.*

From Problem 9.2,

$$\mathbf{k} = \frac{162EI}{5L^3}\begin{bmatrix} 8 & -7 \\ -7 & 8 \end{bmatrix} \qquad \mathbf{m} = \frac{mL}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the given data,

$$\mathbf{k} = \begin{bmatrix} 230.4 & -201.6 \\ -201.6 & 230.4 \end{bmatrix} \qquad \mathbf{m} = \begin{bmatrix} 0.288 & 0 \\ 0 & 0.288 \end{bmatrix}$$

2. *Determine natural frequencies and modes.*

From Problem 10.2,

$$\omega_1 = 9.859\sqrt{\frac{EI}{mL^4}} \qquad \omega_2 = 38.184\sqrt{\frac{EI}{mL^4}}$$

$$\phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

For the given data, ω_n in rads/sec are

$$\omega_1 = 10 \qquad \omega_2 = 38.73$$

3. *Set up modal equations.*

$$M_n\ddot{q}_n + K_n q_n = P_n(t)$$

where

$$M_n = \phi_n^T \mathbf{m}\phi_n \qquad K_n = \omega_n^2 M_n \qquad P_n(t) = \phi_n^T \mathbf{p}(t)$$

For the given data,

$$M_1 = 0.576 \qquad M_2 = 0.576$$

$$P_1(t) = P_{1o}\delta(t) = p_o\delta(t) \qquad P_2(t) = P_{2o}\delta(t) = p_o\delta(t)$$

4. *Solve modal equations.*

$$q_n(t) = \frac{P_{no}}{M_n\omega_n}\sin\omega_n t$$

$$q_1(t) = \frac{10}{0.576\,(10)}\sin 10t = 1.736\sin 10t$$

$$q_2(t) = \frac{10}{0.576\,(38.73)}\sin 38.73t = 0.448\sin 38.73t$$

5. *Combine modal responses.*

$$\mathbf{u}(t) = \sum_{n=1}^{2}\phi_n q_n(t)$$

or

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} 1.736\sin 10t + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} 0.448\sin 38.73t$$

The modal responses and total response are plotted in Figs. P12.13a-b.

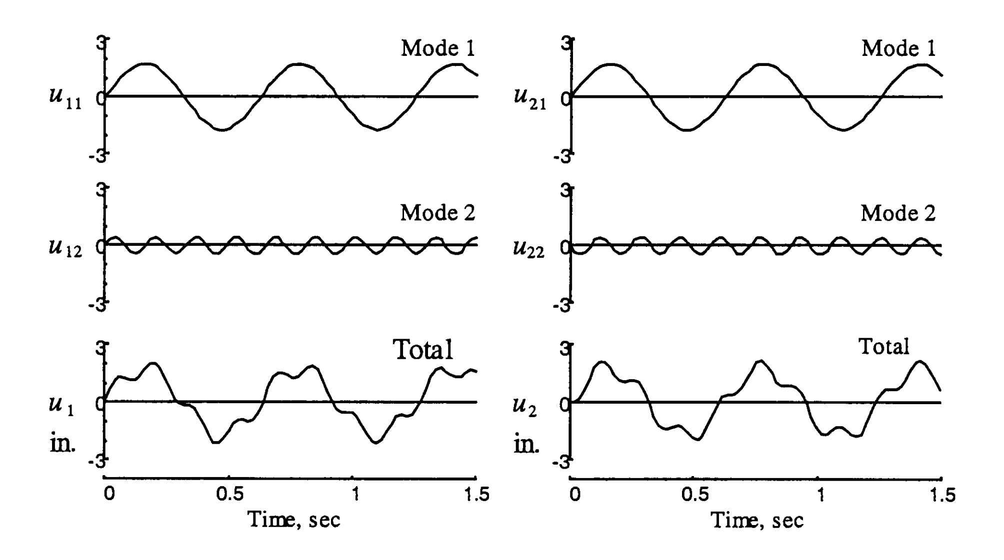


Fig. P12.13a

Fig. P12.13b

Problem 12.14

u_1 u_2

EI $mL/3$ $mL/3$

$$p_1(t) = \begin{cases} 0 & t < 0 \\ 100 \text{ kips} & t \geq 0 \end{cases}$$

1. *Determine natural frequencies and modes and M_n.*

   See Problem 12.13.

2. *Set up modal equations.*

   $$M_n \ddot{q}_n + K_n q_n = P_n(t)$$

where

$$P_n(t) = \phi_n^T \mathbf{p}(t) = p_1(t) = \begin{cases} 0 & t < 0 \\ p_{1o} = 100 & t \geq 0 \end{cases}$$

3. *Solve modal equations.*

$$q_n(t) = \frac{p_{1o}}{K_n}(1 - \cos \omega_n t)$$

$$q_1(t) = \frac{100}{0.576\,(10)^2}(1 - \cos 10t)$$
$$= 1.736\,(1 - \cos 10t)$$

$$q_2(t) = \frac{100}{0.576\,(38.73)^2}(1 - \cos 38.73t)$$
$$= 0.116\,(1 - \cos 38.73t)$$

4. *Combine modal responses.*

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} 1.736\,(1 - \cos 10t) + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} 0.116\,(1 - \cos 38.73t)$$

The modal responses and total response are plotted in Figs. P12.14a-b.

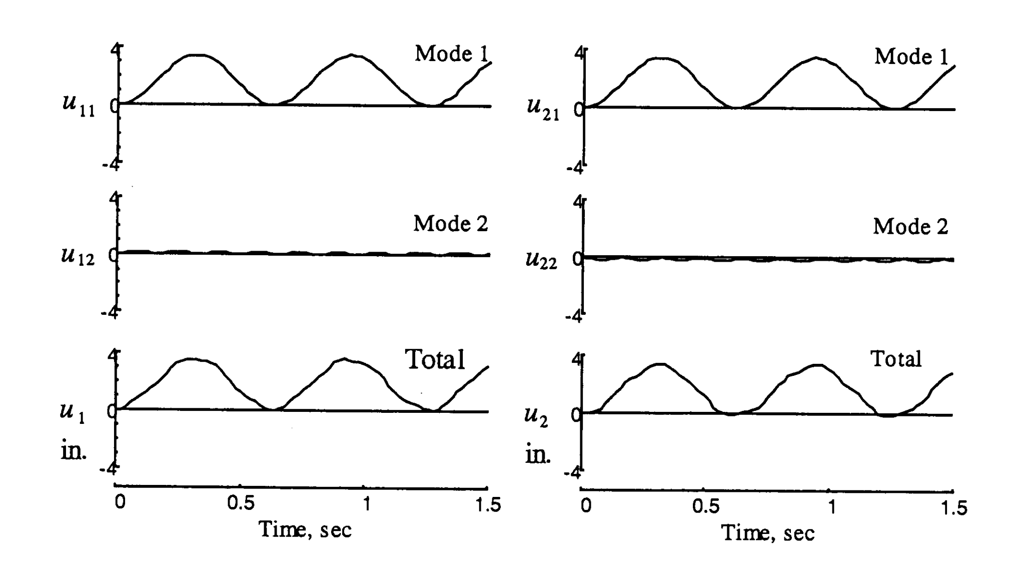
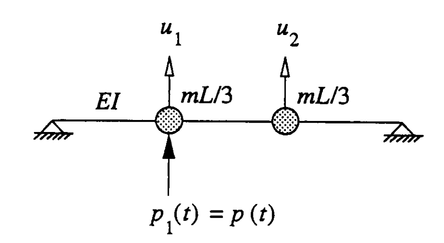
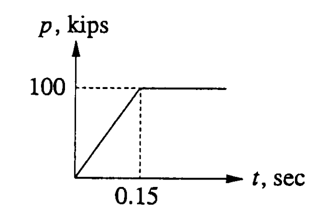


Fig. P12.14a

Fig. P12.14b

Problem 12.15





1. *Determine natural frequencies and modes and M_n.*

See Problem 12.13.

2. *Set up modal equations.*

$$M_n \ddot{q}_n + K_n q_n = P_n(t) \qquad \text{(a)}$$

where

$$P_n(t) = \phi_n^T \mathbf{p}(t) = p_1(t) = \begin{cases} 100\dfrac{t}{t_r} & 0 \le t \le t_r \\ 100 & t \ge t_r \end{cases} \qquad \text{(b)}$$

3. *Solve modal equations.*

The solution of the equation of motion for an SDF system

$$m\ddot{u} + ku = p(t)$$

where $p(t)$ is a step force p_o with rise time t_r is

$$u(t) = \begin{cases} \dfrac{p_o}{k}\left(\dfrac{t}{t_r} - \dfrac{\sin \omega_n t}{\omega_n t_r}\right) & 0 \le t \le t_r \\ \dfrac{p_o}{k}\left\{1 - \dfrac{1}{\omega_n t_r}\left[\sin \omega_n t - \sin \omega_n (t - t_r)\right]\right\} & t \ge t_r \end{cases} \qquad \text{(c)}$$

In Eq. (c), replacing p_o by 100 kips, and k by $K_n = \omega_n^2 M_n$, and substituting for ω_n and $t_r = 0.15$ sec gives the solutions for $q_n(t)$:

$$q_1(t) = \begin{cases} \overbrace{\dfrac{100}{0.576\,(10^2)}}^{1.736}\left(\dfrac{t}{0.15} - \dfrac{\sin 10t}{1.5}\right) & 0 \le t \le 0.15 \text{ sec} \\ 1.736\left\{1 - \dfrac{1}{1.5}\left[\sin 10t - \sin 10\,(t - 0.15)\right]\right\} & t \ge 0.15 \text{ sec} \end{cases} \qquad \text{(d)}$$

$$q_2(t) = \begin{cases} \overbrace{\dfrac{100}{0.576\,(38.73^2)}}^{0.116}\left(\dfrac{t}{0.15} - \dfrac{\sin 38.73t}{5.81}\right) & 0 \le t \le 0.15 \text{ sec} \\ 0.116\left\{1 - \dfrac{1}{5.81}\left[\sin 38.73t - \sin 38.73\,(t - 0.15)\right]\right\} & t \ge 0.15 \text{ sec} \end{cases} \qquad \text{(e)}$$

4. *Combine modal responses.*

The total displacements are

$$\mathbf{u}(t) = \sum_{n=1}^{2} \phi_n q_n(t) \qquad \text{(f)}$$

In Eq. (f),

$$\phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

and $q_n(t)$ is given by Eq. (d) if $0 \le t \le 0.15$ sec and by Eq. (e) if $t \ge 0.15$ sec.

The modal responses and total response are plotted in Figs. P12.15a-b.

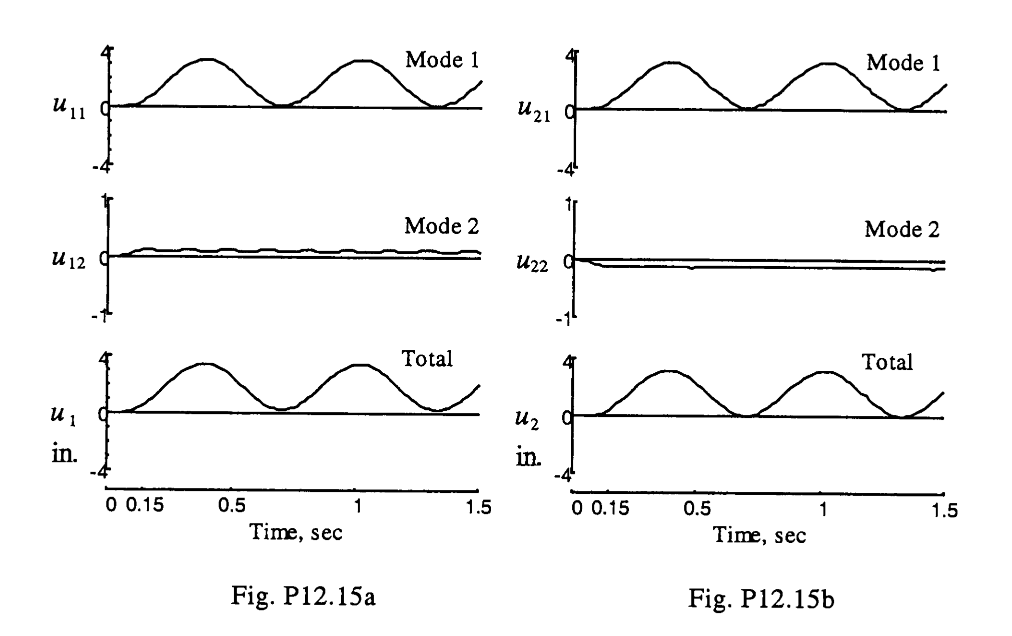
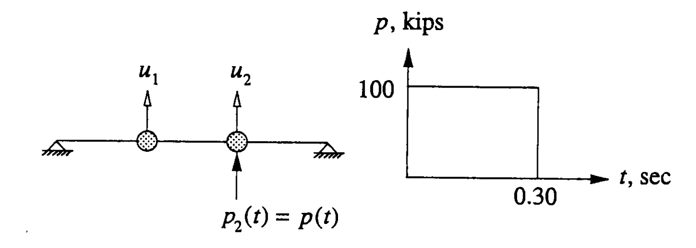


Fig. P12.15a

Fig. P12.15b

Problem 12.16



1. *Determine natural frequencies and modes and M_n.*

See Problem 12.13.

2. *Set up modal equations.*

$$M_n \ddot{q}_n + K_n q_n = P_n(t) \qquad \text{(a)}$$

where

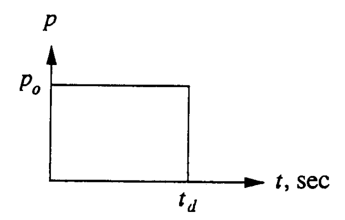
$$\left.\begin{aligned} P_1(t) &= \phi_1^T \mathbf{p}(t) = p_2(t) = \begin{cases} 100 & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \\ P_2(t) &= \phi_2^T \mathbf{p}(t) = -p_2(t) = \begin{cases} -100 & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \end{aligned}\right\} \qquad \text{(b)}$$

3. *Solve modal equations.*

The equation of motion for an SDF system is

$$m\ddot{u} + ku = p(t) \qquad \text{(c)}$$

where $p(t)$ is the rectangular pulse force shown:



The solution of Eq. (c) is

$$u(t) = \begin{cases} \dfrac{p_o}{k}(1 - \cos\omega_n t) & 0 \le t \le t_d \\ u(t_d)\cos\omega_n (t - t_d) + \dfrac{\dot{u}(t_d)}{\omega_n}\sin\omega_n (t - t_d) & t \ge t_d \end{cases} \qquad \text{(d)}$$

For mode 1,

$$p_o = 100; \quad \omega_1 = 10; \quad t_d = 0.3 \text{ sec}$$

$$k = K_1 = \omega_1^2 M_1 = 10^2(0.576) = 57.6$$

Substituting in Eq. (d) for $t \le t_d$ gives

$$q_1(t) = \frac{100}{57.6}(1 - \cos 10t) = 1.736(1 - \cos 10t) \qquad 0 \le t \le 0.3 \qquad \text{(e)}$$

$$q_1(t_d) = 3.455 \qquad \dot{q}_1(t_d) = 2.450$$

Substituting $q_1(t_d)$ and $\dot{q}_1(t_d)$ for $u(t_d)$ and $\dot{u}(t_d)$, respectively, in Eq. (d) gives

$$q_1(t) = 3.455\cos 10(t - 0.3) + \frac{2.45}{10}\sin 10(t - 0.3) \qquad t \ge 0.3 \qquad \text{(f)}$$

For mode 2,

$$p_o = -100; \omega_2 = 38.73; \quad t_d = 0.3 \text{ sec}$$

$$k = K_2 = \omega_2^2 M_2 = (38.73)^2(0.576) = 864.0$$

Substituting in Eq. (d) for $t \le t_d$ gives

$$q_2(t) = \frac{-100}{864.0}(1 - \cos 38.73t) = -0.116(1 - \cos 38.73t) \qquad 0 \le t \le 0.3 \qquad \text{(g)}$$

$$q_2(t_d) = -0.0483 \quad \dot{q}_2(t_d) = 3.648$$

Substituting $q_2(t_d)$ and $\dot{q}_2(t_d)$ for $u(t_d)$ and $\dot{u}(t_d)$, respectively, in Eq. (d) gives

$$q_2(t) = -0.0483\cos 38.73(t - 0.3) + \frac{3.648}{38.73}\sin 38.73(t - 0.3) \qquad t \ge 0.3 \qquad \text{(h)}$$

4. *Combine modal responses.*

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} q_1(t) + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} q_2(t) \qquad 0 \le t \le 0.3 \qquad \text{(i)}$$

$\uparrow$ Eq. (e) $\qquad \uparrow$ Eq. (g)

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} q_1(t) + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} q_2(t) \qquad t \ge 0.3 \qquad \text{(j)}$$

$\uparrow$ Eq. (f) $\qquad \uparrow$ Eq. (h)

The modal responses and total response are plotted in Figs. P12.16a-b.

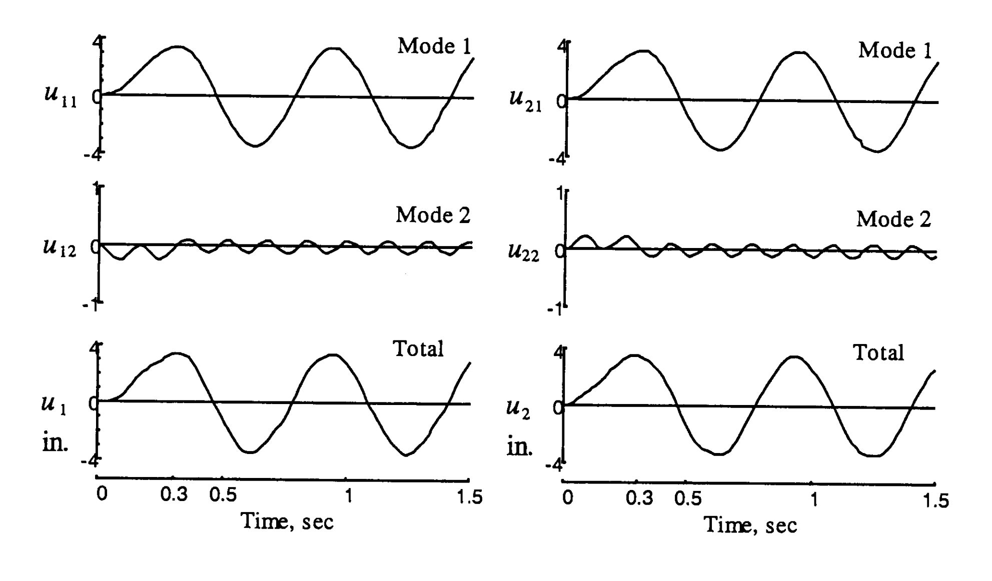
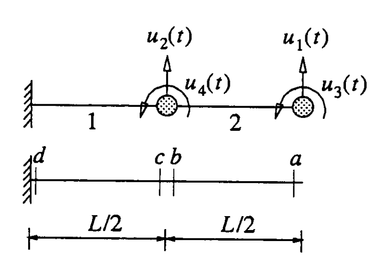


Fig. P12.16a

Fig. P12.16b

Problem 12.17



1. *Determine displacements and rotations.*

From Example 12.6,

$$\begin{aligned} u_1(t) &= q_1(t) + q_2(t) \\ u_2(t) &= 0.3274\, q_1(t) - 1.5274\, q_2(t) \end{aligned} \qquad \text{(a)}$$

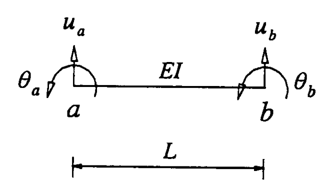
From Eq. (d) of Example 9.8 with $L = 120$ in.,

$$\begin{Bmatrix} u_3(t) \\ u_4(t) \end{Bmatrix} = \begin{bmatrix} 0.0214 & -0.0286 \\ 0.0071 & 0.0071 \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} \qquad \text{(b)}$$

Substituting Eq. (a) in Eq. (b) gives

$$\begin{aligned} u_3(t) &= 0.01207\, q_1(t) + 0.06509\, q_2(t) \\ u_4(t) &= 0.00948\, q_1(t) - 0.00376\, q_2(t) \end{aligned} \qquad \text{(c)}$$

2. *Write force-displacement relations.*



$$\begin{Bmatrix} V_a \\ V_b \\ M_a \\ M_b \end{Bmatrix} = \underbrace{\frac{EI}{L^3} \begin{bmatrix} 12 & -12 & 6L & 6L \\ & 12 & -6L & -6L \\ & & 4L^2 & 2L^2 \\ (sym) & & & 4L^2 \end{bmatrix}}_{\bar{\mathbf{k}}_e} \begin{Bmatrix} u_a \\ u_b \\ \theta_a \\ \theta_b \end{Bmatrix} \qquad \text{(d)}$$

3. *Determine forces in element 1.*

In Eq. (d), substituting $L/2$ for L in $\bar{\mathbf{k}}_e$, $u_a = 0$, $u_b = u_2$, $\theta_a = 0$, and $\theta_b = u_4$, and using Eqs. (b) and (c) gives

$$\begin{Bmatrix} V_d \\ V_c \\ M_d \\ M_c \end{Bmatrix} = \bar{\mathbf{k}}_1 \begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{Bmatrix} = \bar{\mathbf{k}}_1 \begin{Bmatrix} 0 \\ 0.3274\, q_1 - 1.5274\, q_2 \\ 0 \\ 0.00948\, q_1 - 0.00376\, q_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} -6.928\, q_1 + 227.9\, q_2 \\ 6.928\, q_1 - 227.9\, q_2 \\ -666.03\, q_1 + 7019\, q_2 \\ 250.37\, q_1 + 6655.5\, q_2 \end{Bmatrix}$$

Substituting $q_1 = 1.217$ in. and $q_2 = 0.0159$ in. (from Example 12.6) gives

$$\begin{Bmatrix} V_d \\ V_c \\ M_d \\ M_c \end{Bmatrix} = \begin{Bmatrix} -4.8 \\ 4.8 \\ -699 \\ 411 \end{Bmatrix}$$

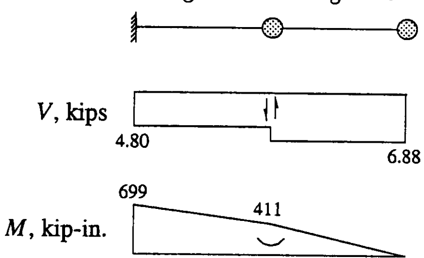
4. *Determine forces in element 2.*

In Eq. (d), substituting $L/2$ for L in $\bar{\mathbf{k}}_e$, $u_a = u_2$, $u_b = u_1$, $\theta_a = u_4$, and $\theta_b = u_3$, and using Eqs. (b) and (c) gives

$$\begin{Bmatrix} V_b \\ V_a \\ M_b \\ M_a \end{Bmatrix} = \bar{\mathbf{k}}_2 \begin{Bmatrix} u_2 \\ u_1 \\ u_4 \\ u_3 \end{Bmatrix} = \bar{\mathbf{k}}_2 \begin{Bmatrix} 0.3274\, q_1 - 1.5274\, q_2 \\ q_1 + q_2 \\ 0.00948\, q_1 - 0.0376\, q_2 \\ 0.01207\, q_1 + 0.06509\, q_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} -4.21\, q_1 - 110.8\, q_2 \\ 4.21\, q_1 + 110.8\, q_2 \\ -251.3\, q_1 - 6650.7\, q_2 \\ -0.97\, q_1 + 4.8\, q_2 \end{Bmatrix} = \begin{Bmatrix} -6.88 \\ 6.88 \\ -411 \\ 0 \end{Bmatrix}$$

5. *Draw shear and bending moment diagrams.*



These results are identical to those obtained in Example 12.6.

Problem 12.18

1. *Determine natural frequencies and modes.*

From Problem 12.13,

$$\omega_1 = 10 \qquad \omega_2 = 38.73$$

$$\phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \phi_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

2. *Determine equivalent static forces for each mode.*

$$\mathbf{f}_n(t) = \omega_n^2 \mathbf{m} \phi_n q_n(t) \quad n = 1,\ 2$$

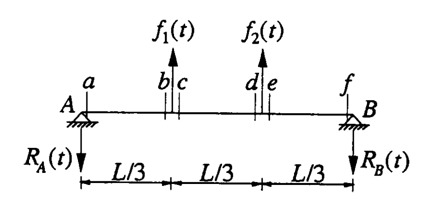
$$\begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}_1 = (10)^2\, 0.288 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} q_1(t) = \begin{Bmatrix} 28.8 \\ 28.8 \end{Bmatrix} q_1(t) \qquad \text{(a)}$$

$$\begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}_2 = (38.73)^2\, 0.288 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} q_2(t) = \begin{Bmatrix} 432 \\ -432 \end{Bmatrix} q_2(t) \qquad \text{(b)}$$

3. *Combine modal forces.*

$$\begin{aligned} f_1(t) &= 28.8\, q_1(t) + 432\, q_2(t) \\ f_2(t) &= 28.8\, q_1(t) - 432\, q_2(t) \end{aligned} \qquad \text{(c)}$$

4. *Determine internal forces.*



By statics, the reactions are

$$\begin{aligned} R_A(t) &= \frac{2}{3} f_1(t) + \frac{1}{3} f_2(t) \\ R_B(t) &= \frac{1}{3} f_1(t) + \frac{2}{3} f_2(t) \end{aligned} \qquad \text{(d)}$$

Static analysis gives the bending moment

$$M_a(t) = M_f(t) = 0$$

$$M_b(t) = M_c(t) = R_A(t)\frac{L}{3} = \frac{L}{9}\left[2f_1(t) + f_2(t)\right] \qquad \text{(e)}$$

$$M_d(t) = M_e(t) = R_B(t)\frac{L}{3} = \frac{L}{9}\left[f_1(t) + 2f_2(t)\right]$$

Static analysis gives the shears:

$$\begin{aligned} V_a(t) &= V_b(t) = \frac{2}{3} f_1(t) + \frac{1}{3} f_2(t) \\ V_f(t) &= V_e(t) = \frac{1}{3} f_1(t) + \frac{2}{3} f_2(t) \\ V_c(t) &= R_A(t) - f_1(t) = -\frac{1}{3} f_1(t) + \frac{1}{3} f_2(t) \\ V_d(t) &= f_2(t) - R_B(t) = \frac{1}{3} f_1(t) + \frac{1}{3} f_2(t) \end{aligned} \qquad \text{(f)}$$

where $f_1(t)$ and $f_2(t)$ are known from Eq. (c).

5. *Determine modal coordinates at* $t = 0.1$ *sec.*

From Problem 12.14,

$$\begin{aligned} q_1(t) &= 1.736\,(1 - \cos 10t) \\ q_2(t) &= 0.116\,(1 - \cos 38.73t) \end{aligned} \qquad \text{(g)}$$

At $t = 0.1$ sec

$$q_1 = 0.798 \text{ in.} \qquad q_2 = -0.202 \text{ in.} \qquad \text{(h)}$$

6. *Determine internal forces at* $t = 0.1$ *sec.*

Substituting Eq. (h) in Eq. (a) and Eq. (b) gives numerical values for the equivalent static forces shown in Figs. P12.18a-b where the shearing forces and bending moments due to each mode are plotted. The combined values are shown in Fig. P12.18c.

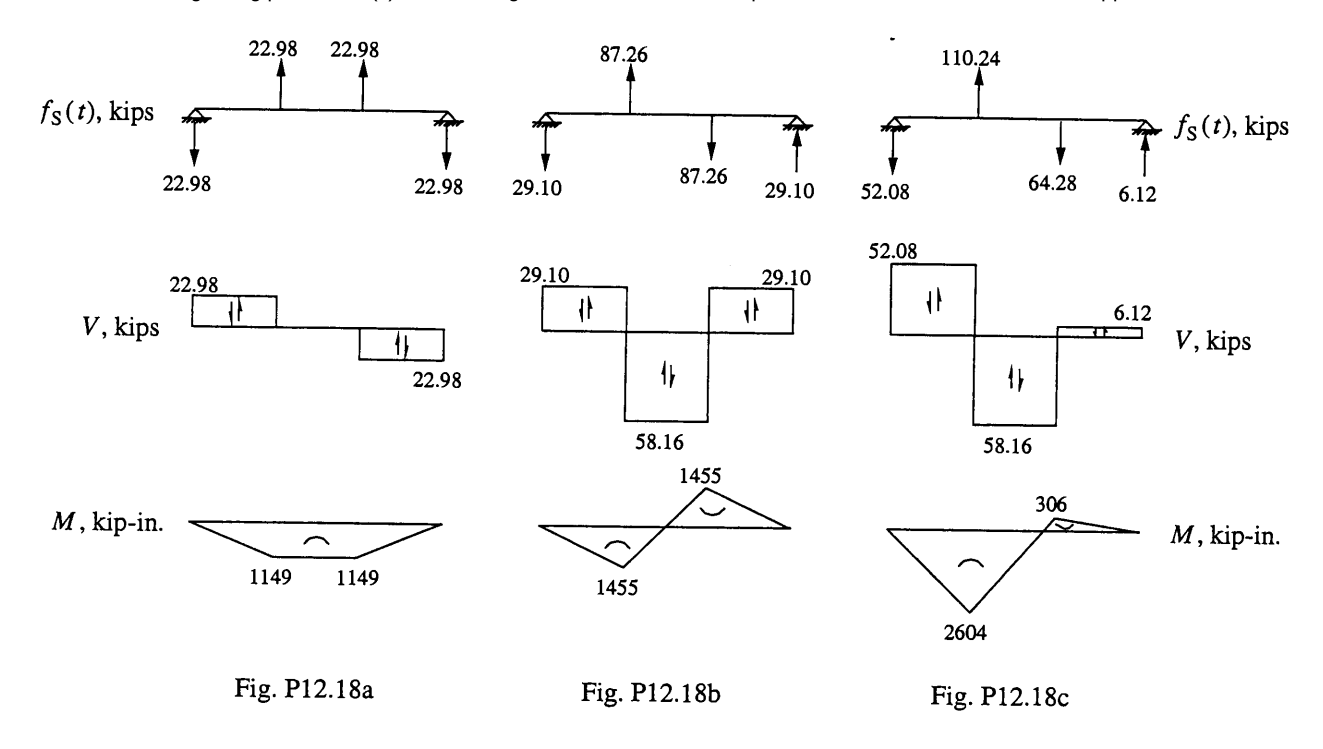


Fig. P12.18a

Fig. P12.18b

Fig. P12.18c

Problem 12.19

$u_1(t)$ $u_2(t)$

$p_1(t) = 100 \sin 25t$

1. *Determine natural frequencies and modes.*

   See Problem 12.13.

2. *Set up modal equations.*

$$M_n \ddot{q}_n + K_n q_n = P_{no} \sin \omega t$$

where

$$P_{no} = \phi_n^T \mathbf{p}_o = 100$$

3. *Solve modal equations.*

$$q_n(t) = \frac{P_{no}}{K_n} \mathcal{C}_n \sin \omega t \qquad \mathcal{C}_n = \frac{1}{1 - (\omega/\omega_n)^2}$$

$$q_1(t) = \frac{100}{(10)^2\ 0.576} \frac{1}{1 - (25/10)^2} \sin 25t$$
$$= -0.331 \sin 25t$$

$$q_2(t) = \frac{100}{(38.73)^2\ 0.576} \frac{1}{1 - (25/38.73)^2} \sin 25t$$
$$= 0.198 \sin 25t$$

4. *Determine displacements and accelerations.*

$$\mathbf{u} = \mathbf{\Phi}\,\mathbf{q} = \begin{Bmatrix} q_1(t) + q_2(t) \\ q_1(t) - q_2(t) \end{Bmatrix} = \begin{Bmatrix} -0.133 \sin 25t \\ -0.529 \sin 25t \end{Bmatrix}$$

$$\ddot{\mathbf{u}} = \mathbf{\Phi}\,\ddot{\mathbf{q}} = \begin{Bmatrix} \ddot{q}_1(t) + \ddot{q}_2(t) \\ \ddot{q}_1(t) - \ddot{q}_2(t) \end{Bmatrix} = (25)^2\,\mathbf{u}$$
$$= \begin{Bmatrix} -83.125 \sin 25t \\ -330.63 \sin 25t \end{Bmatrix}$$

5. *Determine amplitudes of displacements and accelerations.*

$$\mathbf{u}_o = \begin{Bmatrix} 0.133 \\ 0.529 \end{Bmatrix} \text{in.} \quad \ddot{\mathbf{u}}_o = \begin{Bmatrix} 83.125 \\ 330.63 \end{Bmatrix} \text{in./sec}^2$$

Problem 12.20

1. *Determine equivalent static forces.*

From Problem 12.18,

$$\begin{aligned} f_1(t) &= 28.8 q_1(t) + 432 q_2(t) \\ f_2(t) &= 28.8 q_1(t) - 432 q_2(t) \end{aligned} \qquad \text{(a)}$$

2. *Determine modal coordinate amplitudes.*

From Problem 12.19,

$$q_1(t) = -0.331 \sin 25t \quad q_2(t) = 0.198 \sin 25t \qquad \text{(b)}$$

To determine the response amplitude we specialize Eq. (b) when $\sin 25t = 1$; thus

$$q_{1o} = -0.331 \qquad q_{2o} = 0.198$$

Substituting q_{no} for $q_n(t)$ in Eq. (a) gives

$$f_{1o} = 28.8\,(-0.331) + 432\,(0.198) = 76.00 \text{ kips}$$

$$f_{2o} = 28.8\,(-0.331) - 432\,(0.198) = -95.07 \text{ kips}$$

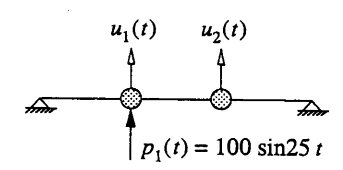
3. *Determine bending moments.*

From Eq. (e) of Problem 12.18,

$$M_{bo} = \frac{150}{9}\left[2\,(76.00) - 95.07\right] = 948.8 \text{ kip-in.}$$

$$M_{do} = \frac{150}{9}\left[(76.00) - 2\,(95.07)\right] = -1902 \text{ kip-in.}$$

Problem 12.21



1. *Determine natural frequencies and modes.*

   See Problem 12.13.

2. *Set up modal equations.*

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = P_{no} \sin \omega t \quad \text{(a)}$$

where

$$P_{no} = \phi_n^T \mathbf{p}_o = 100 \quad \text{(b)}$$

3. *Solve modal equations.*

$$q_n(t) = \frac{P_{no}}{K_n}\left[\mathcal{C}_n \sin \omega t + \mathcal{D}_n \cos \omega t\right] \quad \text{(c)}$$

where

$$\mathcal{C}_n = \frac{1 - (\omega/\omega_n)^2}{[1 - (\omega/\omega_n)^2]^2 + [2\zeta_n \omega/\omega_n]^2} \quad \text{(d)}$$

$$\mathcal{D}_n = \frac{-2\zeta_n \omega/\omega_n}{[1 - (\omega/\omega_n)^2]^2 + [2\zeta_n \omega/\omega_n]^2} \quad \text{(e)}$$

For the first mode,

$$\frac{\omega}{\omega_1} = \frac{25}{10} = 2.5 \qquad \zeta_1 = 0.10$$

$$\frac{P_{1o}}{K_1} = \frac{100}{(10)^2\, 0.576} = 1.736$$

Substituting these numerical values in Eqs. (c)-(e) gives

$$q_1(t) = -0.328 \sin 25t - 0.031 \cos 25t$$

For the second mode:

$$\frac{\omega}{\omega_2} = \frac{25}{38.73} = 0.645 \qquad \zeta_2 = 0.10$$

$$\frac{P_{2o}}{K_2} = \frac{100}{(38.73)^2\, 0.576} = 0.116$$

Substituting these numerical values in Eqs. (c)-(e) gives

$$q_2(t) = 0.189 \sin 25t - 0.042 \cos 25t$$

4. *Determine displacements.*

$$\mathbf{u} = \Phi\,\mathbf{q} = \begin{Bmatrix} q_1(t) + q_2(t) \\ q_1(t) - q_2(t) \end{Bmatrix}$$

$$= \begin{Bmatrix} -0.139 \sin 25t - 0.073 \cos 25t \\ -0.517 \sin 25t + 0.011 \cos 25t \end{Bmatrix}$$

5. *Determine displacement amplitudes.*

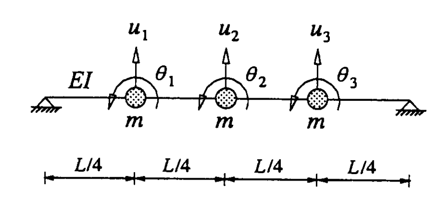
$$\begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} = \begin{Bmatrix} \sqrt{(-0.139)^2 + (-0.073)^2} \\ \sqrt{(-0.517)^2 + (0.011)^2} \end{Bmatrix} = \begin{Bmatrix} 0.157 \\ 0.517 \end{Bmatrix} \text{in.}$$

6. *Determine acceleration amplitudes.*

$$\ddot{\mathbf{u}}_o = \omega^2 \mathbf{u}_o = (25)^2 \mathbf{u}_o$$

$$\begin{Bmatrix} \ddot{u}_{1o} \\ \ddot{u}_{2o} \end{Bmatrix} = (25)^2 \begin{Bmatrix} 0.157 \\ 0.517 \end{Bmatrix} = \begin{Bmatrix} 98.125 \\ 323.12 \end{Bmatrix} \text{in./sec}^2$$

Problem 12.22



$E = 30{,}000$ ksi $\qquad I = 100$ in.^{4}

$L = 150$ in. $\qquad L' = \dfrac{L}{4} = 37.5$ in.

$m = 0.192$ kip$-$sec^2/in.

1. *Define DOFs.*

$$\mathbf{u}_t = \langle u_1 \quad u_2 \quad u_3 \rangle^T \qquad \mathbf{u}_0 = \langle \theta_1 \quad \theta_2 \quad \theta_3 \rangle^T$$

2. *Determine mass matrix in terms of* $\mathbf{u}_t$.

$$\mathbf{m} = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix} = \begin{bmatrix} 0.192 & & \\ & 0.192 & \\ & & 0.192 \end{bmatrix}$$

3. *Determine stiffness matrix.*

$$\mathbf{k} = \frac{EI}{L'^3} \left[\begin{array}{ccc|ccc} 7L'^2 & 2L'^2 & 0 & 3L' & -6L' & 0 \\ & 8L'^2 & 2L'^2 & 6L' & 0 & -6L' \\ & & 7L'^2 & 0 & 6L' & -3L' \\ \hline & & & 15 & -12 & 0 \\ & (sym) & & & 24 & -12 \\ & & & & & 15 \end{array}\right]$$

4. *Determine lateral stiffness matrix.*

$$\begin{bmatrix} \mathbf{k}_{00} & \mathbf{k}_{0t} \\ \mathbf{k}_{t0} & \mathbf{k}_{tt} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_t \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_s \end{bmatrix} \qquad \mathbf{u}_0 = -\mathbf{k}_{00}^{-1}\, \mathbf{k}_{0t}\, \mathbf{u}_t$$

$$\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{t0}\, \mathbf{k}_{00}^{-1}\, \mathbf{k}_{0t}$$

$$\hat{\mathbf{k}}_{tt} = \frac{EI}{L^3} \begin{bmatrix} 630.86 & -603.43 & 246.86 \\ & 877.71 & -603.43 \\ (sym) & & 630.86 \end{bmatrix}$$

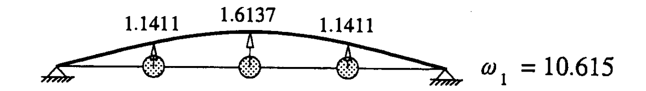
5. *Determine natural frequencies and modes.*

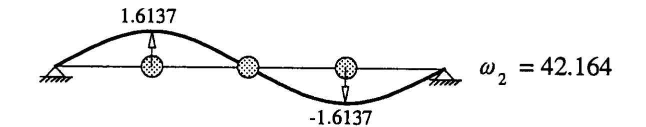
$$\omega_n = 10.615, \quad 42.164, \quad 89.523 \text{ rads/sec}$$

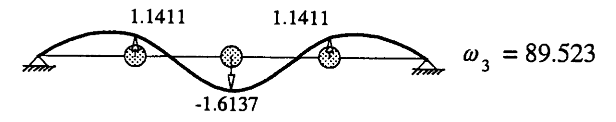
$$\Phi = \begin{bmatrix} 1.1411 & 1.6137 & 1.1411 \\ 1.6137 & 0 & -1.6137 \\ 1.1411 & -1.6137 & 1.1411 \end{bmatrix}$$

The modes are normalized such that

$$M_n = \phi_n^T \mathbf{m}\, \phi_n = 1$$







6. *Determine modal expansion of* **s**.

$$\mathbf{s} = \sum_{n=1}^{3} \mathbf{s}_n = \sum_{n=1}^{3} \Gamma_n\, \mathbf{m}\, \phi_n$$

where

$$\Gamma_n = \frac{\phi_n^T \mathbf{s}}{M_n} = \phi_n^T \mathbf{s}$$

The $\mathbf{s}_n$ values for force distributions $\mathbf{s}_a$ and $\mathbf{s}_b$ are computed and summarized in Table P12.22a-b.

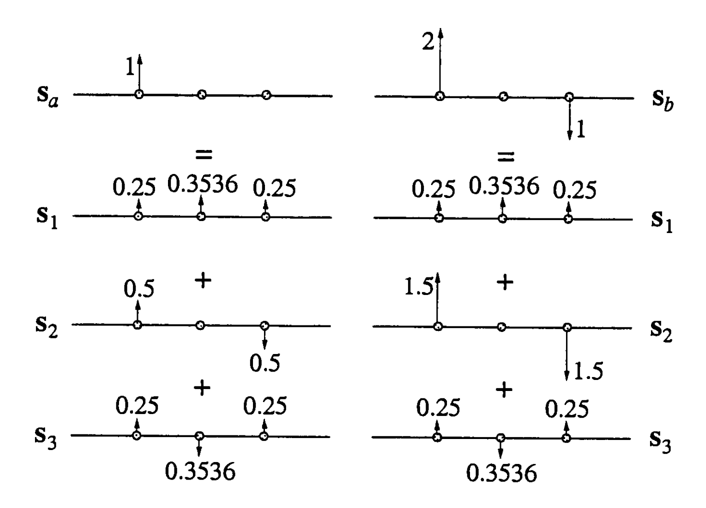
Table P12.22a

$\mathbf{s}_a$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$
1	0.25	0.5	0.25
0	0.3536	0	− 0.3536
0	0.25	− 0.5	0.25

Table P12.22b

$\mathbf{s}_b$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$
2	0.25	1.5	0.25
0	0.3536	0	− 0.3536
− 1	0.25	− 1.5	0.25

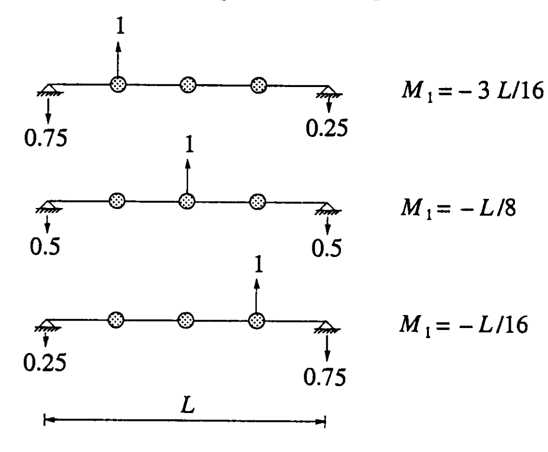
The modal expansions of $\mathbf{s}_a$ and $\mathbf{s}_b$ are shown graphically:



All three modes contribute similarly to force distribution $\mathbf{s}_a$, but the second mode has a predominant component in $\mathbf{s}_b$. Therefore the second mode will probably contribute more than other modes in response of the system to $\mathbf{p}(t) = \mathbf{s}_b p(t)$; however, this observation is tentative until we examine the dynamic response factor for $p(t)$.

7. *Determine modal static responses* M_n^{st}.

First determine M_1 for following load cases:

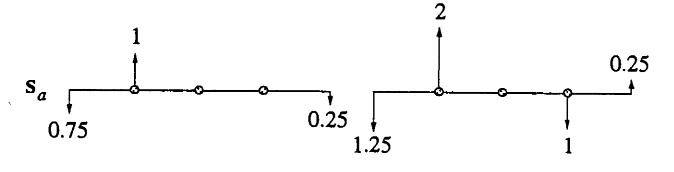


The values of M_n^{st} due to forces $\mathbf{s}_n$ is determined by linear combination of the above three load cases; the results are in Table P12.22c.

Table P12.22c

Mode, n	M_{1n}^{st} due to $\mathbf{s}_a$	M_{1n}^{st} due to $\mathbf{s}_b$
1	$-0.1067L$	$-0.1067L$
2	$-0.0625L$	$-0.1875L$
3	$-0.0183L$	$-0.0183L$
$\sum_{n=1}^{3} M_{1n}^{st}$	$-0.1875L$ $= -28.125$ k-in.	$-0.3125L$ $= -46.875$ k-in.

Next we determine M_1^{st}:



$$M_1^{st} = -\frac{3L}{16} = -0.1875L \quad M_1^{st} = -\frac{5L}{16} = -0.3125L$$

We have demonstrated that $\sum_{n=1}^{3} M_{1n}^{st} = M_1^{st}$.

8. *Determine modal contribution factors, their cumulative values and error* e_J.

Table P12.22d

Mode, n or no. of modes, J	Force distribution $\mathbf{s}_a$		
	$\overline{M}_{1n}$	$\sum_{n=1}^{J} \overline{M}_{1n}$	e_J
1	0.5691	0.5691	0.4309
2	0.3333	0.9024	0.0976
3	0.0976	1.0	0

Table P12.22e

Mode, n or no. of modes, J	Force distribution $\mathbf{s}_b$		
	$\overline{M}_{1n}$	$\sum_{n=1}^{J} \overline{M}_{1n}$	e_J
1	0.3414	0.3414	0.6586
2	0.6000	0.9414	0.0586
3	0.0586	1.0	0

In case of $\mathbf{s}_a$, the modal contribution factor is largest for the first mode and progressively decreases for the second and third modes (Table P12.22d). In case of $\mathbf{s}_b$, the modal contribution factor is largest for the second mode (Table P12.22e). As a corollary, for the $\mathbf{s}_b$ case e_1 is larger but e_2 is smaller, both relative to the $\mathbf{s}_a$ case.

9. *Determine response to half-cycle sine pulse.*

The peak modal response equation (12.11.2) is specialized for $r \equiv M_1$ to obtain

$$\left(M_{1n}\right)_o = p_o M_1^{st} \overline{M}_{1n} R_{dn}$$

Table P12.22f

Mode, n	Spectral values		Force Distribution $\mathbf{s}_a$		Force Distribution $\mathbf{s}_b$	
	$\frac{T_n}{t_d}$	R_{dn}	$\overline{M}_{1n}$	$\frac{(M_{1n})_o}{p_o M_1^{st}}$	$\overline{M}_{1n}$	$\frac{(M_{1n})_o}{p_o M_1^{st}}$
1	1	1.73	0.5691	0.985	0.3414	0.591
2	0.252	1.14	0.3333	0.380	0.6000	0.684
3	0.119	1.06	0.0976	0.103	0.0586	0.062

10. *Comments.*

(a) For force distribution $\mathbf{s}_a$, the modal responses decrease for higher modes, as suggested by the modal contribution factors $\overline{M}_{1n}$; the decrease is more rapid, however, because R_{dn} also decreases with mode number. For force distribution $\mathbf{s}_b$, the largest contribution is from the second mode, again as suggested by $\overline{M}_{1n}$.

(b) No. The peak value of the total response can not be determined from the peak modal responses because the modal peaks occur at different time instants. Furthermore, the SRSS or CQC modal combination rules do not apply to pulse excitations.

Problem 12.23

From Problem 9.13 the mass stiffness matrices are:

$$\mathbf{m} = m\begin{bmatrix} 5 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{k} = \frac{3EI}{10L^3}\begin{bmatrix} 28 & 6 & -6 \\ 6 & 7 & 3 \\ -6 & 3 & 7 \end{bmatrix}$$

From Problem 10.23 the natural frequencies and modes of the system are:

$$\omega_1 = 0.5259\sqrt{\frac{EI}{mL^3}}, \omega_2 = 1.6135\sqrt{\frac{EI}{mL^3}},$$

$$\omega_3 = 1.7321\sqrt{\frac{EI}{mL^3}}$$

$$\phi_1 = \begin{Bmatrix} 1 \\ -1.9492 \\ 1.9492 \end{Bmatrix}, \qquad \phi_2 = \begin{Bmatrix} 1 \\ 1.2826 \\ -1.2826 \end{Bmatrix}, \qquad \phi_3 = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

The generalized masses

$$\mathbf{M}_n = \phi_n^T \mathbf{m} \phi_n$$

for the three modes are

$$M_1 = 12.60, \qquad M_2 = 8.29, \qquad M_3 = 2.0$$

Part a

1. *Determine modal expansion of* $\mathbf{s}$*:*

$$\mathbf{s} = \sum_{n=1}^{3} \mathbf{s}_n = \sum_{n=1}^{3} \Gamma_n \mathbf{m} \phi_n$$

where

$$\Gamma_n = \frac{\phi_n^T \mathbf{s}}{M_n}$$

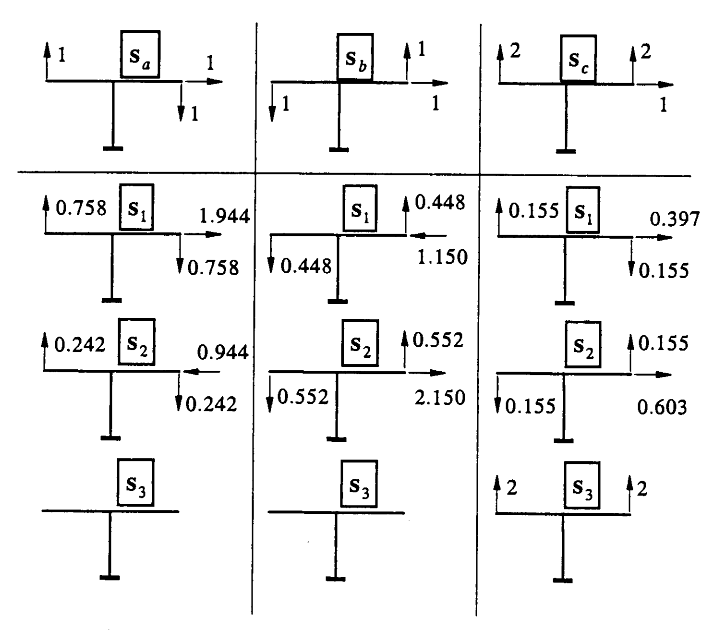
The $\mathbf{s}_n$ values for force distributions $\mathbf{s}_a$, $\mathbf{s}_b$ and $\mathbf{s}_c$ are computed and summarized:

$\mathbf{s}_a$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$
1	1.944	−0.944	0
−1	− 0.758	− 0.242	0
1	0.758	0.242	0

$\mathbf{s}_b$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$
1	−1.150	2.150	0
1	0.448	0.552	0
−1	− 0.448	− 0.552	0

$\mathbf{s}_b$	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$
1	0.397	0.603	0
2	− 0.155	0.155	2
2	0.155	− 0.155	2

The model expansions $\mathbf{s}_a$, $\mathbf{s}_b$ and $\mathbf{s}_c$ are shown graphically



Note that because $\mathbf{s}_a$ and $\mathbf{s}_b$ are anti-symmetric they do not have any component in the symmetric third mode, whereas, the main component of $\mathbf{s}_c$ is in the third mode.

Part b

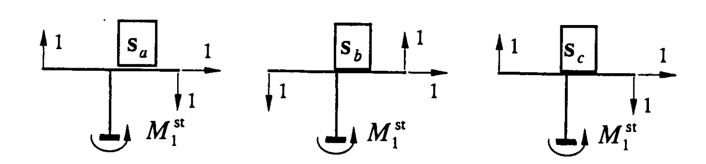
2a. *Determine the modal static responses of* M_{1n}^{st}.

First determine M_1 for the following load cases:

The values of M_{1n}^{st} due to forces $\mathbf{s}_n$ is determined by linear combination of the above three load cases; the results are summarized below:

Mode, n	M_{1n}^{st} due to $\mathbf{s}_a$	M_{1n}^{st} due to $\mathbf{s}_b$	M_{1n}^{st} due to $\mathbf{s}_c$
1	3.460 L	-2.047 L	0.706 L
2	-0.460 L	1.047 L	0.294 L
3	0	0	0
$\sum_{n=1}^{3} M_{1n}^{st}$	$3L$	$-L$	L

2b. *Determine* M_1^{st}:



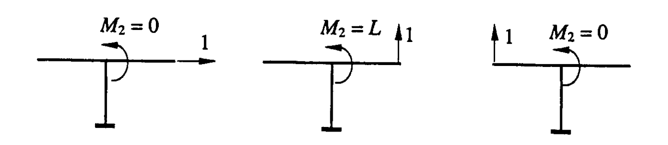
$M_1^{st} = L + L + L = 3L$ $\quad$ $M_1^{st} = L - L - L = -L$ $\quad$ $M_1^{st} = L - 2L + 2L = L$

2c. *We have demonstrated that:*

$$M_1^{st} = \sum M_{1n}^{st}$$

3a. *Determine the modal static responses of* M_{2n}^{st}.

First determine M_2 for the following load cases:



The values of M_{2n}^{st} due to forces $\mathbf{s}_n$ is determined by linear combination of the above three load cases; the results are summarized below:

Mode, n	M_{2n}^{st} due to $\mathbf{s}_a$	M_{2n}^{st} due to $\mathbf{s}_b$	M_{2n}^{st} due to $\mathbf{s}_c$
1	-0.758 L	0.448 L	-0.155 L
2	-0.242 L	0.552 L	0.155 L
3	0	0	2 L
$\sum_{n=1}^{3} M_{1n}^{st}$	$-L$	L	2 L

3b. *Determine* M_{2n}^{st}:

$M_2^{st} = -L$ $\quad$ $M_2^{st} = L$ $\quad$ $M_2^{st} = 2L$

We have demonstrated that:

$$M_2^{st} = \sum M_{2n}^{st}$$

Part c

4. *Determine modal contribution factors, their cumulative values and error* e_J.

These results for M_1 are as follows:

Mode, n or number of modes, J	Force distribution $\mathbf{s}_a$		
	$\bar{M}_{1n}$	$\sum_{n=1}^{J} \bar{M}_{1n}$	e_J
1	0.883	0.883	0.117
2	0.117	1.0	0
3	0	1.0	0

Mode, n ornumber of modes, J	Force distribution $\mathbf{s}_b$		
	$\bar{M}_{1n}$	$\sum_{n=1}^{J} \bar{M}_{1n}$	e_J
1	0.662	0.662	0.338
2	0.338	1.0	0
3	0	1.0	0

Mode, n or number of modes, J	Force distribution $\mathbf{s}_c$		
	$\bar{M}_{1n}$	$\sum_{n=1}^{J} \bar{M}_{1n}$	e_J
1	0.706	0.706	0.294
2	0.294	1.0	0
3	0	1.0	0

These results for M_2 are as follows:

Mode, n or number of modes, J	Force distribution $\mathbf{s}_a$		
	$\overline{M}_{2n}$	$\sum_{n=1}^{J} \overline{M}_{2n}$	e_J
1	0.758	0.758	0.242
2	0.242	1.0	0
3	0	1.0	0

Mode, n or number of modes, J	Force distribution $\mathbf{s}_b$		
	$\overline{M}_{2n}$	$\sum_{n=1}^{J} \overline{M}_{2n}$	e_J
1	0.448	0.448	0.552
2	0.552	1.0	0
3	0	1.0	0

Mode, n or number of modes, J	Force distribution $\mathbf{s}_c$		
	$\overline{M}_{2n}$	$\sum_{n=1}^{J} \overline{M}_{2n}$	e_J
1	0. 067	0. 067	0.933
2	0. 067	0.134	0.866
3	0.866	1.0	0

Observe that:

1. For force distribution $\mathbf{s}_a$ and $\mathbf{s}_b$, the third modal contribution factor is zero for responses M_1 and M_2. It will be zero for every response quantity, because the third mode, being symmetric, does not contribute to the antisymmetric force distributions $\mathbf{s}_a$ and $\mathbf{s}_b$ (see Part a).

2. For force distribution $\mathbf{s}_c$, the third modal contribution factor for response M_2, is dominant relative to the other two modes; however, it is zero for response M_1, because a symmetric mode will not contribute to M_1.

Part d

5. *Determine response to rectangular pulse.*

The peak modal response Eq. (12.11.2) is specialized for $r \equiv M_1$ to obtain:

$$(M_{1n})_o = p_0 M_{1n}^{st} R_{dn}$$

Mode, n	Spectral Values		Force Distribution, $\mathbf{s}_a$		Force Distribution, $\mathbf{s}_b$		Force Distribution, $\mathbf{s}_c$	
	$\frac{T_n}{t_d}$	R_{dn}	$\overline{M}_{1n}$	$\frac{(M_{1n})_o}{p_0 M_1^{st}}$	$\overline{M}_{1n}$	$\frac{(M_{1n})_o}{p_0 M_1^{st}}$	$\overline{M}_{1n}$	$\frac{(M_{1n})_o}{p_0 M_1^{st}}$
1	5.0	0.691	0.883	0.610	0.662	0.457	0.706	0.502
2	1.63	2.0	0.117	0.234	0.338	0.676	0.294	0.587
3	1.51	2.0	0	0	0	0	0	0

Similarly for

$$(M_{2n})_o = p_0 M_{2n}^{st} R_{dn}$$

Mode, n	Spectral Values		Force Distribution, $\mathbf{s}_a$		Force Distribution, $\mathbf{s}_b$		Force Distribution, $\mathbf{s}_c$	
	$\frac{T_n}{t_d}$	R_{dn}	$\overline{M}_{1n}$	$\frac{(M_{1n})_o}{p_0 M_1^{st}}$	$\overline{M}_{1n}$	$\frac{(M_{1n})_o}{p_0 M_1^{st}}$	$\overline{M}_{1n}$	$\frac{(M_{1n})_o}{p_0 M_1^{st}}$
1	5.0	0.691	0.758	0.524	0.448	0.310	0.067	0.046
2	1.63	2.0	0.242	0.484	0.552	1.104	0.067	0.046
3	1.51	2.0	0	0	0	0	0.866	1.732

Part e

6. *Comments.*

The peak modal responses determined in Part (d) demonstrate that the relative response contributions of the three modes depend on the numerical values of the modal contribution factors ($\overline{M}_{1n}$ and $\overline{M}_{2n}$) and on the dynamic response factors (R_{dn}). As an early example for the force distribution $\mathbf{s}_b$, the second mode contribution to M_1 is larger than the first mode contribution, although $\overline{M}_{11} > \overline{M}_{12}$, because R_{d2} is almost three times R_{d1}.

Part f

7. No. The peak value of the total modal response can not be determined from the peak modal responses because the modal peaks occur at different time instants. Furthermore, the SRSS or CQC modal combination rules do not apply to pulse excitations.

Problem 12.24

***Part a**: Classical modal analysis*

The nth-mode response is given by Eq. (12.10.1), specialized for $r \equiv M_1$:

$$M_{1n}(t) = M_1^{st}\,\overline{M}_{1n}\left[\omega_n^2 D_n(t)\right] \qquad \text{(a)}$$

where $D_n(t)$, the solution of Eq. (12.9.2), is

$$D_n(t) = \frac{p_o}{\omega_n^2}\frac{1}{1-(T_n/2t_d)^2}\left[\sin\left(\pi\frac{t}{t_d}\right) - \left(\frac{T_n}{2t_d}\right)\sin\left(2\pi\frac{t}{T_n}\right)\right]$$

$$t \le t_d \qquad \text{(b)}$$

$$D_n(t) = D_n(t_d)\cos\omega_n(t-t_d) + \frac{\dot{D}_n(t_d)}{\omega_n}\sin\omega_n(t-t_d)$$

$$t \ge t_d \qquad \text{(c)}$$

For force distribution $\mathbf{s}_b$, from Problem 12.22,

$$M_1^{st} = -0.3125L$$

$$\overline{M}_{1n} = 0.3414,\ 0.6000,\ \text{and}\ 0.0586$$

In Eq. (a) we substitute these data, and numerical values for $\omega_n^2 D_n(t)$ from Eqs. (b) and (c) in witch

$$t_d = T_1 = \frac{2\pi}{10.615} = 0.5919\,\text{sec}\ \ \text{(from Problem 12.22)}$$

The resulting bending moment due to each mode and the total response is plotted in Fig. P12.24a.

Part b: Static correction method

Specializing Eq. (12.12.6) for M_1 with $N_d = 1$:

$$M_1(t) = M_1^{st}\left\{p(t) + \overline{M}_{11}\left[\omega_1^2 D_1(t) - p(t)\right]\right\}$$

(d)

where $M_1^{st} = -0.3125L$, $\overline{M}_{11} = 0.3414$, $D_1(t)$ was computed in Part a, and

$$p(t) = \begin{cases} p_o \sin\left(\dfrac{\pi t}{t_d}\right) & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(e)}$$

Eq. (d) is plotted in Fig. P12.24b, where it is compared with the exact solution by classical modal analysis. The static correction method provides reasonable results; the error is primarily because the second mode has significant dynamic effects, as indicated by $R_{d2} = 1.14$ being significantly larger than one.

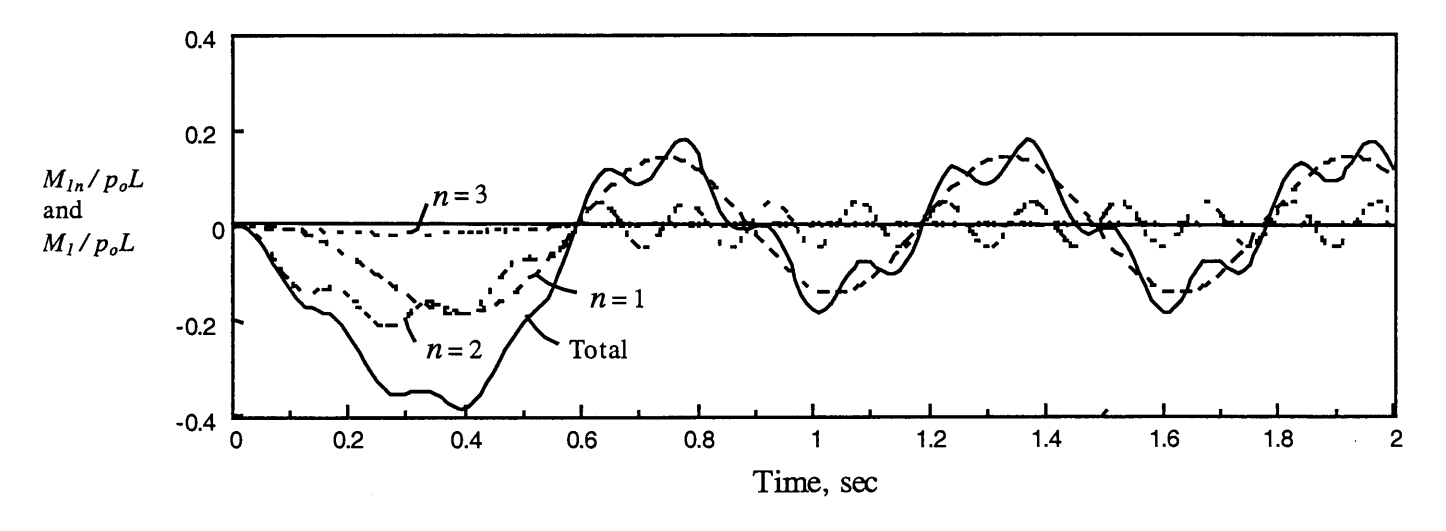


Fig. P12.24a

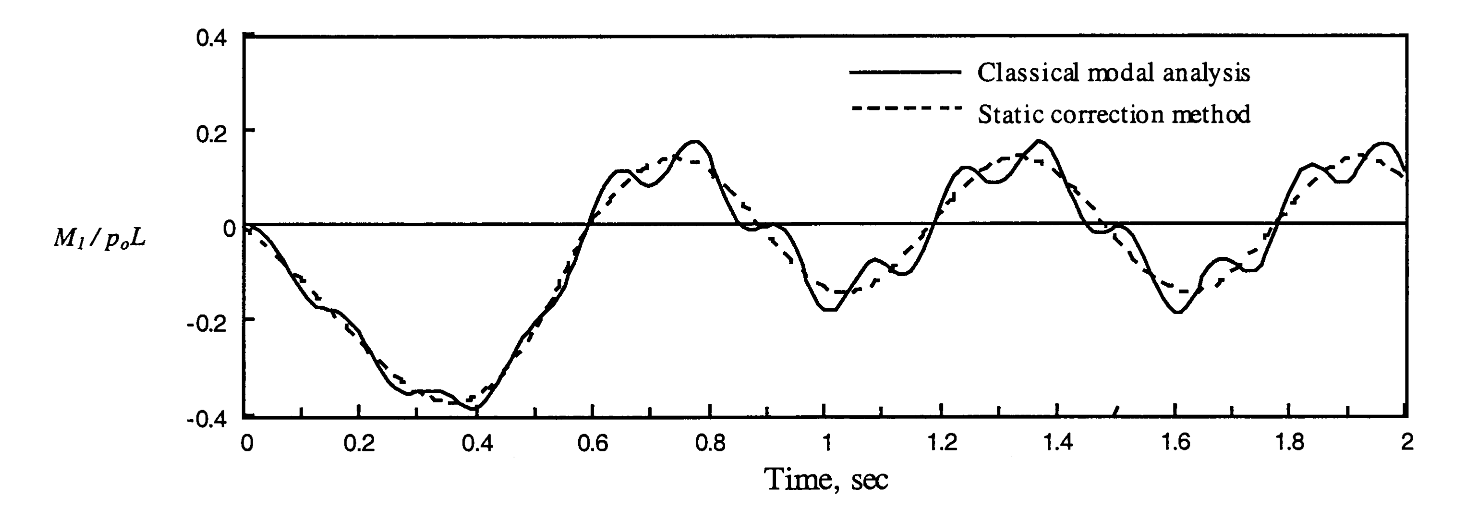


Fig. P12.24b

Problem 13.1

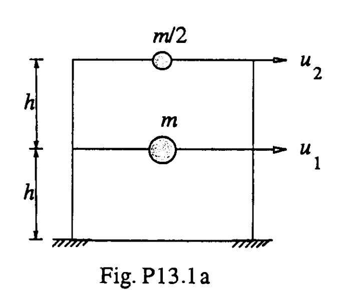


Fig. P13.1a

Stiffness and mass matrices (from Problem 9.5):

$$\mathbf{k} = k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{m} = m\begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix}$$

Vibration properties (from Problem 10.6):

$$\omega_1 = 0.765\sqrt{\frac{k}{m}} \qquad \omega_2 = 1.848\sqrt{\frac{k}{m}}$$

$$\phi_1 = \langle 0.707 \quad 1\rangle^T \qquad \phi_2 = \langle -0.707 \quad 1\rangle^T$$

Part a

The modal quantities given by Eq. (13.2.3) are

$$M_1 = m \qquad M_2 = m$$

$$K_1 = 0.586k \qquad K_2 = 3.414k$$

$$L_1^h = 1.207m \qquad L_2^h = -0.207m$$

$$\Gamma_1 = \frac{L_1^h}{M_1} = 1.207 \qquad \Gamma_2 = \frac{L_2^h}{M_2} = -0.207$$

Substituting Γ_n, $\mathbf{m}$, and ϕ_n in Eq. (13.2.4) gives

$$\mathbf{s}_1 = \Gamma_1\,\mathbf{m}\,\phi_1 = m\begin{Bmatrix} 0.854 \\ 0.604 \end{Bmatrix}$$

$$\mathbf{s}_2 = \Gamma_2\,\mathbf{m}\,\phi_2 = m\begin{Bmatrix} 0.146 \\ -0.104 \end{Bmatrix}$$

The modal expansion of effective earthquake forces is shown in Fig. P13.1b.

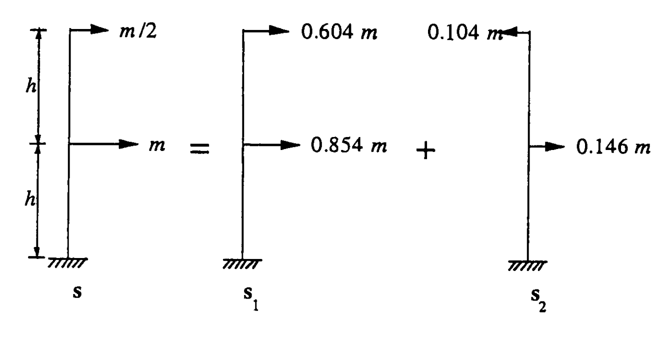


Fig. P13.1b

Part b

Substituting Γ_n and ϕ_{jn} in Eq. (13.2.5) gives floor displacements due to each mode:

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}_1 = 1.207\begin{Bmatrix} 0.707 \\ 1 \end{Bmatrix} D_1(t) = \begin{Bmatrix} 0.854 \\ 1.207 \end{Bmatrix} D_1(t)$$

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}_2 = -0.207\begin{Bmatrix} -0.707 \\ 1 \end{Bmatrix} D_2(t) = \begin{Bmatrix} 0.146 \\ -0.207 \end{Bmatrix} D_2(t)$$

Combining the contributions of the two modes gives

$$u_1(t) = 0.854\,D_1(t) + 0.146\,D_2(t)$$

$$u_2(t) = 1.207\,D_1(t) - 0.207\,D_2(t)$$

Part c

The modal static responses V_{jn}^{st} for the story shears are determined in Fig. P13.1c.

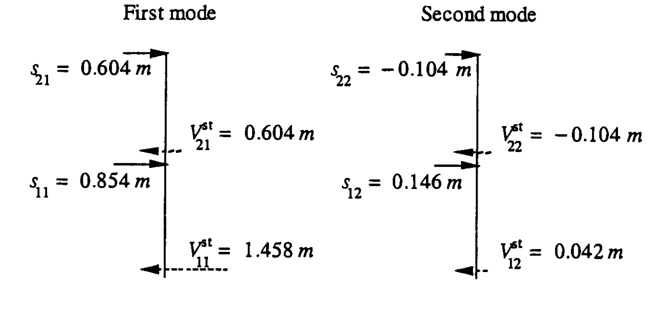


Fig. P13.1c

Substituting V_{jn}^{st} in Eq. (13.2.8) gives the modal responses:

$$V_{11}(t) = 1.458\,mA_1(t) \quad V_{12}(t) = 0.042\,mA_2(t)$$

$$V_{21}(t) = 0.604\,mA_1(t) \quad V_{22}(t) = -0.104\,mA_2(t)$$

Combining the modal responses gives the total responses:

$$V_1(t) = V_{11}(t) + V_{12}(t) = 1.458\,mA_1(t) + 0.042\,mA_2(t)$$

$$V_2(t) = V_{21}(t) + V_{22}(t) = 0.604\,mA_1(t) - 0.104\,mA_2(t)$$

Part d

Static analysis of the structure for external floor forces $\mathbf{s}_n$ gives the modal static responses M_{bn}^{st} and M_{1n}^{st} for M_b and M_1, the overturning moments at the base and the first floor, respectively:

$$M_{b1}^{st} = mh[0.854\,(1) + 0.604\,(2)] = 2.062mh$$

$$M_{b2}^{st} = mh[0.146\,(1) - 0.104\,(2)] = -0.062mh$$

$$M_{11}^{st} = 0.604mh$$

$$M_{12}^{st} = -0.104mh$$

Substituting M_{bn}^{st} and M_{1n}^{st} in Eq. (13.2.8) gives the modal responses:

$$M_{b1}(t) = 2.062\,mhA_1(t) \qquad M_{b2}(t) = -0.062\,mhA_2(t)$$

$$M_{11}(t) = 0.604\,mhA_1(t) \qquad M_{12}(t) = -0.104\,mhA_2(t)$$

Combining the modal responses gives the total response:

$$M_b(t) = M_{b1}(t) + M_{b2}(t) = 2.062\,mhA_1(t) - 0.062\,mhA_2(t)$$

$$M_1(t) = M_{11}(t) + M_{12}(t) = 0.604\,mhA_1(t) - 0.104\,mhA_2(t)$$

Problem 13.2

System properties:

$$m = \frac{w}{g} = \frac{100}{386} = 0.2591 \text{ kip-sec}^2/\text{in.}$$

$$k = \frac{24EI}{h^3} = \frac{24\,(727)\,(29 \times 10^3)}{(12 \times 12)^3} = 169.5 \text{ kips/in.}$$

Vibration properties (from Problem 10.6):

$$\omega_1 = 0.765\sqrt{\frac{k}{m}} = 19.565$$

$$\omega_2 = 1.848\sqrt{\frac{k}{m}} = 47.263$$

$$T_1 = 0.321 \text{ sec} \qquad T_2 = 0.133 \text{ sec}$$

$$\phi_1 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.707 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1.389 \\ 1.965 \end{Bmatrix}$$

$$\phi_2 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.707 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 1.389 \\ -1.965 \end{Bmatrix}$$

Modal properties:

Table P13.2a

Mode	M_n	L_n^h	L_n^θ/h
1	1.0	0.614	0.869
2	1.0	0.105	– 0.149

Part a

The displacements $D_n(t)$ and pseudo-accelerations $A_n(t)$ of the two modal SDF systems are calculated using the procedure of Section 5.2 with $\Delta t = 0.01$ sec and are shown in Figs. P13.2a and P13.2b.

Part b

The modal static responses for the various response quantities are given in Table P13.2b.

Table P13.2b

Mode n	1	2
u_{1n}^{st}	2.23×10^{-3}	6.529×10^{-5}
u_{2n}^{st}	3.15×10^{-3}	-9.237×10^{-5}
V_{bn}^{st}/m	1.458	0.042
V_{2n}^{st}/m	0.604	– 0.104
M_{bn}^{st}/mh	2.062	– 0.062
M_{1n}^{st}/mh	0.604	– 0.104

Step 5c of Section 13.2.4 is implemented to determine the contribution of the n[th] mode to selected response quantities — floor displacements, story shears, and story overturning moments:

$$r_n(t) = r_n^{st} A_n(t)$$

where r_n^{st} and $A_n(t)$ are both known. These results for roof displacement $u_2(t)$, base shear $V_b(t)$, and base overturning moment $M_b(t)$ are plotted in Figs. P13.2c-e, where their peak values are noted.

Part c

The modal contributions to each response quantity are combined at each time instant to obtain the total responses shown in Figs. P13.2c-e. Table P13.2c summarizes the peak values of the total responses.

Table P13.2c

Floor or Story	Displacement, in.	Shear, kips	Overturning Moment, kip-ft
2	0.964	49.56	594.65
1	0.679	115.11	1959.25

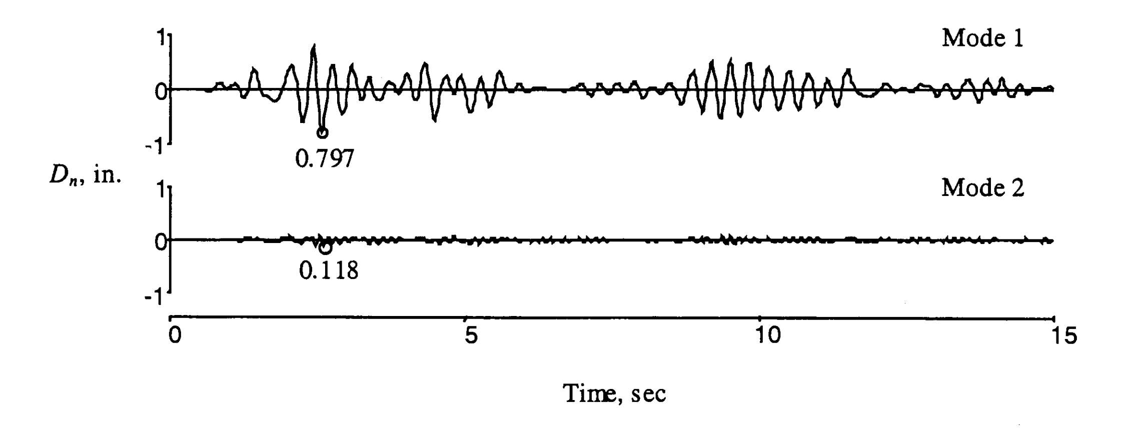


Fig. P13.2a

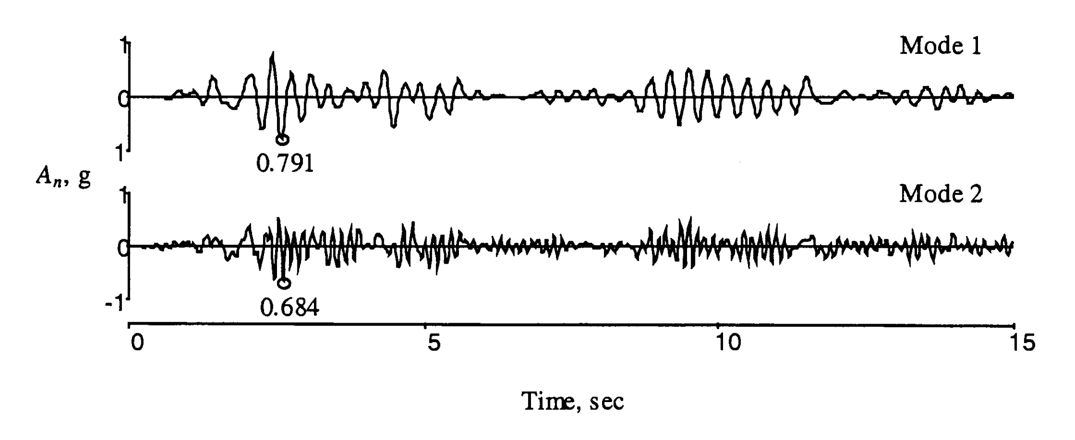


Fig. P13.2b

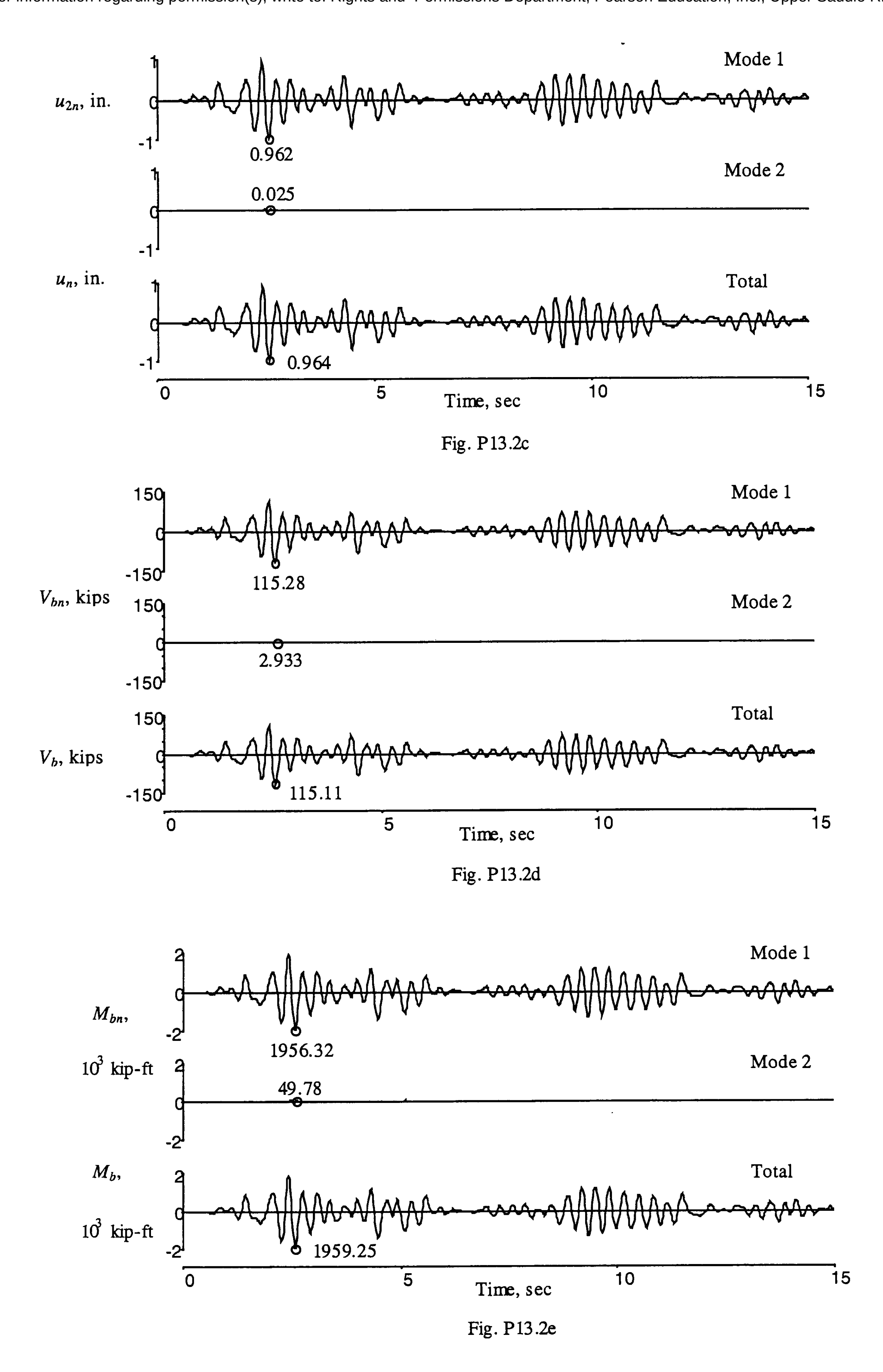


Fig. P13.2c

Fig. P13.2d

Fig. P13.2e

Problem 13.3

From Problem 13.2:

$$\omega_1 = 0.765\sqrt{\frac{k}{m}} \qquad \omega_2 = 1.848\sqrt{\frac{k}{m}}$$

$$\phi_1 = \langle 0.707 \quad 1\rangle^T \qquad \phi_2 = \langle -0.707 \quad 1\rangle^T$$

$$L_1^h = \sum_{j=1}^{2} m_j\phi_{j1} = m(0.707) + \frac{m}{2}(1) = 1.207m$$

$$L_2^h = \sum_{j=1}^{2} m_j\phi_{j2} = m(-0.707) + \frac{m}{2}(1) = -0.207m$$

$$L_1^\theta = \sum_{j=1}^{2} h_j m_j\phi_{j1} = h\,m(0.707) + 2h\frac{m}{2}(1) = 1.707mh$$

$$L_2^\theta = \sum_{j=1}^{2} h_j m_j\phi_{j2} = hm(-0.707) + 2h\left(\frac{m}{2}\right)(1)$$
$$= 0.293mh$$

$$L_2^\theta = \sum_{j=1}^{2} h_j m_j\phi_{j2} = h\,m(-0.707) + 2h\left(\frac{m}{2}\right)(1)$$
$$= 0.293mh$$

$$M_1 = \sum_{j=1}^{2} m_j\phi_{j1}^2 = m(0.707)^2 + \frac{m}{2}(1)^2 = m$$

$$M_2 = \sum_{j=1}^{2} m_j\phi_{j2}^2 = m(-0.707)^2 + \frac{m}{2}(1)^2 = m$$

From Eq. (13.2.9a), the effective modal masses are

$$M_1^* = \frac{(L_1^h)^2}{M_1} = 1.457m \qquad M_2^* = \frac{(L_2^h)^2}{M_2} = 0.043m$$

From Eq. (13.2.9a), the effective modal heights are

$$h_1^* = \frac{L_1^\theta}{L_1^h} = 1.414h \qquad h_2^* = \frac{L_2^\theta}{L_2^h} = -1.415h$$

Verify Eq. (13.2.14):

$$\left.\begin{aligned}\sum_{n=1}^{2} M_n^* &= 1.457m + 0.043m = 1.5m\\ \sum_{n=1}^{2} m_j &= m + \frac{m}{2} = 1.5m\end{aligned}\right\} \Rightarrow$$

$$\sum_{n=1}^{2} M_n^* = \sum_{n=1}^{2} m_j$$

Verify Eq. (13.2.17):

$$\left.\begin{aligned}\sum_{n=1}^{2} h_n^* M_n^* &= 1.414h\,(1.457m) + (-1.415h)\,(0.043m) = 2mh\\ \sum_{n=1}^{2} h_j m_j &= h\,m + 2h\left(\frac{m}{2}\right) = 2mh\end{aligned}\right\}$$

$$\Rightarrow \sum_{n=1}^{2} h_n^* M_n^* = \sum_{n=1}^{2} h_j m_j$$

Problem 13.4

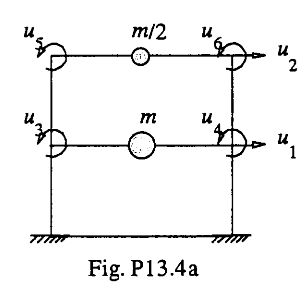


Fig. P13.4a

Mass and lateral stiffness matrices (from Problem 9.6):

$$\mathbf{m} = m\begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix} \qquad \hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 37.15 & -15.12 \\ -15.12 & 10.19 \end{bmatrix}$$

Vibration properties (from Problem 10.10):

$$\omega_1 = 2.407\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 7.193\sqrt{\frac{EI}{mh^3}}$$

$$\phi_1 = \begin{Bmatrix} 0.482 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1.037 \\ 1 \end{Bmatrix}$$

Modal properties:

$$M_1 = m(0.482)^2 + \frac{m}{2}(1)^2 = 0.732m$$

$$M_2 = m(-1.037)^2 + \frac{m}{2}(1)^2 = 1.575m$$

$$L_1^h = m(0.482) + \frac{m}{2}(1) = 0.982m$$

$$L_2^h = m(-1.037) + \frac{m}{2}(1) = -0.537m$$

$$\Gamma_1 = \frac{L_1^h}{M_1} = 1.341 \qquad \Gamma_2 = \frac{L_2^h}{M_2} = -0.341$$

Part a

From Eq. (13.2.5), the floor displacements due to the first mode are

$$\mathbf{u}_1(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}_1 = 1.341\begin{Bmatrix} 0.482 \\ 1 \end{Bmatrix} D_1(t) = \begin{Bmatrix} 0.647 \\ 1.341 \end{Bmatrix} D_1(t) \tag{a}$$

The joint rotations associated with $\mathbf{u}_1$ are

$$\mathbf{u}_{01} = \mathbf{T}\,\mathbf{u}_1$$

where $\mathbf{T}$ is available in the solution to Problem 10.10:

$$\mathbf{u}_{01}(t) = \begin{Bmatrix} u_3(t) \\ u_4(t) \\ u_5(t) \\ u_6(t) \end{Bmatrix}_1 = \frac{1}{h}\begin{bmatrix} -0.164 & -0.411 \\ -0.164 & -0.411 \\ 0.904 & -0.740 \\ 0.904 & -0.740 \end{bmatrix}\begin{Bmatrix} 0.647 \\ 1.341 \end{Bmatrix} D_1(t)$$

$$= \frac{1}{h}\begin{Bmatrix} -0.657 \\ -0.657 \\ -0.407 \\ -0.407 \end{Bmatrix} D_1(t) \tag{b}$$

The floor displacements due to the second mode are

$$\mathbf{u}_2(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}_2 = -0.341\begin{Bmatrix} -1.037 \\ 1 \end{Bmatrix} D_2(t) = \begin{Bmatrix} 0.353 \\ -0.341 \end{Bmatrix} D_2(t) \tag{c}$$

The joint rotations associated with $\mathbf{u}_2$, $\mathbf{u}_{02} = \mathbf{T}\,\mathbf{u}_2$, can be computed following Eq. (b) to obtain

$$\mathbf{u}_{02}(t) = \begin{Bmatrix} u_3(t) \\ u_4(t) \\ u_5(t) \\ u_6(t) \end{Bmatrix}_2 = \frac{1}{h}\begin{Bmatrix} 0.082 \\ 0.082 \\ 0572 \\ 0572 \end{Bmatrix} D_2(t) \tag{d}$$

Combining the modal responses gives the total floor displacements $\mathbf{u}(t)$ and total joint rotations $\mathbf{u}_0(t)$:

$$\mathbf{u}(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 0.647\,D_1(t) + 0.353\,D_2(t) \\ 1.341\,D_1(t) - 0.341\,D_2(t) \end{Bmatrix} \tag{e}$$

$$\mathbf{u}_0(t) = \begin{Bmatrix} u_3(t) \\ u_4(t) \\ u_5(t) \\ u_6(t) \end{Bmatrix} = \frac{1}{h}\begin{Bmatrix} -0.657\,D_1(t) + 0.082\,D_2(t) \\ -0.657\,D_1(t) + 0.082\,D_2(t) \\ -0.407\,D_1(t) + 0572\,D_2(t) \\ -0.407\,D_1(t) + 0572\,D_2(t) \end{Bmatrix} \tag{f}$$

Part b

The bending moments at the ends of a flexural element are related to the nodal displacements by

$$M_a = \frac{4EI}{L}\theta_a + \frac{2EI}{L}\theta_b + \frac{6EI}{L^2}u_a - \frac{6EI}{L^2}u_b \tag{g}$$

$$M_b = \frac{2EI}{L}\theta_a + \frac{4EI}{L}\theta_b + \frac{6EI}{L^2}u_a - \frac{6EI}{L^2}u_b \tag{h}$$

For a first story column, $L = h$ and the nodal displacements are as shown in Fig. P13.4b

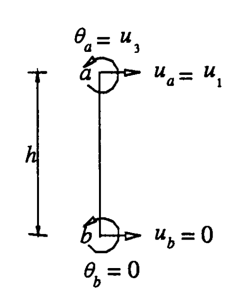


Fig. P13.4b

Substituting these u_a, u_b, θ_a and θ_b and Eqs. (e)-(f) in Eqs. (g)-(h) gives:

$$M_a = \frac{4EI}{h} u_3 + \frac{2EI}{h}(0) + \frac{6EI}{h^2} u_1 - \frac{6EI}{h^2}(0)$$

$$= \frac{EI}{h^2}\left[1.254\, D_1(t) + 2.446\, D_2(t)\right] \qquad \text{(i)}$$

Relate $D_n(t)$ to $A_n(t)$:

$$D_1(t) = \frac{A_1(t)}{\omega_1^2} = \frac{mh^3}{EI\,(2.407)^2} A_1(t)$$

$$D_2(t) = \frac{A_2(t)}{\omega_2^2} = \frac{mh^3}{EI\,(7.193)^2} A_2(t) \qquad \text{(j)}$$

Substituting Eq. (j) in Eq. (i) gives

$$M_a = mh\left[0.216\, A_1(t) + 0.0473\, A_2(t)\right] \qquad \text{(k)}$$

Similarly,

$$M_b = \frac{2EI}{h} u_3 + \frac{6EI}{h^2} u_1$$

$$= mh\left[0.443\, A_1(t) + 0.0441\, A_2(t)\right] \qquad \text{(l)}$$

For the second floor beam, $L = 2h$ and the nodal displacements are as shown in Fig. P13.4c.

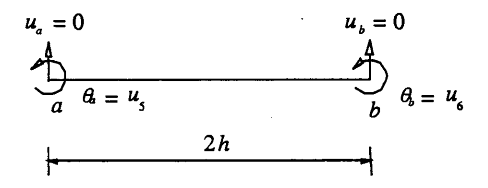


Fig. P13.4c

Substituting these u_a, u_b, θ_a and θ_b and Eqs. (e)-(f) in Eqs. (g)-(h) gives:

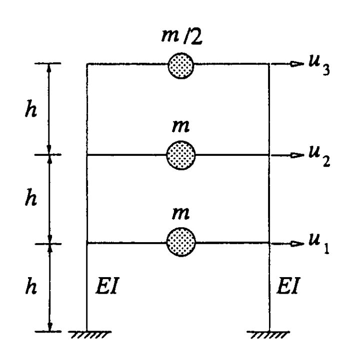
$$M_a = \frac{4EI}{2h} u_5 + \frac{2EI}{2h} u_6 \qquad \text{(m)}$$

$$= mh\left[-0.211\, A_1(t) + 0.0332\, A_2(t)\right]$$

$$M_b = \frac{4EI}{2h} u_6 + \frac{2EI}{2h} u_5 \qquad \text{(n)}$$

$$= mh\left[-0.211\, A_1(t) + 0.0332\, A_2(t)\right]$$

Problem 13.5



Mass and stiffness matrices (from Problem 9.7):

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} \qquad \mathbf{k} = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

where $k = 24EI/h^3$.

Vibration properties (from Problem 10.11):

$$\omega_1^2 = (2-\sqrt{3})\,\frac{k}{m}; \quad \omega_2^2 = 2\frac{k}{m}; \quad \omega_3^2 = (2+\sqrt{3})\frac{k}{m}$$

$$\phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix}$$

The influence vector is

$$\boldsymbol{\iota} = \mathbf{1}$$

The first-mode properties are computed from Eq. (13.2.3):

$$L_1^h = \sum_{j=1}^{3} m_j \phi_{j1} = m(0.5) + m(0.866) + \frac{m}{2}(1) = 1.866m$$

$$M_1 = \sum_{j=1}^{3} m_j \phi_{j1}^2 = m(0.5)^2 + m(0.866)^2 + \frac{m}{2}(1)^2 = 1.5m$$

$$\Gamma_1 = \frac{L_1^h}{M_1} = 1.244$$

Similar calculations for the second and third modes give:

$$L_2^h = -0.5m \qquad L_3^h = 0.134m$$

$$M_2 = 1.5m \qquad M_3 = 1.5m$$

$$\Gamma_2 = -0.3333 \qquad \Gamma_3 = 0.0893$$

Part a

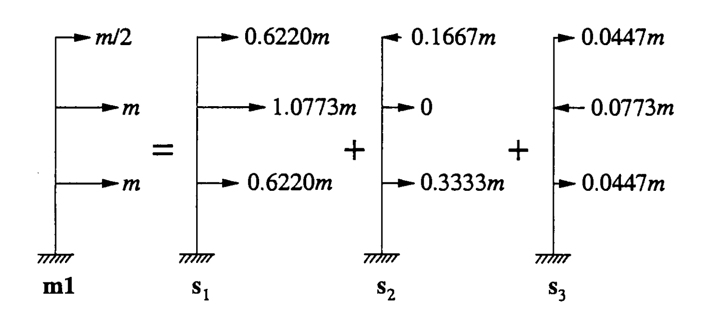
Substituting Γ_n, $\mathbf{m}$, and ϕ_n in Eq. (13.2.4) gives

$$\mathbf{s}_1 = 1.244m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}\begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} = m\begin{Bmatrix} 0.6220 \\ 1.0773 \\ 0.6220 \end{Bmatrix}$$

$$\mathbf{s}_2 = -0.3333m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} = m\begin{Bmatrix} 0.3333 \\ 0 \\ -0.1667 \end{Bmatrix}$$

$$\mathbf{s}_3 = 0.0893m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}\begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} = m\begin{Bmatrix} 0.0447 \\ -0.0773 \\ 0.0447 \end{Bmatrix}$$

The modal expansion of **m1** is shown next:



Part b

The floor displacements due to the n^th mode are

$$\mathbf{u}_n = \Gamma_n\,\phi_n\,D_n(t)$$

Substituting for Γ_n and ϕ_n gives

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}_1 = 1.244\begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} D_1(t) = \begin{Bmatrix} 0.6220 \\ 1.0774 \\ 1.2440 \end{Bmatrix} D_1(t)$$

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}_2 = -0.3333\begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} D_2(t) = \begin{Bmatrix} 0.3333 \\ 0 \\ -0.3333 \end{Bmatrix} D_2(t)$$

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}_3 = 0.0893\begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} D_3(t) = \begin{Bmatrix} 0.0447 \\ -0.0774 \\ 0.0893 \end{Bmatrix} D_3(t)$$

Combining the modal responses gives the floor displacements:

$$u_1(t) = 0.6220\,D_1(t) + 0.3333\,D_2(t) + 0.0447\,D_3(t)$$

$$u_2(t) = 1.0774\,D_1(t) - 0.0774\,D_3(t)$$

$$u_3(t) = 1.2440\,D_1(t) - 0.3333\,D_2(t) + 0.0893\,D_3(t)$$

Part c

Static analysis of the frame for external floor forces $\mathbf{s}_n$ gives V_{in}^{st}, $i = 1, 2, 3$:

$$V_{31}^{st} = 0.6220m \qquad V_{32}^{st} = -0.1667m \qquad V_{33}^{st} = 0.0447m$$

$$V_{21}^{st} = 1.6993m \qquad V_{22}^{st} = -0.1667m \qquad V_{23}^{st} = -0.0326m$$

$$V_{11}^{st} = 2.3213m \qquad V_{12}^{st} = 0.1667m \qquad V_{13}^{st} = 0.0121m$$

The total story shears are

$$V_j(t) = \sum_{n=1}^{3} V_{jn}(t) = \sum_{n=1}^{3} V_{jn}^{st} A_n(t)$$

Substituting values of V_{jn}^{st} gives

$$V_1(t) = m\left[2.3213\, A_1(t) + 0.1667\, A_2(t) + 0.0121\, A_3(t)\right]$$

$$V_2(t) = m\left[1.6993\, A_1(t) - 0.1667\, A_2(t) - 0.0326\, A_3(t)\right]$$

$$V_3(t) = m\left[0.6220\, A_1(t) - 0.1667\, A_2(t) + 0.0447\, A_3(t)\right]$$

Part d

Static analysis of the frame for external floor forces $\mathbf{s}_n$ gives M_{bn}^{st}:

$$M_{b1}^{st} = mh\left[0.6220\,(1) + 1.0773\,(2) + 0.6220(3)\right] = 4.6426mh$$

$$M_{b2}^{st} = mh\left[0.3333\,(1) + 0\,(2) - 0.1667\,(3)\right] = -0.1667mh$$

$$M_{b3}^{st} = mh\left[0.0447(1) - 0.0773(2) + 0.0447(3)\right] = 0.0242mh$$

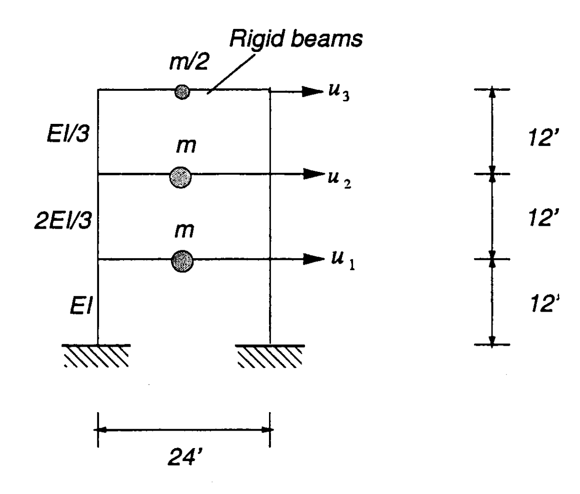
The base overturning moment response is

$$M_b(t) = \sum_{n=1}^{3} M_{bn}(t) = \sum_{n=1}^{3} M_{bn}^{st} A_n(t)$$

Substituting values of M_{bn}^{st} gives

$$M_b(t) = mh\left[4.6420\, A_1(t) - 0.1667\, A_2(t) + 0.0242\, A_3(t)\right]$$

Problem 13.6



Mass and stiffness matrices (from Problem 9.8)

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix}$$

$$\mathbf{k} = k\begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

where $k = 8EI/h^3$ and h = story height

Vibration properties (from Problem 10.12):

$$\omega_1 = 2.241\sqrt{\frac{EI}{mh^3}};\ \omega_2 = 4.899\sqrt{\frac{EI}{mh^3}};\ \omega_3 = 7.14\sqrt{\frac{EI}{mh^3}}$$

$$\phi_1 = \begin{bmatrix} 0.314 \\ 0.686 \\ 1 \end{bmatrix};\ \phi_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix};\ \phi_3 = \begin{bmatrix} 3.189 \\ -2.186 \\ 1 \end{bmatrix}$$

The first mode properties are computed from Eq. (13.2.3):

$$M_1 = 1.069m$$

$$L_1^h = \sum_{j=1}^{3} m_j \phi_{j1} = 1.5m$$

$$\Gamma_1 = \frac{L_1}{M_1} = 1.403$$

Similar calculations for the second and third modes give:

$$M_2 = m$$

$$L_2^h = \sum_{j=1}^{3} m_j \phi_{j2} = -0.5m$$

$$\Gamma_2 = \frac{L_2}{M_2} = -0.5$$

$$M_3 = 15.46m$$

$$L_3^h = \sum_{j=1}^{3} m_j \phi_{j3} = 1.503m$$

$$\Gamma_3 = \frac{L_3}{M_3} = 0.0972$$

Part a

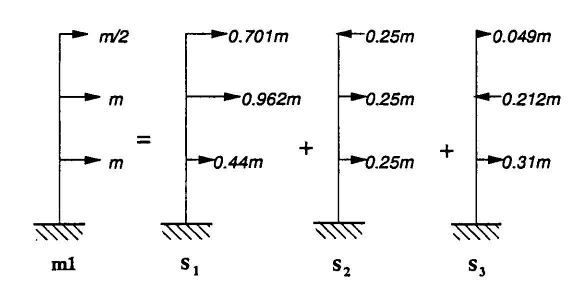
Substituting Γ_n, m and ϕ_n in Eq. (13.2.4) gives

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m} \phi_1 = 1.403m\begin{bmatrix} 0.314 \\ 0.686 \\ 0.5 \end{bmatrix} = m\begin{bmatrix} 0.44 \\ 0.962 \\ 0.701 \end{bmatrix}$$

$$\mathbf{s}_2 = \Gamma_2 \mathbf{m} \phi_2 = -0.5m\begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = m\begin{bmatrix} 0.25 \\ 0.25 \\ -0.25 \end{bmatrix}$$

$$\mathbf{s}_3 = \Gamma_3 \mathbf{m} \phi_3 = 0.0972m\begin{bmatrix} 3.189 \\ -2.186 \\ 0.5 \end{bmatrix} = m\begin{bmatrix} 0.31 \\ -0.212 \\ 0.049 \end{bmatrix}$$

The modal expansion of **m1** is shown next:



Part b

Equation (13.2.5) gives floor displacements due to each mode:

$$u_{jn}(t) = \Gamma_n \phi_{jn} D_n(t)$$

Substituting for Γ_n and ϕ_n gives:

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}_1 = \begin{bmatrix} 0.44 \\ 0.962 \\ 1.403 \end{bmatrix} D_1(t)$$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}_2 = \begin{bmatrix} 0.25 \\ 0.25 \\ -0.25 \end{bmatrix} D_2(t)$$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}_3 = \begin{bmatrix} 0.31 \\ -0.212 \\ 0.0972 \end{bmatrix} D_3(t)$$

Combining the modal responses gives the floor displacements:

$$u_1(t) = 0.44 D_1(t) + 0.25 D_2(t) + 0.31 D_3(t)$$

$$u_2(t) = 0.962 D_1(t) + 0.25 D_2(t) - 0.212 D_3(t)$$

$$u_3(t) = 1.403 D_1(t) - 0.25 D_2(t) + 0.0972 D_3(t)$$

Part c

Static analysis of the frame for external floor forces $\mathbf{s}_n$ gives V_{in}^{st}, $i = 1, 2, 3$:

$$V_{31}^{st} = 0.701m \qquad V_{32}^{st} = -0.25m \qquad V_{33}^{st} = 0.049m$$

$$V_{21}^{st} = 1.663m \qquad V_{22}^{st} = 0 \qquad V_{23}^{st} = -0.163m$$

$$V_{11}^{st} = 2.103m \qquad V_{12}^{st} = 0.25m \qquad V_{13}^{st} = 0.147m$$

The total story shears are:

$$V_j(t) = \sum_{n=1}^{3} V_{jn}(t) = \sum_{n=1}^{3} V_{jn}^{st} A_n(t)$$

Substituting values of V_{jn}^{st} gives

$$V_1(t) = 2.103mA_1(t) + 0.25mA_2(t) + 0.147mA_3(t)$$
$$V_2(t) = 1.663mA_1(t) + 0 \qquad - 0.163mA_3(t)$$
$$V_3(t) = 0.701mA_1(t) - 0.25mA_2(t) + 0.049mA_3(t)$$

Part d

Static analysis of the frame for external floor forces $\mathbf{s}_n$ gives M_{bn}^{st}:

$$M_{b1}^{st} = mh[0.44(1) + 0.962(2) + 0.701(3)] = 4.467mh$$

$$M_{b2}^{st} = mh[0.25(1) + 0.25(2) - 0.25(3)] = 0$$

$$M_{b3}^{st} = mh[0.31(1) - 0.212(2) + 0.049(3)] = 0.033mh$$

The base overturning moment response is:

$$M_b(t) = \sum_{n=1}^{3} M_{bn}(t) = \sum_{n=1}^{3} M_{bn}^{st} A_n(t)$$

Substituting values of M_{bn}^{st} gives

$$M_b(t) = 4.467mhA_1(t) + 0.033mhA_3(t)$$

Static analysis of the frame for external floor forces $\mathbf{s}_n$ gives M_{1n}^{st}:

$$M_{11}^{st} = mh(0.962(1) + 0.701(2)) = 2.364mh$$

$$M_{12}^{st} = mh(0.25(1) - 0.25(2)) = -0.25mh$$

$$M_{13}^{st} = mh(-0.212(1) + 0.049(2)) = -0.114mh$$

The first floor overturning moment response is

$$M_1(t) = \sum_{n=1}^{3} M_{1n}(t) = \sum_{n=1}^{3} M_{1n}^{st} A_n(t)$$

Substituting values of M_{1n}^{st} gives

$$M_1(t) = 2.364mhA_1(t) - 0.25mhA_2(t) - 0.114mhA_3(t)$$

Static analysis of the frame for external floor forces $\mathbf{s}_n$ gives M_{2n}^{st}:

$$M_{21}^{st}(t) = 0.701mh$$

$$M_{22}^{st} = -0.25mh$$

$$M_{23}^{st}(t) = 0.049mh$$

The second floor overturning moment rseponse is

$$M_2(t) = \sum_{n=1}^{3} M_{2n}(t) = \sum_{n=1}^{3} M_{2n}^{st} A_n(t)$$

Substituting values of M_{2n}^{st} gives

$$M_2(t) = 0.701mhA_1(t) - 0.25mhA_2(t) + 0.049mhA_3(t)$$

$$\left.\begin{array}{l} \sum_{n=1}^{2} h_n^* M_n^* = 1.414h\,(1.457m) + (-1.415h)\,(0.043m) = 2mh \\ \sum_{n=1}^{2} h_j m_j = h\,m + 2h\left(\frac{m}{2}\right) = 2mh \end{array}\right\}$$

$$\Rightarrow \sum_{n=1}^{2} h_n^* M_n^* = \sum_{n=1}^{2} h_j m_j$$

Problem 13.7

System properties:

$$m = \frac{100}{g} = \frac{100}{386.4} = 0.2588 \text{ kip-sec}^2/\text{in.}$$

$$k = \frac{24EI}{h^3} = \frac{24(29 \times 10^3)(1400)}{(12 \times 12)^3} = 326.32 \text{ kips/in.}$$

Vibration properties (from Problem 10.11):

$$\omega_1^2 = (2-\sqrt{3})\frac{k}{m}; \quad \omega_2^2 = 2\frac{k}{m}; \quad \omega_3^2 = (2+\sqrt{3})\frac{k}{m}$$

$$T_1 = 0.3418 \text{ sec} \qquad T_2 = 0.1251 \text{ sec} \quad T_3 = 0.0716 \text{ sec}$$

$$\phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix}$$

Modal properties (from Problem 13.5):

$$\Gamma_1 = 1.244 \qquad \Gamma_2 = -0.3333 \qquad \Gamma_3 = 0.0893$$

Part a

The displacements $D_n(t)$ and pseudo-acceleration $A_n(t)$ of the three modal SDF systems (with T_n given above and $\zeta_n = 0.05$) are calculated using the numerical procedure of Section 5.2 with $\Delta t = 0.01\text{sec}$. The results are shown in Figs. P13.7a-b.

Part b

The modal static responses for the various response quantities are given in Table P13.7a (also see Problem 13.5).

Table P13.7a

Mode n	1	2	3
u_{3n}^{st}	3.682×10^{-3}	0.132×10^{-3}	0.019×10^{-3}
V_{bn}^{st}/m	2.3213	0.1667	0.0121
V_{2n}^{st}/m	1.6993	-0.1667	-0.0326
V_{1n}^{st}/m	0.6220	-0.1667	0.0447
M_{bn}^{st}/mh	4.6426	-0.1667	0.0242

Step 5c of Section 13.2.4 is implemented to determine the contribution of the n[th] mode to selected response quantities:

$$r_n(t) = r_n^{st} A_n(t)$$

where r_n^{st} and $A_n(t)$ are both known. These results for roof displacement $u_3(t)$, base shear $V_b(t)$, and base overturning moment $M_b(t)$ are plotted in Figs. P13.7c-e where their peak values are noted.

Part c

The modal contributions to each response quantity are combined at each time instant to obtain Figs. P13.7c-e. Table P13.7b summarizes the peak values of the total responses.

Table P13.7b

Floor or story	Displacement, in.	Shear, kips	Overturning moment, kip-ft
3	1.103	52.22	626.6
2	0.957	138.08	2267.5
1	0.580	189.29	4320.8

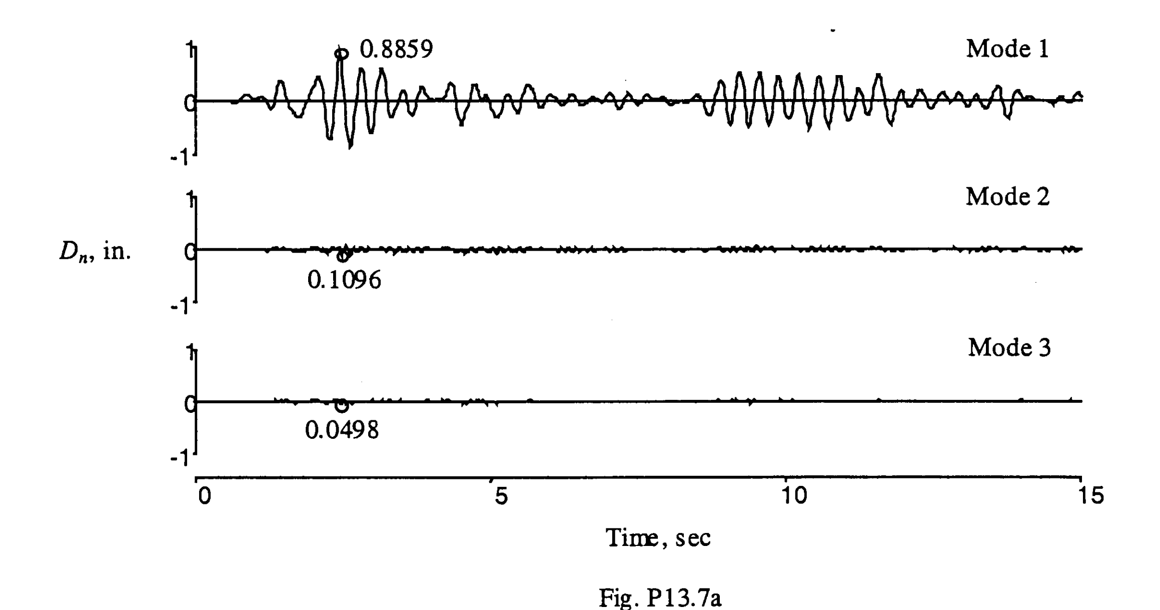


Fig. P13.7a

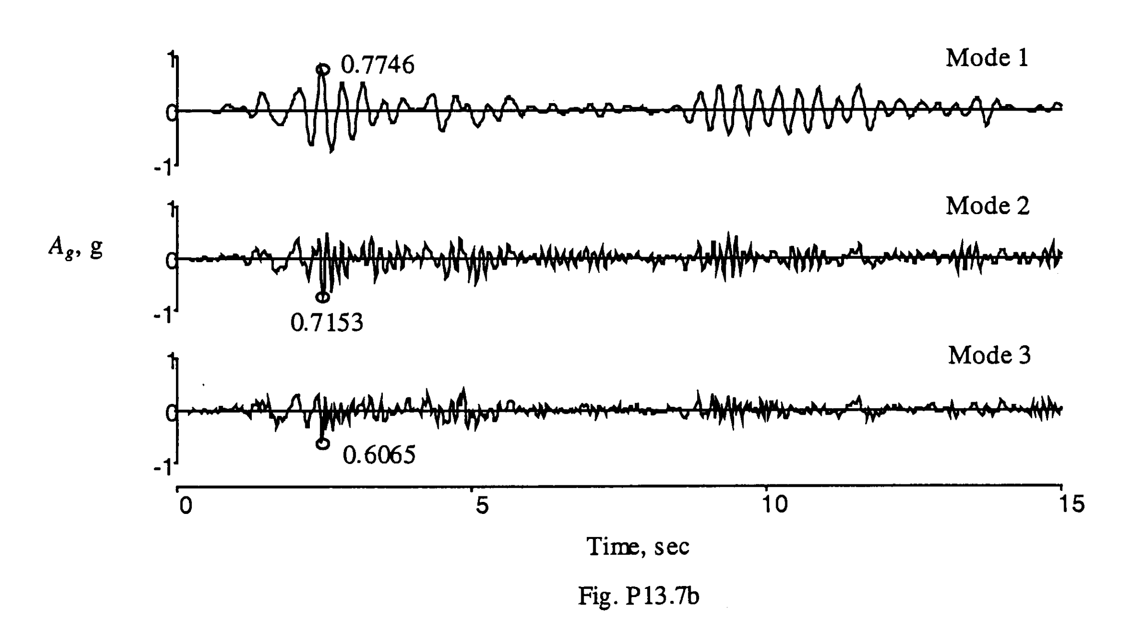


Fig. P13.7b

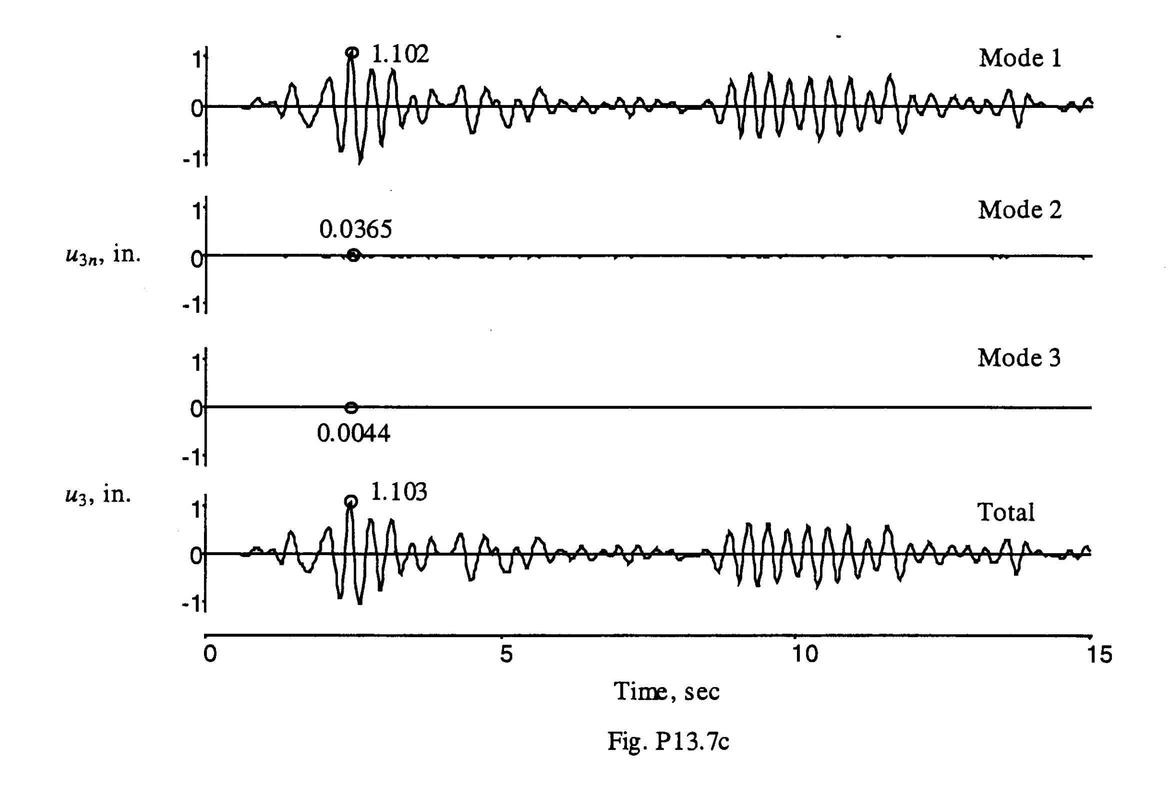


Fig. P13.7c

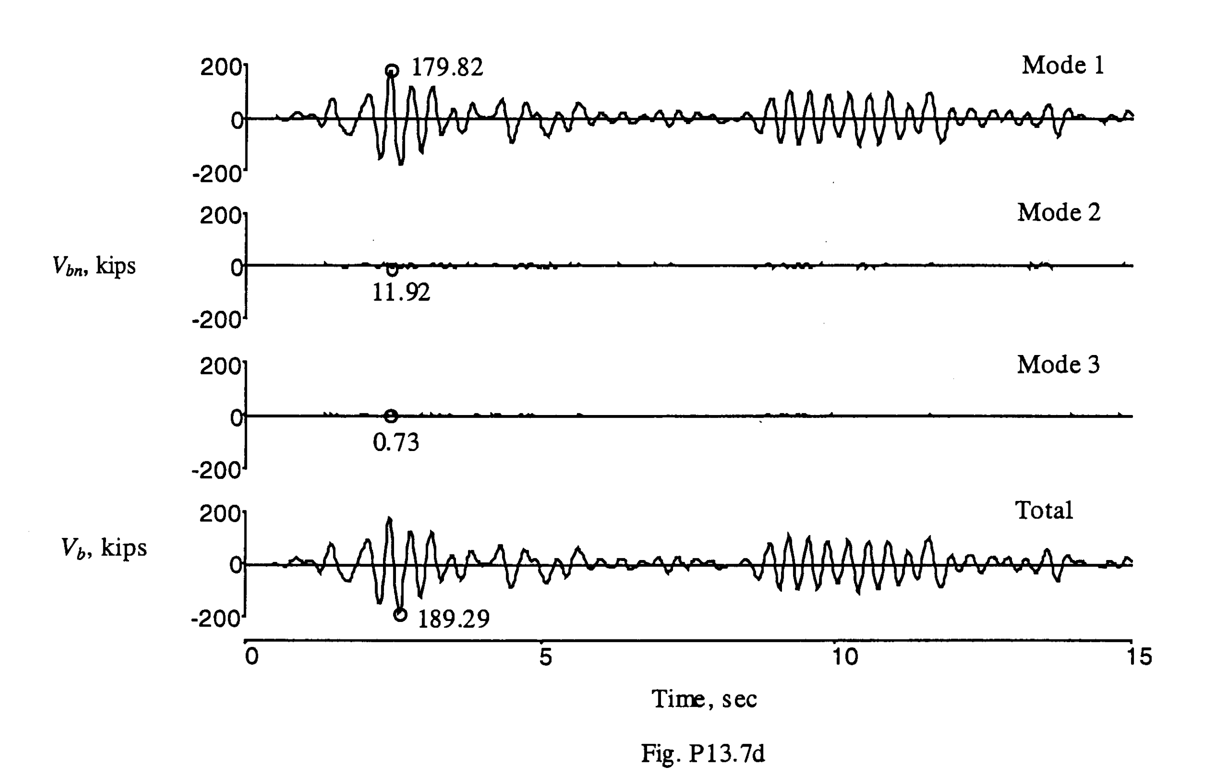


Fig. P13.7d

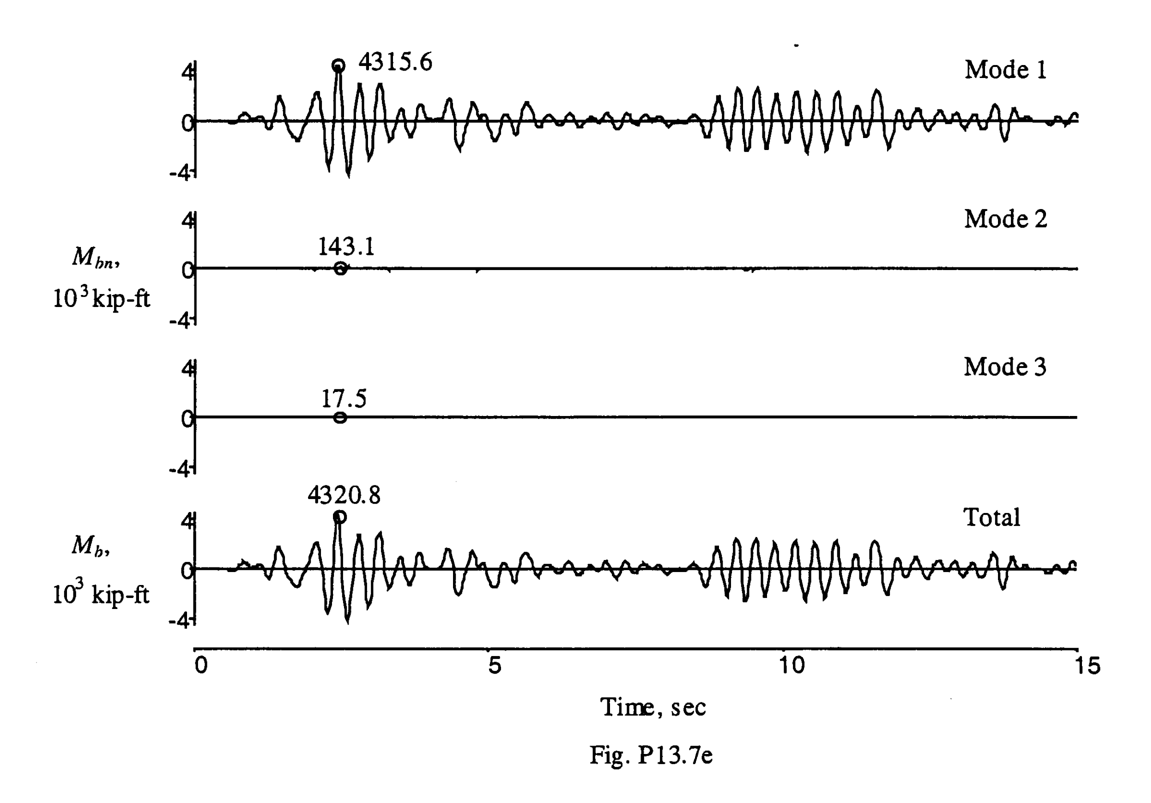

4315.6
Mode 1
Mode 2
143.1
Mode 3
17.5
4320.8
Total
M_{bn}, 10^3 kip-ft
M_b, 10^3 kip-ft
4
0
-4
0
5
10
15
Time, sec

Fig. P13.7e

Problem 13.8

System properties:

$$m = \frac{100}{g} = \frac{100}{386.4} = 0.2588 \text{ kip} - \text{sec}^2/\text{in.}$$

$$k = \frac{8EI}{h^3} = \frac{8(29\times10^3)(1400)}{(12\times12)^3} = 108.77 \text{ kips/in.}$$

Vibration properties (from Problem 10.12):

$$\omega_1^2 = 0.6277\frac{k}{m}; \quad \omega_2^2 = 3\frac{k}{m}; \quad \omega_3^2 = 6.372\frac{k}{m}$$

$$T_1 = 0.3868 \text{ sec} \quad T_2 = 0.1769 \text{ sec} \quad T_3 = 0.1214 \text{ sec}$$

$$\phi_1 = \begin{Bmatrix} 0.314 \\ 0.686 \\ 1 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} -0.5 \\ -0.5 \\ 1 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 3.186 \\ -2.186 \\ 1 \end{Bmatrix}$$

Modal properties from (from Problem 13.6):

$$\Gamma_1 = 1.403 \quad \Gamma_2 = -0.5 \quad \Gamma_3 = 0.0972$$

Part a

The displacements $D_n(t)$ and pseudo-acceleration $A_n(t)$ of the three modal SDF systems (with T_n given above and $\varsigma_n = 0.05$) are calculated using the numerical procedure of Section 5.2 with $\Delta t = 0.02$ sec. The results are shown in Figs. P13.8a-b.

Part b

The modal static responses for the various response quantities are given in Table P13.8a (also see Problem 13.6).

Table P13.8a

Mode n	1	2	3
u_{3n}^{st}	5.3710×10^{-3}	-0.3965×10^{-3}	0.0363×10^{-3}
V_{bn}^{st}/m	2.103	0.25	0.147
V_{2n}^{st}/m	1.663	0	-0.163
V_{3n}^{st}/m	0.701	-0.25	0.049
M_{bn}^{st}/mh	4.467	0	0.033

Step 5c of section 13.2.4 is implemented to determine the contribution of the n[th] mode to selected response quantities:

$$r_n(t) = r_n^{st} A_n(t)$$

where r_n^{st} and $A_n(t)$ are both known. These results for roof displacement $u_3(t)$, base shear $V_b(t)$, and base overturning moment $M_b(t)$ are plotted in Figs. P13.8c-e where their peak values are noted.

Part c

The modal contributions to each response quantity are combined at each time instant to obtain Figs. 13.8c-e. Table P13.8b summarizes the peak values of the total responses.

Table P13.8b

Floor or Story	Displacement, in.	Shear kips.	Overturning moment. kip-ft.
3	1.4332	54.85	658.2
2	1.1085	126.17	2136.0
1	0.5281	172.23	3965.9

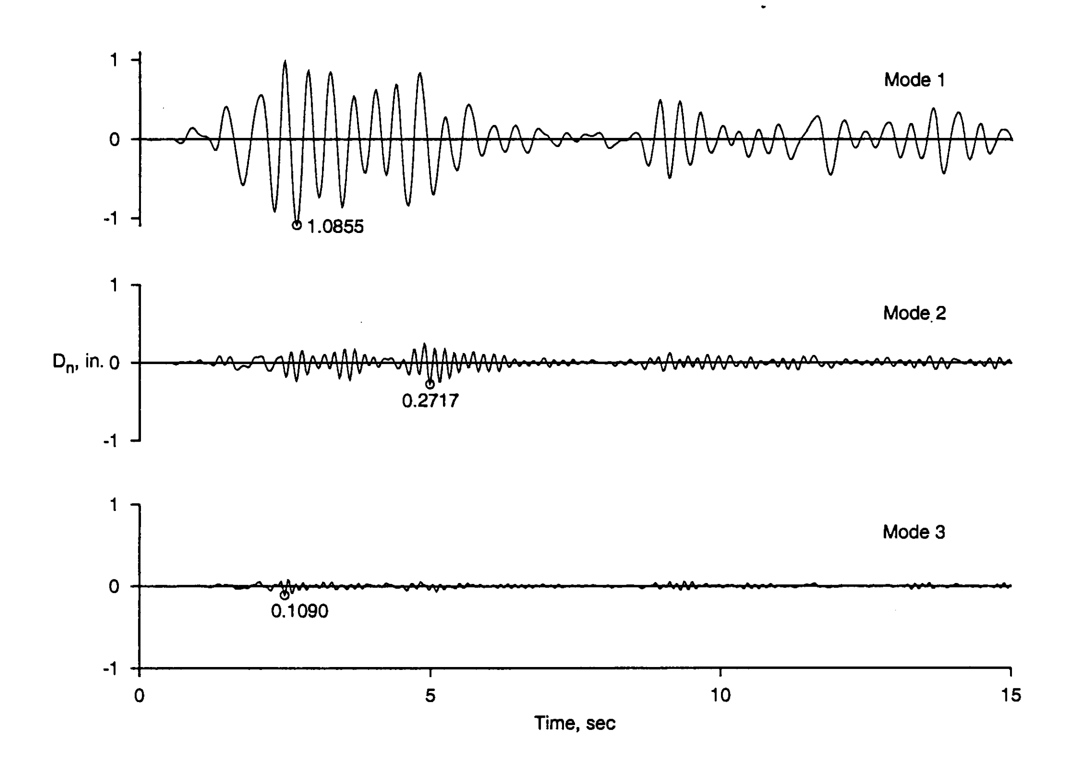


Fig. P13.8a

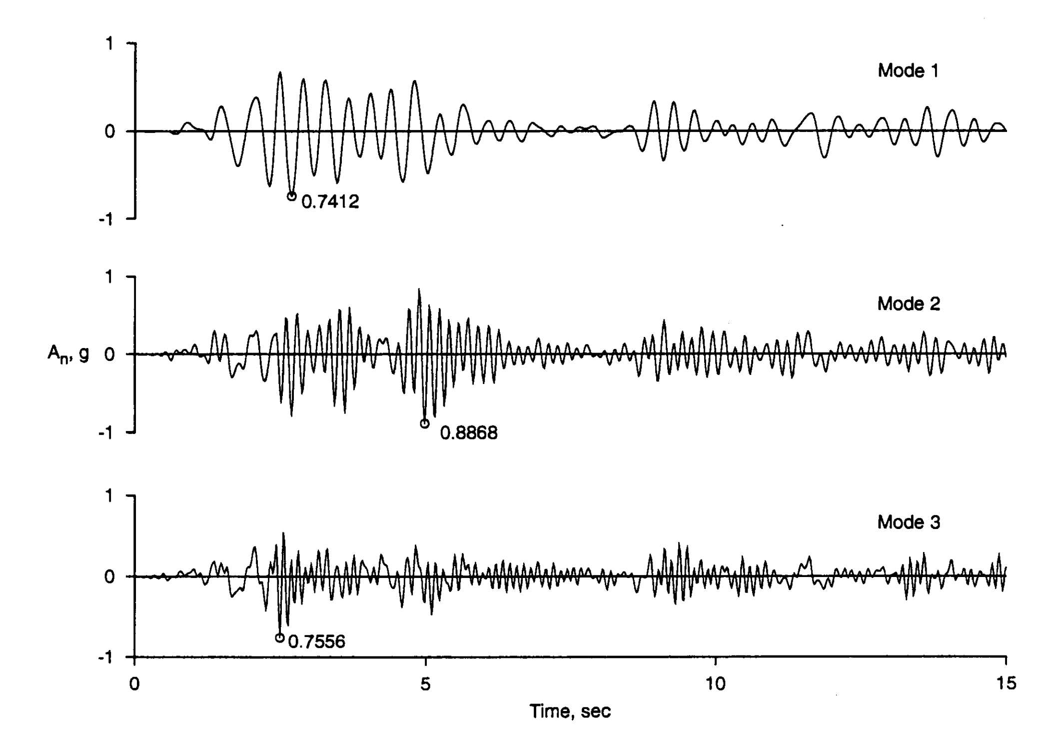


Fig.. P13.8b

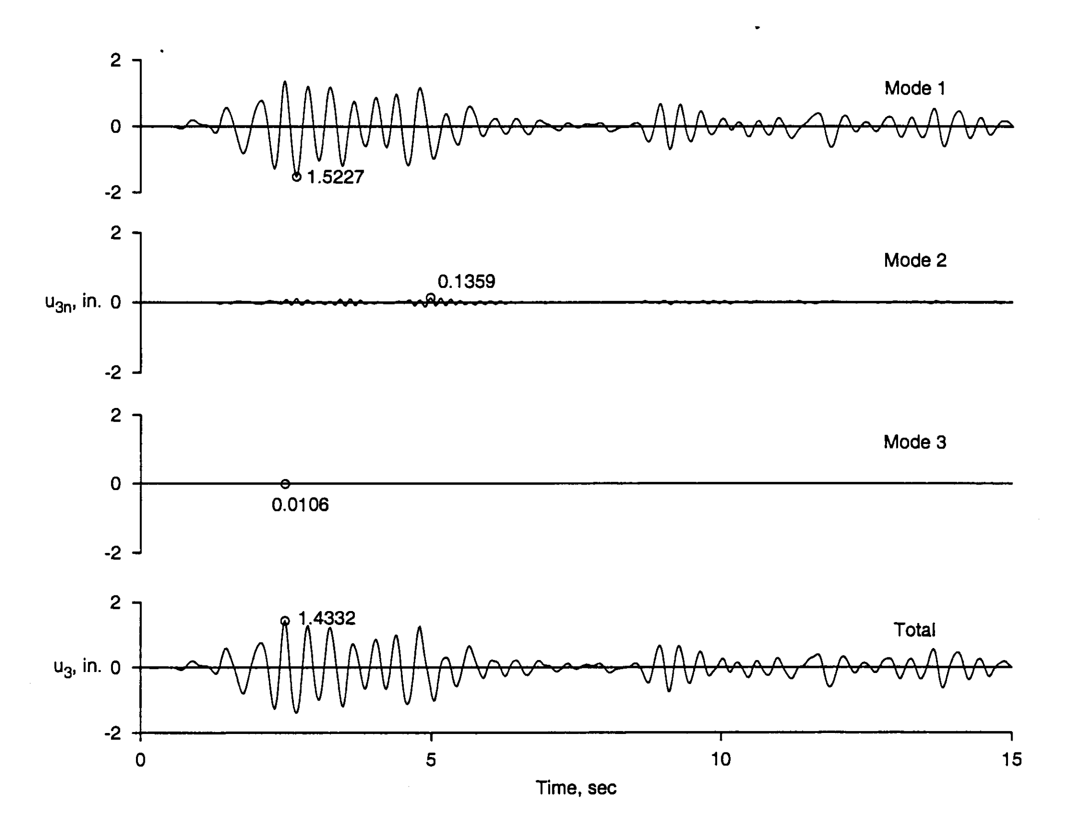


Fig. P13.8c

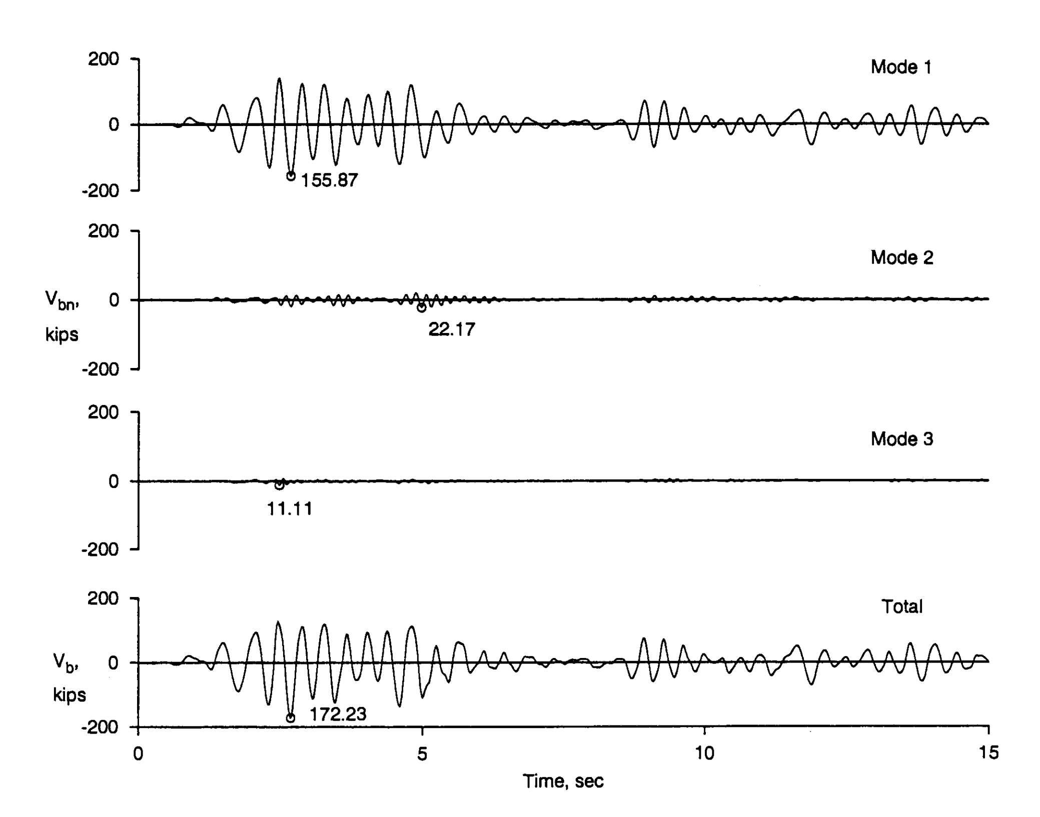


Fig. P13.8d

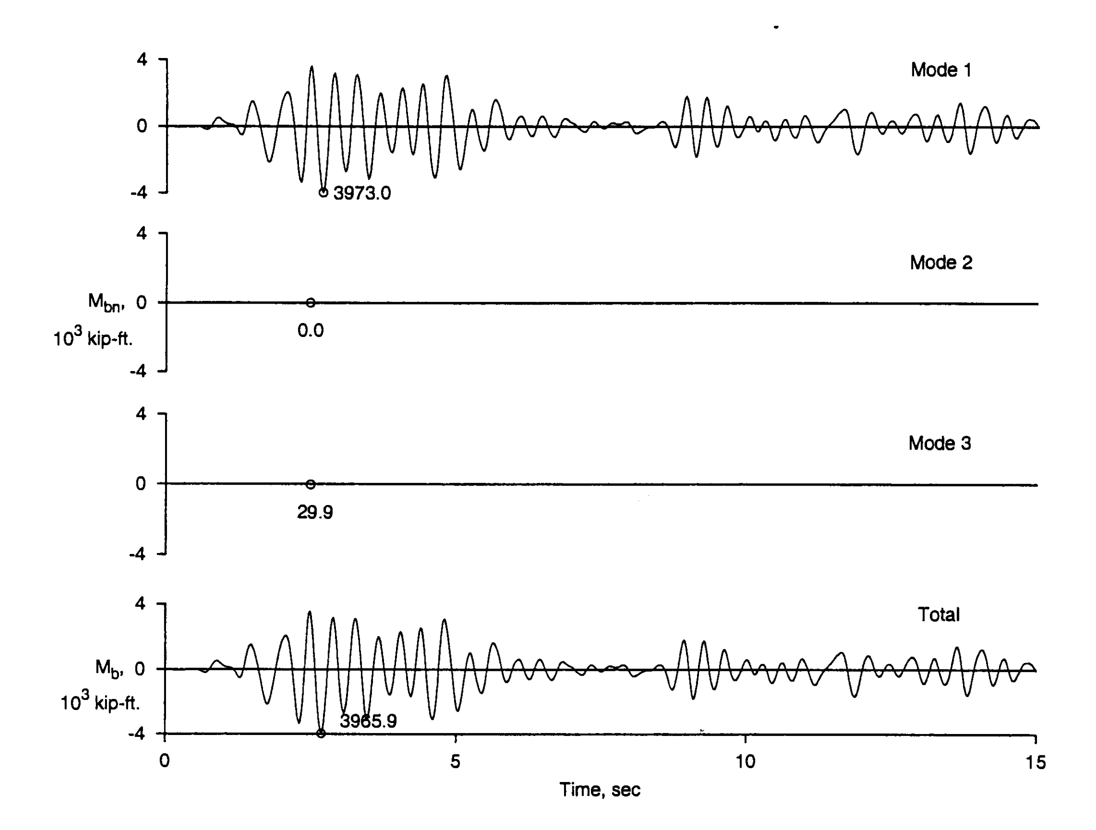

Mode 1
3973.0
Mode 2
M_{bn}, 10^3 kip-ft.
0.0
Mode 3
29.9
Total
M_b, 10^3 kip-ft.
3965.9
Time, sec

Fig. P13.8e

Problem 13.9

The floor masses, and the height of each floor above the base are

$$m_1 = m \qquad m_2 = m \qquad m_3 = \frac{m}{2}$$

$$h_1 = h \qquad h_2 = 2h \qquad h_3 = h$$

The natural modes and generalized masses (from Problem 13.5) are:

$$\phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \qquad \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix}$$

$$M_1 = 1.5m \qquad M_2 = 1.5m \qquad M_3 = 1.5m$$

Substituting for m_j and ϕ_{jn} in Eq. (13.2.3) gives L_n^h:

$$L_1^h = \sum_{j=1}^{3} m_j \phi_{j1} = m[1 \times 0.5 + 1 \times 0.866 + 0.5 \times 1] = 1.866m$$

$$L_2^h = \sum_{j=1}^{3} m_j \phi_{j2} = m[1 \times (-1) + 0 + 0.5 \times 1] = -0.5m$$

$$L_3^h = \sum_{j=1}^{3} m_j \phi_{j3} = m[1 \times 0.5 + 1 \times (-0.866) + 0.5 \times 1] = 0.134m$$

Substituting for m_j, h_j and ϕ_{jn} in Eq. (13.2.9b) gives L_n^θ:

$$L_1^\theta = \sum_{j=1}^{3} h_j m_j \phi_{j1} = mh[1 \times 1 \times 0.5 + 2 \times 1 \times 0.866 + 3 \times 0.5 \times 1] = 3.732mh$$

$$L_2^\theta = \sum_{j=1}^{3} h_j m_j \phi_{j2} = mh[1 \times 1 \times (-1) + 0 + 3 \times 0.5 \times 1] = 0.5mh$$

$$L_3^\theta = \sum_{j=1}^{3} h_j m_j \phi_{j3} = mh[1 \times 1 \times 0.5 + 2 \times 1 \times (-0.866) + 3 \times 0.5 \times 1] = 0.268mh$$

The effective modal masses and effective modal heights are given by Eq. (13.2.9a):

$$M_1^* = \frac{(L_1^h)^2}{M_1} = \frac{m(1.866)^2}{15} = 2.3213m$$

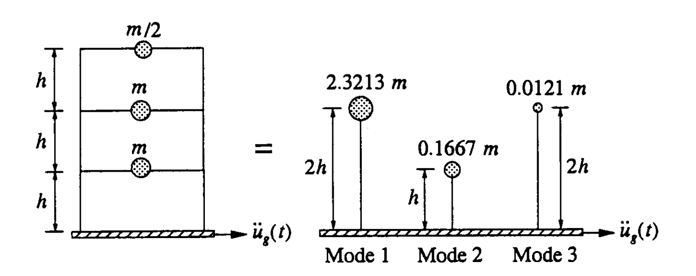
$$M_2^* = \frac{(L_2^h)^2}{M_2} = \frac{m(-0.5)^2}{1.5} = 0.1667m$$

$$M_3^* = \frac{(L_3^h)^2}{M_3} = \frac{m(0.134)^2}{1.5} = 0.0121m$$

$$h_1^* = \frac{L_1^\theta}{L_1^h} = h\frac{3.732}{1.866} = 2h$$

$$h_2^* = \frac{L_2^\theta}{L_2^h} = h\frac{0.5}{-0.5} = -h$$

$$h_3^* = \frac{L_3^\theta}{L_3^h} = h\frac{0.268}{0.134} = 2h$$



Verify Eq. (13.2.14):

$$\sum_{n=1}^{3} M_n^* = 2.3213m + 0.1667m + 0.0121m = 2.5m$$

$$\sum_{n=1}^{3} m_j = m + m + \frac{m}{2} = 2.5m$$

$$\therefore \sum_{n=1}^{3} M_n^* = \sum_{n=1}^{3} m_j$$

Verify Eq. (13.2.17):

$$\sum_{n=1}^{3} h_n^* M_n^* = 2h(2.3123m) + (-h)(0.1667m) + 2h(0.0120m) = 4.5mh$$

$$\sum_{n=1}^{3} h_j m_j = h(m) + 2h(m) + 3h(0.5m) = 4.5mh$$

$$\therefore \sum_{n=1}^{3} h_n^* M_n^* = \sum_{n=1}^{3} h_j m_j$$

Problem 13.10

The floor masses, and the height of each floor above the base are:

$$m_1 = m \qquad m_2 = m \qquad m_3 = \frac{m}{2}$$

$$h_1 = h \qquad h_2 = 2h \qquad h_3 = 3h$$

The natural modes and generalized masses (from Problem 13.6) are:

$$\phi_1 = \begin{bmatrix} 0.314 \\ 0.686 \\ 1 \end{bmatrix}; \phi_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}; \phi_3 = \begin{bmatrix} 3.189 \\ -2.186 \\ 1 \end{bmatrix}$$

$$M_1 = 1.069m \qquad M_2 = m \qquad M_3 = 15.46m$$

Substituting for m_j and ϕ_{jn} in Eq. (13.2.3) gives L_n^h:

$$L_1^h = \sum_{j=1}^{3} m_j \phi_{j1} = m(0.314 + 0.686 + 0.5) = 1.5m$$

$$L_2^h = \sum_{j=1}^{3} m_j \phi_{j2} = m(-0.5 - 0.5 + 0.5) = -0.5m$$

$$L_3^h = \sum_{j=1}^{3} m_j \phi_{j3} = m(3.189 - 2.186 + 0.5) = 1.503m$$

Substituting for m_j, h_j, and ϕ_{jn} in Eq. (13.2.9b) gives L_n^θ:

$$L_1^\theta = \sum_{j=1}^{3} h_j m_j \phi_{j1} = 3.186mh$$

$$L_2^\theta = \sum_{j=1}^{3} h_j m_j \phi_{j2} = 0$$

$$L_3^\theta = \sum_{j=1}^{3} h_j m_j \phi_{j3} = 0.317mh$$

The effective modal masses and effective modal heights are given by Eq. (13.2.9a):

$$M_1^* = \frac{\left(L_1^h\right)^2}{M_1} = \frac{(1.5m)^2}{1.069m} = 2.104m$$

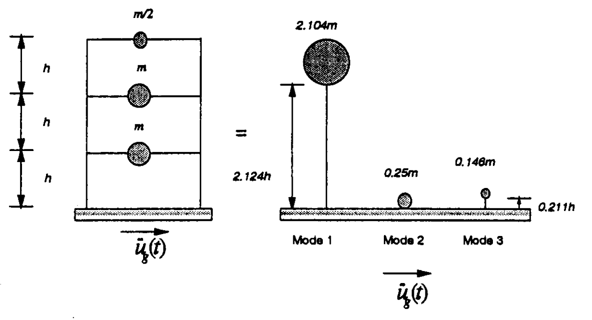
$$M_2^* = \frac{\left(L_2^h\right)^2}{M_2} = \frac{(-0.5m)^2}{m} = 0.25m$$

$$M_3^* = \frac{\left(L_3^h\right)^2}{M_3} = \frac{(1.503m)^2}{15.46m} = 0.146m$$

$$h_1^* = \frac{L_1^\theta}{L_1^h} = \frac{3.186mh}{1.5m} = 2.124h$$

$$h_2^* = \frac{L_2^\theta}{L_2^h} = \frac{0}{-0.5m} = 0$$

$$h_3^* = \frac{L_3^\theta}{L_3^h} = \frac{0.317mh}{1.503m} = 0.211h$$



Verify Eq. (13.2.14):

$$\sum_{n=1}^{3} M_n^* = m(2.104 + 0.25 + 0.146) = 2.5m$$

$$\sum_{j=1}^{3} m_j = m(1 + 1 + 0.5) = 2.5m$$

$$\therefore \sum_{n=1}^{3} M_n^* = \sum_{n=1}^{3} m_j$$

Verify Eq. (13.2.17):

$$\sum_{n=1}^{3} h_n^* M_n^* = mh(2.104 \times 2.124 + 0.25 \times 0 + 0.146 \times 0.211) = 4.5mh$$

$$\sum_{j=1}^{3} h_j m_j = mh(1 + 2 + 3 \times 0.5) = 4.5mh$$

$$\therefore \sum_{n=1}^{3} h_n^* M_n^* = \sum_{n=1}^{3} h_j m_j$$

Problem 13.11

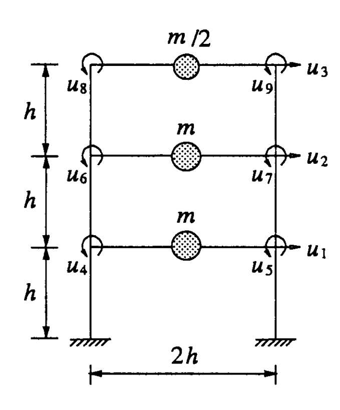


Fig. P13.11a

Mass and lateral stiffness matrices (from Problem 9.9):

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}$$

$$\hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 40.85 & -23.26 & 5.11 \\ -23.26 & 31.09 & -14.25 \\ 5.11 & -14.25 & 10.06 \end{bmatrix}$$

Natural frequencies and modes (from Problem 10.19):

$$\omega_1 = 1.4576\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 4.7682\sqrt{\frac{EI}{mh^3}}$$

$$\omega_3 = 8.1980\sqrt{\frac{EI}{mh^3}}$$

$$\phi_1 = \begin{Bmatrix} 0.3156 \\ 0.7451 \\ 1 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} -0.7409 \\ -0.3572 \\ 1 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 1.2546 \\ -1.2024 \\ 1 \end{Bmatrix}$$

From Eq. (13.2.3), for the first mode:

$$L_1^h = 0.3156m + 0.7451m + 1\left(\frac{m}{2}\right) = 1.5607m$$

$$M_1 = (0.3156)^2 m + (0.7451)^2 m + (1)^2\left(\frac{m}{2}\right)$$

$$= 1.1548m$$

$$\Gamma_1 = \frac{L_1^h}{M_1} = 1.3515$$

Computed similarly, these quantities for the second and third modes are

$$L_2^h = -0.5981m \qquad L_3^h = 0.5522m$$

$$M_2 = 1.1765m \qquad M_3 = 3.5198m$$

$$\Gamma_2 = -0.5083 \qquad \Gamma_3 = 0.1569$$

Part a

From Eq. (13.2.5) the floor displacements due to the first mode are

$$\mathbf{u}_1(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}_1 = 1.3515\begin{Bmatrix} 0.3156 \\ 0.7451 \\ 1 \end{Bmatrix} D_1(t) = \begin{Bmatrix} 0.4265 \\ 1.0070 \\ 1.3515 \end{Bmatrix} D_1(t) \qquad \text{(a)}$$

The joint rotations associated with $\mathbf{u}_1$ are

$$\mathbf{u}_{01}(t) = \mathbf{T}\,\mathbf{u}_1(t)$$

where $\mathbf{T}$ was determined in solving Problem 10.19. Thus:

$$\mathbf{u}_{01}(t) = \frac{1}{h}\begin{bmatrix} -0.1084 & -0.5342 & 0.0774 \\ -0.1084 & -0.5342 & 0.0774 \\ 0.5961 & -0.0619 & -0.4258 \\ 0.5961 & -0.0619 & -0.4258 \\ -0.1703 & 0.8748 & -0.7355 \\ -0.1703 & 0.8748 & -0.7355 \end{bmatrix}\begin{Bmatrix} 0.4625 \\ 1.0070 \\ 1.3515 \end{Bmatrix} D_1(t)$$

$$= \frac{1}{h}\begin{Bmatrix} -0.4796 \\ -0.4796 \\ -0.3836 \\ -0.3836 \\ -0.1856 \\ -0.1856 \end{Bmatrix} D_1(t) \qquad \text{(b)}$$

Similarly, the floor displacements due to the second and third modes are

$$\mathbf{u}_2(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}_2 = -0.5083\begin{Bmatrix} -0.7409 \\ -0.3572 \\ 1 \end{Bmatrix} D_2(t)$$

$$= \begin{Bmatrix} 0.3766 \\ 01816 \\ -0.5083 \end{Bmatrix} D_2(t) \qquad \text{(c)}$$

$$\mathbf{u}_3(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}_3 = 0.1569 \begin{Bmatrix} 1.2546 \\ -1.2024 \\ 1 \end{Bmatrix} D_3(t)$$

$$= \begin{Bmatrix} 0.1968 \\ -0.1887 \\ 0.1569 \end{Bmatrix} D_3(t) \qquad \text{(d)}$$

The joint rotations associated with $\mathbf{u}_2$ and $\mathbf{u}_3$ can be computed following Eq. (b):

$$\mathbf{u}_{02}(t) = \frac{1}{h} \begin{Bmatrix} -0.1772 \\ -0.1772 \\ 0.4297 \\ 0.4297 \\ 0.4686 \\ 0.4686 \end{Bmatrix} D_2(t) \qquad \text{(e)}$$

$$\mathbf{u}_{03}(t) = \frac{1}{h} \begin{Bmatrix} 0.0916 \\ 0.0916 \\ 0.0622 \\ 0.0622 \\ -0.3140 \\ -0.3140 \end{Bmatrix} D_3(t) \qquad \text{(f)}$$

Combining modal responses gives the total floor displacements:

$$\left.\begin{aligned} u_1(t) &= 0.4265\, D_1(t) + 0.3766\, D_2(t) + 0.1968\, D_3(t) \\ u_2(t) &= 1.0070\, D_1(t) + 0.1816\, D_2(t) - 0.1887\, D_3(t) \\ u_3(t) &= 1.3515\, D_1(t) - 0.5083\, D_2(t) + 0.1569\, D_3(t) \end{aligned}\right\} \qquad \text{(g)}$$

Combining modal contributions to joint rotations gives

$$\mathbf{u}_0(t) = \mathbf{u}_{01}(t) + \mathbf{u}_{02}(t) + \mathbf{u}_{03}(t) \qquad \text{(h)}$$

Part b

The bending moments at the ends of a flexural element are related to the nodal displacements by

$$M_a = \frac{4EI}{L}\theta_a + \frac{2EI}{L}\theta_b + \frac{6EI}{L^2}u_a - \frac{6EI}{L^2}u_b \qquad \text{(i)}$$

$$M_b = \frac{2EI}{L}\theta_a + \frac{4EI}{L}\theta_b + \frac{6EI}{L^2}u_a - \frac{6EI}{L^2}u_b \qquad \text{(j)}$$

For a first story column, $L = h$ and the nodal displacements are shown in Fig. P13.11b.

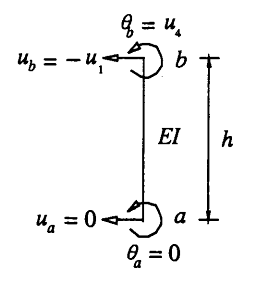


Fig. P13.11b

Substituting these u_a, u_b, θ_a and θ_b, and Eqs. (g)-(h) in Eqs. (i) and (j) gives

$$M_a = \frac{EI}{h^2}\left[1.5999\, D_1(t) + 1.9054\, D_2(t) + 1.3643\, D_3(t)\right]$$

$$M_b = \frac{EI}{h^2}\left[0.6408\, D_1(t) + 1.5511\, D_2(t) + 1.5475\, D_3(t)\right]$$

Substituting $D_n(t) = A_n(t)/\omega_n^2$ and ω_n in terms of E, I, m, and h gives

$$M_a = mh\left[0.7526\, A_1(t) + 0.08381\, A_2(t) + 0.02030\, A_3(t)\right]$$

$$M_b = mh\left[0.3014\, A_1(t) + 0.06823\, A_2(t) + 0.02302\, A_3(t)\right]$$

For the second floor beam, $L = 2h$ and the nodal displacements are shown in Fig. P13.11c.

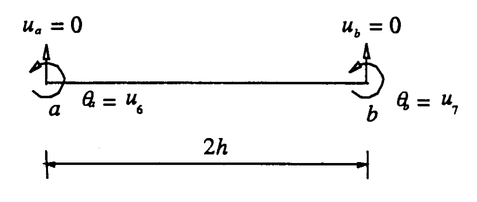


Fig. P13.11c

Substituting these u_a, u_b, θ_a and θ_b, and Eqs. (g)-(h) in Eqs. (i) and (j) gives

$$M_a = M_b$$
$$= \frac{EI}{h^2}\left[-1.1508\, D_1(t) + 1.2892\, D_2(t) + 0.1867\, D_3(t)\right]$$

Substituting $D_n(t) = A_n(t)/\omega_n^2$ and ω_n in terms of E, I, m, and h gives

$$M_a = M_b$$
$$= mh\left[-0.5414\, A_1(t) + 0.0567\, A_2(t) + 0.00278\, A_3(t)\right]$$

Problem 13.12

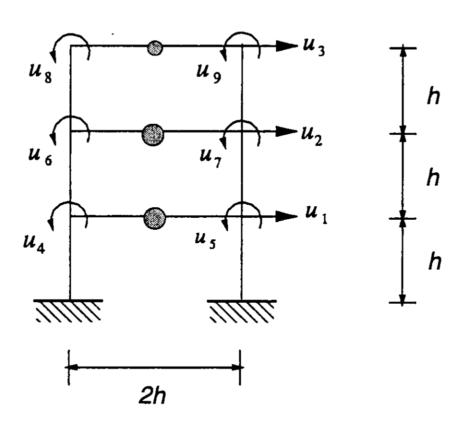


Fig. P13.12(a)

Mass and lateral stiffness matrices (from Problem 9.10)

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}$$

$$\hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 39.38 & -22.68 & 5.486 \\ & 27.13 & -11.75 \\ & & 7.418 \end{bmatrix}$$

Natural frequencies and modes (from Problem 10.20):

$$\omega_1 = 1.197\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 4.178\sqrt{\frac{EI}{mh^3}} \quad \omega_3 = 7.903\sqrt{\frac{EI}{mh^3}}$$

$$\phi_1 = \begin{bmatrix} 0.273 \\ 0.698 \\ 1 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -0.706 \\ -0.441 \\ 1 \end{bmatrix} \quad \phi_3 = \begin{bmatrix} 1.529 \\ -1.315 \\ 1 \end{bmatrix}$$

From Eq. (13.2.3), the modal properties are:

First Mode:

$$M_1 = (0.273)^2 m + (0.698)^2 m + (1)^2 (0.5m) = 1.06m$$

$$L_1^h = 0.273m + 0.698m + 1(0.5m) = 1.47m$$

$$\Gamma_1 = \frac{L_1^h}{M_1} = 1.386$$

Computed similarly, these quantities for the second and third modes are

$$M_2 = 1.193m \qquad M_3 = 4.567m$$

$$L_2^h = -0.647m \qquad L_3^h = 0.714m$$

$$\Gamma_2 = \frac{L_2^h}{M_2} = -0.542 \qquad \Gamma_3 = \frac{L_3^h}{M_3} = 0.156$$

Part a

From Eq. (13.2.5) the floor displacements due to the first mode are:

$$\mathbf{u}_1(t) = 1.386\begin{Bmatrix} 0.273 \\ 0.698 \\ 1 \end{Bmatrix} D_1(t) = \begin{Bmatrix} 0.377 \\ 0.966 \\ 1.386 \end{Bmatrix} D_1(t) \qquad \text{(a)}$$

The joint rotations associated with $\mathbf{u}_1$ are

$$\mathbf{u}_{01}(t) = \mathbf{T}\mathbf{u}_1(t)$$

where $\mathbf{T}$ was determined in solving Problem 10.20. Thus:

$$\mathbf{u}_{01}(t) = \frac{1}{h}\begin{bmatrix} -0.1512 & -0.6084 & 0.0962 \\ -0.1512 & -0.6084 & 0.0962 \\ 0.7184 & -0.11 & -0.457 \\ 0.7184 & -0.11 & -0.457 \\ -0.2612 & 1.131 & -0.925 \\ -0.2612 & 1.131 & -0.925 \end{bmatrix}\begin{Bmatrix} 0.377 \\ 0.966 \\ 1.386 \end{Bmatrix} D_1(t)$$

$$= \frac{1}{h}\begin{bmatrix} -0.511 \\ -0.511 \\ -0.469 \\ -0.469 \\ -0.288 \\ -0.288 \end{bmatrix} D_1(t) \qquad \text{(b)}$$

Similarly, the floor displacements due to the second and third modes are

$$\mathbf{u}_2(t) = -0.542\begin{Bmatrix} -0.706 \\ -0.441 \\ 1 \end{Bmatrix} D_2(t) = \begin{Bmatrix} 0.383 \\ 0.239 \\ -0.542 \end{Bmatrix} D_2(t) \qquad \text{(c)}$$

$$\mathbf{u}_3(t) = 0.156\begin{Bmatrix} 1.529 \\ -1.315 \\ 1 \end{Bmatrix} D_3(t) = \begin{Bmatrix} 0.238 \\ -0.205 \\ 0.156 \end{Bmatrix} D_3(t) \qquad \text{(d)}$$

The joint rotations associated with $\mathbf{u}_2$ and $\mathbf{u}_3$ can be computed following Eq. (b):

$$\mathbf{u}_{02}(t) = \frac{1}{h}\begin{bmatrix} -0.256 \\ -0.256 \\ 0.497 \\ 0.497 \\ 0.671 \\ 0.671 \end{bmatrix} D_2(t) \qquad \text{(e)}$$

$$\mathbf{u}_{03}(t) = \frac{1}{h}\begin{bmatrix} 0.104 \\ 0.104 \\ 0.122 \\ 0.122 \\ -0.438 \\ -0.438 \end{bmatrix} D_3(t) \qquad \text{(f)}$$

Combining modal responses gives the total floor displacements:

$$u_1(t) = 0.377 D_1(t) + 0.383 D_2(t) + 0.238 D_3(t)$$
$$u_2(t) = 0.966 D_1(t) + 0.239 D_2(t) - 0.205 D_3(t) \qquad \text{(g)}$$
$$u_3(t) = 1.386 D_1(t) - 0.542 D_2(t) + 0.156 D_3(t)$$

Combining modal contributions to joint rotations gives

$$\mathbf{u}_0(t) = \mathbf{u}_{01}(t) + \mathbf{u}_{02}(t) + \mathbf{u}_{03}(t) \qquad \text{(h)}$$

$$u_4(t) = u_5(t) = -0.511 D_1(t) - 0.256 D_2(t) + 0.104 D_3(t)$$
$$u_6(t) = u_7(t) = -0.469 D_1(t) + 0.239 D_2(t) - 0.205 D_3(t)$$
$$u_8(t) = u_9(t) = -0.288 D_1(t) + 0.671 D_2(t) - 0.438 D_3(t)$$

Part b

The bending moments at the ends of a flexural element are related to the nodal displacements by:

$$M_a = \frac{4EI}{L}\theta_a + \frac{2EI}{L}\theta_b + \frac{6EI}{L^2}u_a - \frac{6EI}{L^2}u_b \qquad \text{(i)}$$

$$M_b = \frac{2EI}{L}\theta_a + \frac{4EI}{L}\theta_b + \frac{6EI}{L^2}u_a - \frac{6EI}{L^2}u_b \qquad \text{(j)}$$

For a first story column, $L=h$ and the nodal displacements are shown in Fig. P13.12(b):

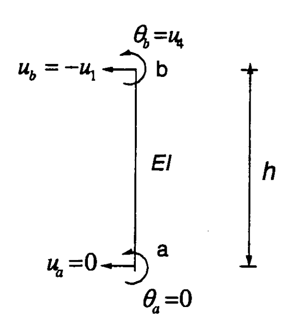


Fig. P13.12(b)

Substituting u_a, u_b, θ_a, and θ_b, and Eqs. (g)-(h) in Eqs. (i) and (j) gives:

$$M_a = \frac{EI}{h^2}\left[0.998 D_1(t) + 1.787 D_2(t) + 1.624 D_3(t)\right]$$

$$M_b = \frac{EI}{h^2}\left[0.216 D_1(t) + 1.274 D_2(t) + 1.811 D_3(t)\right]$$

Substituting $D_n(t) = A_n(t)/\omega_n^2$ and ω_n in terms of *E, I, m, h gives*

$$M_a = mh\left[0.697 A_1(t) + 0.102 A_2(t) + 0.026 A_3(t)\right]$$

$$M_b = mh\left[0.151 A_1(t) + 0.073 A_2(t) + 0.029 A_3(t)\right]$$

For the second floor beam, $L=2h$ and the nodal displacements are shown in Fig. P13.12(c):

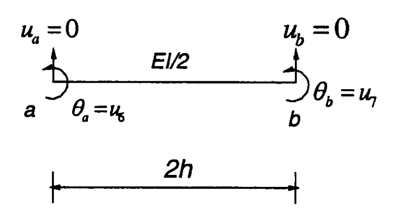


Fig. P13.12(c)

Substituting u_a, u_b, θ_a, and θ_b, and Eqs. (g)-(h) in Eqs. (i) and (j) gives:

$$M_a = M_b = \frac{EI}{h^2}\left[-1.407 D_1(t) + 0.717 D_2(t) - 0.615 D_3(t)\right]$$

Substituting $D_n(t) = A_n(t)/\omega_n^2$ and ω_n in terms of *E, I, m,* *h* gives

$$M_a = M_b = mh\left[-1.407 D_1(t) + 0.717 D_2(t) - 0.615.026 D_3(t)\right]$$

Problem 13.13

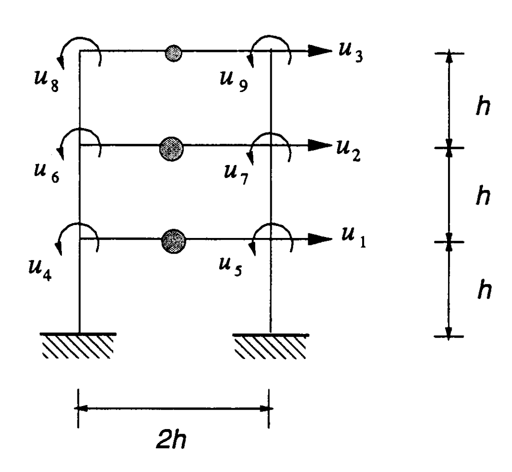


Fig. P13.13(a)

Mass and lateral stiffness matrices (from Problem 9.11)

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}$$

$$\hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 33.36 & -14.91 & 1.942 \\ & 15.96 & -5.489 \\ & & 3.923 \end{bmatrix}$$

Natural frequencies and modes (from Problem 10.21):

$$\omega_1 = 1.329\sqrt{\frac{EI}{mh^3}} \quad \omega_2 = 3.514\sqrt{\frac{EI}{mh^3}} \quad \omega_3 = 6.562\sqrt{\frac{EI}{mh^3}}$$

$$\phi_1 = \begin{bmatrix} 0.234 \\ 0.639 \\ 1 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -0.512 \\ -0.591 \\ 1 \end{bmatrix} \quad \phi_3 = \begin{bmatrix} 3.324 \\ -2.032 \\ 1 \end{bmatrix}$$

From Eq. (13.2.3), the modal properties are:

First mode:

$$M_1 = (0.234)^2 m + (0.639)^2 m + (1)^2 (0.5m) = 0.963m$$

$$L_1^h = 0.234m + 0.639m + 1(0.5m) = 1.373m$$

$$\Gamma_1 = \frac{L_1^h}{M_1} = 1.426$$

Computed similarly, these quantities for the second and third modes are

$$M_2 = 1.11m \qquad M_3 = 15.678m$$

$$L_2^h = -0.603m \qquad L_3^h = 1.792m$$

$$\Gamma_2 = \frac{L_2^h}{M_2} = -0.543 \qquad \Gamma_3 = \frac{L_3^h}{M_3} = 0.114$$

Part a

From Eq. (13.2.5) the floor displacements due to the first mode are:

$$\mathbf{u}_1(t) = 1.426\begin{Bmatrix} 0.234 \\ 0.639 \\ 1 \end{Bmatrix} D_1(t) = \begin{Bmatrix} 0.333 \\ 0.911 \\ 1.426 \end{Bmatrix} D_1(t) \qquad \text{(a)}$$

The joint rotations associated with $\mathbf{u}_1$ are

$$\mathbf{u}_{01}(t) = \mathbf{T}\mathbf{u}_1(t)$$

where $\mathbf{T}$ was determined in solving Problem 10.21. Thus:

$$\mathbf{u}_{01}(t) = \frac{1}{h}\begin{bmatrix} -0.3006 & -0.3695 & 0.0313 \\ -0.3006 & -0.3695 & 0.0313 \\ 0.6795 & -0.321 & -0.227 \\ 0.6795 & -0.321 & -0.227 \\ -0.1942 & 0.9489 & -0.7923 \\ -0.1942 & 0.9489 & -0.7923 \end{bmatrix}\begin{Bmatrix} 0.333 \\ 0.911 \\ 1.426 \end{Bmatrix} D_1(t)$$

$$= \frac{1}{h}\begin{bmatrix} -0.392 \\ -0.392 \\ -0.390 \\ -0.390 \\ -0.333 \\ -0.333 \end{bmatrix} D_1(t) \qquad \text{(b)}$$

Similarly, the floor displacements due to the second and third modes are

$$\mathbf{u}_2(t) = -0.543\begin{Bmatrix} -0.512 \\ -0.591 \\ 1 \end{Bmatrix} D_2(t) = \begin{Bmatrix} 0.278 \\ 0.321 \\ -0.543 \end{Bmatrix} D_2(t) \qquad \text{(c)}$$

$$\mathbf{u}_3(t) = 0.114\begin{Bmatrix} 3.324 \\ -2.032 \\ 1 \end{Bmatrix} D_3(t) = \begin{Bmatrix} 0.379 \\ -0.232 \\ 0.114 \end{Bmatrix} D_3(t) \qquad \text{(d)}$$

The joint rotations associated with $\mathbf{u}_2$ and $\mathbf{u}_3$ can be computed following Eq. (b):

$$\mathbf{u}_{02}(t) = \frac{1}{h}\begin{bmatrix} -0.219 \\ -0.219 \\ 0.209 \\ 0.209 \\ 0.681 \\ 0.681 \end{bmatrix} D_2(t) \qquad \text{(e)}$$

$$\mathbf{u}_{03}(t) = \frac{1}{h}\begin{bmatrix} -0.025 \\ -0.025 \\ 0.306 \\ 0.306 \\ -0.384 \\ -0.384 \end{bmatrix} D_3(t) \qquad \text{(f)}$$

Combining modal responses gives the total floor displacements:

$$u_1(t) = 0.333 D_1(t) + 0.278 D_2(t) + 0.379 D_3(t)$$
$$u_2(t) = 0.911 D_1(t) + 0.321 D_2(t) - 0.232 D_3(t) \qquad \text{(g)}$$
$$u_3(t) = 1.426 D_1(t) - 0.543 D_2(t) + 0.114 D_3(t)$$

Combining modal contributions to joint rotations gives

$$\mathbf{u}_0(t) = \mathbf{u}_{01}(t) + \mathbf{u}_{02}(t) + \mathbf{u}_{03}(t) \qquad \text{(h)}$$

$$u_4(t) = u_5(t) = -0.392 D_1(t) - 0.219 D_2(t) - 0.025 D_3(t)$$
$$u_6(t) = u_7(t) = -0.39 D_1(t) + 0.209 D_2(t) + 0.306 D_3(t)$$
$$u_8(t) = u_9(t) = -0.33 D_1(t) + 0.681 D_2(t) - 0.384 D_3(t)$$

Part b

The bending moments at the ends of a flexural element are related to the nodal displacements by:

$$M_a = \frac{4EI}{L}\theta_a + \frac{2EI}{L}\theta_b + \frac{6EI}{L^2}u_a - \frac{6EI}{L^2}u_b \qquad \text{(i)}$$

$$M_b = \frac{2EI}{L}\theta_a + \frac{4EI}{L}\theta_b + \frac{6EI}{L^2}u_a - \frac{6EI}{L^2}u_b \qquad \text{(j)}$$

For a first story column, $L=h$ and the nodal displacements are shown in Fig. P13.13(b):

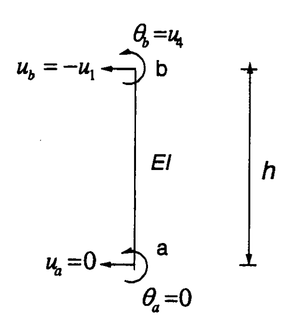


Fig.P13.13(b)

Substituting u_a, u_b, θ_a, and θ_b, and Eqs. (g)-(h) in Eqs. (i) and (j) gives:

$$M_a = \frac{EI}{h^2}\left[2.75 D_1(t) + 1.235 D_2(t) + 1.765 D_3(t)\right]$$

$$M_b = \frac{EI}{h^2}\left[0.43 D_1(t) + 0.79 D_2(t) + 2.153 D_3(t)\right]$$

Substituting $D_n(t) = A_n(t)/\omega_n^2$ and ω_n in terms of *E, I, m, h gives*

$$M_a = mh\left[1.555 A_1(t) + 0.1 A_2(t) + 0.041 A_3(t)\right]$$

$$M_b = mh\left[0.243 A_1(t) + 0.064 A_2(t) + 0.05 A_3(t)\right]$$

For the second floor beam, $L=2h$ and the nodal displacements are shown in Fig. P13.13(c):

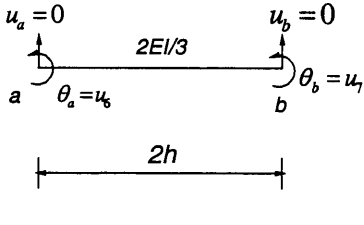


Fig. P13.13(c)

Substituting u_a, u_b, θ_a, and θ_b, and Eqs. (g)-(h) in Eqs. (i) and (j) gives:

$$M_a = M_b = \frac{EI}{h^2}\left[-1.17 D_1(t) + 0.628 D_2(t) + 0.918 D_3(t)\right]$$

Substituting $D_n(t) = A_n(t)/\omega_n^2$ and ω_n in terms of *E, I, m, h* gives

$$M_a = M_b = mh\left[-0.66 A_1(t) + 0.05 A_2(t) + 0.021 A_3(t)\right]$$

Problem 13.14

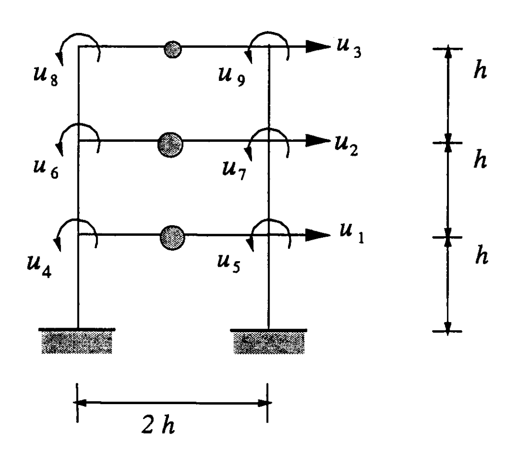


Fig. P13.14(a)

Mass and lateral stiffness matrices (from Problem 9.12)

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.5 \end{bmatrix}$$

$$\hat{\mathbf{k}}_{tt} = \frac{EI}{h^3}\begin{bmatrix} 30.77 & -14.01 & 2.43 \\ & 13.82 & -4.80 \\ Symm & & 2.92 \end{bmatrix}$$

Natural frequencies and modes (from Problem 10.22):

$$\omega_1 = 1.043\sqrt{\frac{EI}{mh^3}} \qquad \omega_2 = 3.081\sqrt{\frac{EI}{mh^3}} \qquad \omega_3 = 6.314\sqrt{\frac{EI}{mh^3}}$$

$$\phi_1 = \begin{bmatrix} 0.200 \\ 0.597 \\ 1 \end{bmatrix} \qquad \phi_2 = \begin{bmatrix} -0.545 \\ -0.656 \\ 1 \end{bmatrix} \qquad \phi_3 = \begin{bmatrix} 3.220 \\ -1.916 \\ 1 \end{bmatrix}$$

From Eq. (13.2.3), for the first mode:

$$M_1 = (0.200)^2 m + (0.597)^2 m + (1)^2 (0.5m) = 0.896m$$

$$L_1^h = 0.200m + 0.597m + 1(0.5m) = 1.296m$$

$$\Gamma_1 = \frac{L_1^h}{M_1} = 1.447$$

Computed similarly, these quantities for the second and third modes are

$$M_2 = 1.227m \qquad M_3 = 14.541m$$

$$L_2^h = -0.701m \qquad L_3^h = 1.804m$$

$$\Gamma_2 = -0.571 \qquad \Gamma_3 = 0.124$$

Part a:

From Eq. (13.2.5) the floor displacements due to the first mode are:

$$\mathbf{u}_1(t) = 1.447\begin{Bmatrix} 0.200 \\ 0.597 \\ 1 \end{Bmatrix} D_1(t) = \begin{Bmatrix} 0.289 \\ 0.863 \\ 1.447 \end{Bmatrix} D_1(t) \qquad \text{(a)}$$

The joint rotations associated with $\mathbf{u}_1$ are

$$\mathbf{u}_{01}(t) = \mathbf{T}\mathbf{u}_1(t) \qquad \text{(b)}$$

where $\mathbf{T}$ was determined in solving Problem 10.22. Thus:

$$\mathbf{u}_{01}(t) = \frac{1}{h}\begin{bmatrix} -0.4005 & -0.4152 & 0.0458 \\ -0.4005 & -0.4152 & 0.0458 \\ 0.9530 & -0.4569 & -0.2803 \\ 0.9530 & -0.4569 & -0.2803 \\ -0.3465 & 1.2570 & -0.9890 \\ -0.3465 & 1.2570 & -0.9890 \end{bmatrix}\begin{Bmatrix} 0.289 \\ 0.863 \\ 1.447 \end{Bmatrix} D_1(t)$$

$$= \frac{1}{h}\begin{bmatrix} -0.408 \\ -0.408 \\ -0.525 \\ -0.525 \\ -0.446 \\ -0.446 \end{bmatrix} D_1(t)$$

Similarly, the floor displacements due to the second and third modes are

$$\mathbf{u}_2(t) = -0.571\begin{Bmatrix} -0.545 \\ -0.656 \\ 1 \end{Bmatrix} D_2(t) = \begin{Bmatrix} 0.312 \\ 0.374 \\ -0.571 \end{Bmatrix} D_2(t) \qquad \text{(c)}$$

$$\mathbf{u}_3(t) = 0.124\begin{Bmatrix} 3.220 \\ -1.916 \\ 1 \end{Bmatrix} D_3(t) = \begin{Bmatrix} 0.399 \\ -0.238 \\ 0.124 \end{Bmatrix} D_3(t) \qquad \text{(d)}$$

The joint rotations associated with $\mathbf{u}_2$ and $\mathbf{u}_3$ can be computed following Eq. (b):

$$\mathbf{u}_{02}(t) = \frac{1}{h}\begin{bmatrix} -0.306 \\ -0.306 \\ 0.286 \\ 0.286 \\ 0.928 \\ 0.928 \end{bmatrix} D_2(t) \quad \mathbf{u}_{03}(t) = \frac{1}{h}\begin{bmatrix} -0.056 \\ -0.056 \\ 0.455 \\ 0.455 \\ -0.560 \\ -0.560 \end{bmatrix} D_3(t) \qquad \text{(e)}$$

Combining modal responses gives the total floor displacements:

$$\begin{aligned} u_1(t) &= 0.289D_1(t)+0.312D_2(t)+0.399D_3(t) \\ u_2(t) &= 0.863D_1(t)+0.374D_2(t)-0.238D_3(t) \\ u_3(t) &= 1.447D_1(t)-0.571D_2(t)+0.124D_3(t) \end{aligned} \qquad \text{(f)}$$

Combining modal contributions to joint rotations gives $\mathbf{u}_0(t)=\mathbf{u}_{01}(t)+\mathbf{u}_{02}(t)+\mathbf{u}_{03}(t)$

$$u_4(t)=u_5(t)=\frac{1}{h}[-0.408D_1(t)-0.306D_2(t)-0.056D_3(t)]$$

$$u_6(t)=u_7(t)=\frac{1}{h}[-0.525D_1(t)+0.286D_2(t)+0.455D_3(t)] \qquad \text{(g)}$$

$$u_8(t)=u_9(t)=\frac{1}{h}[-0.446D_1(t)+0.928D_2(t)-0.560D_3(t)]$$

Part b:

The bending moments at the ends of a flexural element are related to the nodal displacements by:

$$M_a=\frac{4EI}{L}\theta_a+\frac{2EI}{L}\theta_b+\frac{6EI}{L^2}u_a-\frac{6EI}{L^2}u_b \qquad \text{(h)}$$

$$M_b=\frac{2EI}{L}\theta_a+\frac{4EI}{L}\theta_b+\frac{6EI}{L^2}u_a-\frac{6EI}{L^2}u_b \qquad \text{(i)}$$

For a first story column, $L=h$ and the nodal displacements are shown in Fig. P13.14(b):

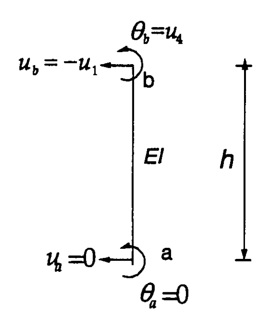


Fig. P13.14(b)

Substituting u_a, u_b, θ_a, and θ_b, and Eqs. (f)-(g) in Eqs. (h) and (i) gives:

$$M_a=\frac{EI}{h^2}[0.918D_1(t)+1.256D_2(t)+2.286D_3(t)]$$

$$M_b=\frac{EI}{h^2}[0.102D_1(t)+0.644D_2(t)+2.174D_3(t)]$$

Substituting $D_n(t)=A_n(t)/\omega_n^2$ and ω_n in terms of *E, I, m,* and *h* gives

$$M_a=mh\,[\,0.844A_1(t)+0.132A_2(t)+0.057A_3(t)\,]$$

$$M_b=mh\,[\,0.094A_1(t)+0.068A_2(t)+0.055A_3(t)\,]$$

For the second floor beam, $L=2h$ and the nodal displacements are shown in Fig. P13.14(c):

$u_a=0$ $u_b=0$
4EI/3
a b
$\theta_a=u_6$ $\theta_b=u_7$
2h

Fig. P13.14(c)

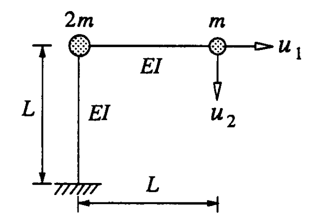
Substituting u_a, u_b, θ_a, and θ_b, and Eqs. (f)-(g) in Eqs. (h) and (i) gives:

$$M_a=M_b=\frac{EI}{h^2}[-0.525D_1(t)+0.286D_2(t)+0.455D_3(t)]$$

Substituting $D_n(t)=A_n(t)/\omega_n^2$ and ω_n in terms of *E, I, m,* and *h* gives

$$M_a=M_b=mh[-0.482A_1(t)+0.030A_2(t)+0.011A_3(t)]$$

Problem 13.15



Part a

From Example 9.6, the mass and stiffness matrices are:

$$\mathbf{m} = m\begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \qquad \mathbf{k} = \frac{6EI}{7L^2}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}$$

The natural frequencies and modes of the system (from Example 10.3) are:

$$\omega_1 = 0.6987\sqrt{\frac{EI}{mL^3}} \qquad \omega_1 = 1.874\sqrt{\frac{EI}{mL^3}}$$

$$\phi_1 = \begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix}$$

The equations of motion are given by Eqs. (13.1.1) and (13.1.2) where, for vertical ground motion taken positive downward, the influence vector is:

$$\iota = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Substituting for $\mathbf{m}$, ι and ϕ_n in Eq. (13.1.5) gives the modal quantities:

$$L_1 = \phi_1^T \mathbf{m}\, \iota = 2.097m$$

$$M_1 = \phi_1^T \mathbf{m}\, \phi_1 = 7.397m$$

$$\Gamma_1 = \frac{L_1}{M_1} = 0.2834$$

$$L_2 = \phi_2^T \mathbf{m}\, \iota = -1.431m$$

$$M_2 = \phi_2^T \mathbf{m}\, \phi_2 = 5.048m$$

$$\Gamma_2 = \frac{L_2}{M_2} = -0.2834$$

The effective earthquake forces are given by Eq. (13.1.2):

$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\,\iota\,\ddot{u}_g(t) = -\begin{bmatrix} 3m & \\ & m \end{bmatrix}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\ddot{u}_g(t)$$

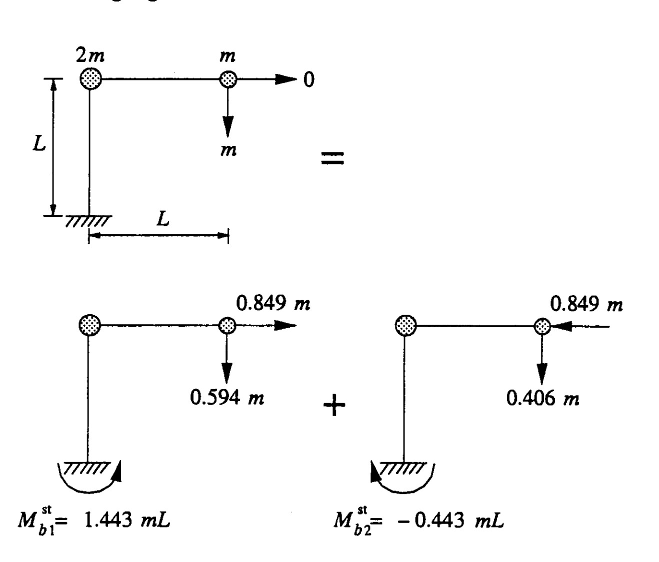
$$= -\begin{Bmatrix} 0 \\ m \end{Bmatrix}\ddot{u}_g(t)$$

Substituting Γ_n, $\mathbf{m}$ and ϕ_n in Eq. (13.1.6) gives

$$\mathbf{s}_1 = \Gamma_1\,\mathbf{m}\,\phi_1 = 0.2834\begin{bmatrix} 3m & \\ & m \end{bmatrix}\begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} = \begin{Bmatrix} 0.849m \\ 0.594m \end{Bmatrix}$$

$$\mathbf{s}_2 = \Gamma_2\,\mathbf{m}\,\phi_2 = -0.2834\begin{bmatrix} 3m & \\ & m \end{bmatrix}\begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix} = \begin{Bmatrix} -0.849m \\ 0.406m \end{Bmatrix}$$

The modal expansion of effective forces is shown in the following figure.



Part b

The modal displacements from Eq. (13.1.10) are

$$\mathbf{u}_1(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}_1 = \Gamma_1\,\phi_1\,D_1(t)$$

$$= 0.2834\begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} D_1(t) = \begin{Bmatrix} 0.283 \\ 0.594 \end{Bmatrix} D_1(t)$$

$$\mathbf{u}_2(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}_2 = \Gamma_2\,\phi_2\,D_2(t)$$

$$= -0.2834\begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix} D_2(t) = \begin{Bmatrix} -0.283 \\ 0.406 \end{Bmatrix} D_2(t)$$

Combining the modal displacements gives the total displacements:

$$u_1(t) = 0.283\,D_1(t) - 0.283\,D_2(t)$$

$$u_2(t) = 0.594\,D_1(t) + 0.406\,D_2(t)$$

Part c

Using M_{bn}^{st} shown in the figure, the modal responses for M_b are

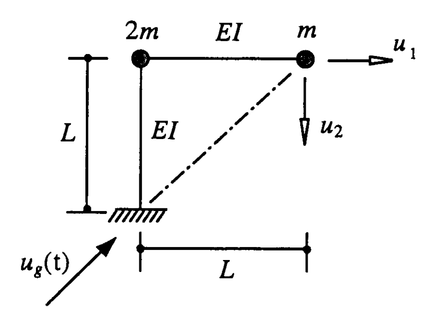
$$M_{b1}(t) = M_{b1}^{st} A_1(t) = 1.443\, mL\, A_1(t)$$

$$M_{b2}(t) = M_{b2}^{st} A_2(t) = -0.443\, mL\, A_2(t)$$

The total bending moment is

$$\begin{aligned} M_b(t) &= M_{b1}(t) + M_{b2}(t) \\ &= 1.443\, mL\, A_1(t) - 0.443\, mL\, A_2(t) \end{aligned}$$

Problem 13.16



Part a

From Example 9.6, the mass and stiffness matrices are:

$$\mathbf{m} = m\begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \qquad \mathbf{k} = \frac{6EI}{7L^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}$$

The natural frequencies and modes of the system (from Example 10.3) are:

$$\omega_1 = 0.6987\sqrt{\frac{EI}{mL^3}} \qquad \omega_2 = 1.874\sqrt{\frac{EI}{mL^3}}$$

$$\phi_1 = \begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix}$$

The equations of motion are given by Eqs. (13.1.1) and (13.1.2) where, for the ground motion assumed to be positive in the direction shown above, the influence vector is:

$$\iota = \begin{Bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{Bmatrix}$$

Substituting for $\mathbf{m}$, ι and ϕ_n in Eq. (13.1.5) gives the modal quantities:

$$L_1 = \phi_1^T \mathbf{m}\iota = 0.639m \qquad L_2 = \phi_2^T \mathbf{m}\iota = 3.133m$$

$$M_1 = \phi_1^T \mathbf{m}\phi_1 = 7.397m \qquad M_2 = \phi_2^T \mathbf{m}\phi_2 = 5.048m$$

$$\Gamma_1 = \frac{L_1}{M_1} = 0.0863 \qquad \Gamma_2 = \frac{L_2}{M_2} = 0.6206$$

The effective earthquake forces are given by Eq. (13.1.2):

$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\,\iota\,\ddot{u}_g(t) = -\begin{bmatrix} 3m & \\ & m \end{bmatrix}\begin{Bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{Bmatrix}\ddot{u}_g(t)$$

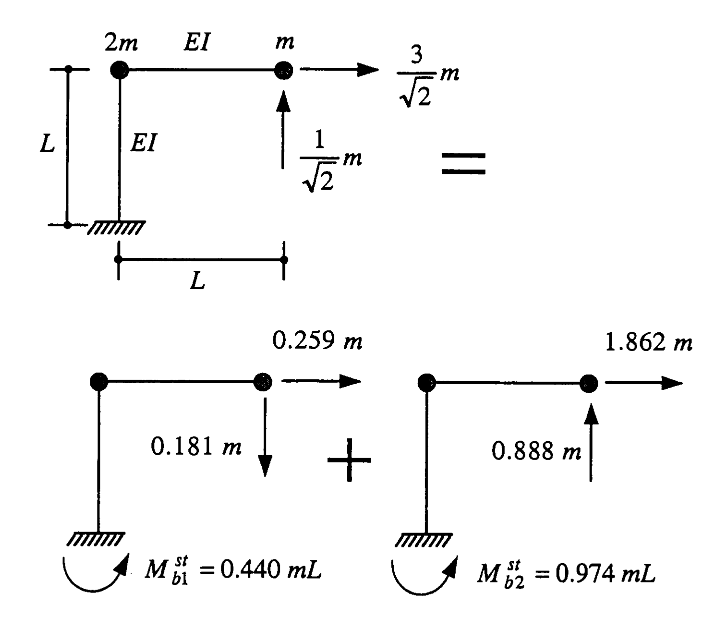
$$= \frac{-m}{\sqrt{2}}\begin{Bmatrix} 3 \\ -1 \end{Bmatrix}\ddot{u}_g(t)$$

Substituting Γ_n, $\mathbf{m}$ and ϕ_n in Eq. (13.1.6) gives:

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m}\phi_1 = \begin{Bmatrix} 0.259m \\ 0.181m \end{Bmatrix}$$

$$\mathbf{s}_2 = \Gamma_2 \mathbf{m}\phi_2 = \begin{Bmatrix} 1.862m \\ -0.888m \end{Bmatrix}$$

The modal expansion of the spatial distribution $\mathbf{m}\iota$ of the effective forces is shown in the following figure.



Part b

The modal displacements from Eq. (13.1.10) are:

$$\mathbf{u}_1(t) = \Gamma_1\phi_1 D_1(t) = \begin{Bmatrix} 0.086 \\ 0.181 \end{Bmatrix} D_1(t)$$

$$\mathbf{u}_2(t) = \Gamma_2\phi_2 D_2(t) = \begin{Bmatrix} 0.621 \\ -0.888 \end{Bmatrix} D_2(t)$$

Combining the modal displacements gives the total displacements:

$$\mathbf{u}_1(t) = 0.086D_1(t) + 0.621D_2(t)$$

$$\mathbf{u}_2(t) = 0.181D_1(t) - 0.888D_2(t)$$

Part c

Using M_{bn}^{st} shown in the figure above, the modal responses for M_b are:

$$M_{b1}(t) = M_{b1}^{st} A_1(t) = 0.440\, m\, L\, A_1(t)$$

$$M_{b2}(t) = M_{b2}^{st} A_2(t) = 0.974\, m\, L\, A_2(t)$$

Thus, the total bending moment is:

$$\begin{aligned} M_b(t) &= M_{b1}(t) + M_{b2}(t) \\ &= 0.440\, m\, L\, A_1(t) + 0.974\, m\, L\, A_2(t) \end{aligned}$$

Problem 13.17

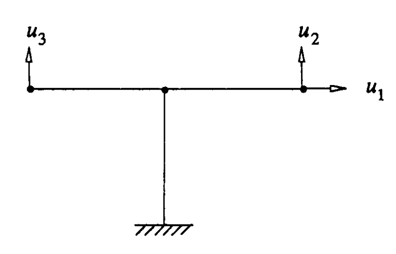


Fig. P13.17a

Part a

The equations of motion are given by Eqs. (13.1.1) and (13.1.2). The mass and stiffness matrices (from Problem 9.13) are

$$\mathbf{m} = m\begin{bmatrix} 5 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \mathbf{k} = \frac{3EI}{10L^3}\begin{bmatrix} 28 & 6 & -6 \\ 6 & 7 & 3 \\ -6 & 3 & 7 \end{bmatrix}$$

The influence vector due to horizontal ground motion is (from Problem 9.13)

$$\boldsymbol{\iota} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

The natural frequencies and modes of the system (from Problem 10.23) are

$$\omega_1 = 0.526\sqrt{\frac{EI}{mL^3}} \qquad \omega_2 = 1.614\sqrt{\frac{EI}{mL^3}}$$

$$\omega_3 = 1.732\sqrt{\frac{EI}{mL^3}}$$

$$\phi_1 = \begin{Bmatrix} 1 \\ -1.949 \\ 1.949 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} 1 \\ 1.283 \\ -1.283 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

Substituting for $\mathbf{m}$, $\boldsymbol{\iota}$ and ϕ_n in Eq. (13.1.5) gives the first-mode quantities:

$$L_1 = \phi_1^T \mathbf{m} \boldsymbol{\iota} = \langle 1 \quad -1.949 \quad 1.949 \rangle m \begin{bmatrix} 5 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 5m$$

$$M_1 = \phi_1^T \mathbf{m} \phi_1 = \langle 1 \quad -1.949 \quad 1.949 \rangle m \begin{bmatrix} 5 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ -1.949 \\ 1.949 \end{Bmatrix}$$

$$= 12.597m$$

$$\Gamma_1 = \frac{L_1}{M_1} = 0.397$$

Similar calculations for the second and third modes gives:

$$L_2 = \phi_2^T \mathbf{m} \boldsymbol{\iota} = 5m \qquad L_3 = \phi_3^T \mathbf{m} \boldsymbol{\iota} = 0$$

$$M_2 = \phi_2^T \mathbf{m} \phi_2 = 8.292m \qquad M_3 = \phi_3^T \mathbf{m} \phi_3 = 2m$$

$$\Gamma_2 = \frac{L_2}{M_2} = 0.603 \qquad \Gamma_3 = \frac{L_3}{M_3} = 0$$

The effective earthquake forces are given by Eq. (13.1.4):

$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\boldsymbol{\iota}\,\ddot{u}_g(t) = -\begin{bmatrix} 5m & & \\ & m & \\ & & m \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}\ddot{u}_g(t)$$

$$= -\begin{Bmatrix} 5m \\ 0 \\ 0 \end{Bmatrix}\ddot{u}_g(t)$$

Substituting Γ_n, $\mathbf{m}$ and ϕ_n in Eq. (13.1.6) gives

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m} \phi_1 = 0.397\begin{bmatrix} 5m & & \\ & m & \\ & & m \end{bmatrix}\begin{Bmatrix} 1 \\ -1.949 \\ 1.949 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1.985m \\ -0.774m \\ 0.774m \end{Bmatrix}$$

$$\mathbf{s}_2 = \Gamma_2 \mathbf{m} \phi_2 = 0.603\begin{bmatrix} 5m & & \\ & m & \\ & & m \end{bmatrix}\begin{Bmatrix} 1 \\ 1.283 \\ -1.283 \end{Bmatrix}$$

$$= \begin{Bmatrix} 3.015m \\ 0.774m \\ -0.774m \end{Bmatrix}$$

$$\mathbf{s}_3 = \Gamma_3 \mathbf{m} \phi_3 = 0\begin{bmatrix} 5m & & \\ & m & \\ & & m \end{bmatrix}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The modal expansion of the spatial distribution of effective forces is shown in Fig. P13.17b. The effective forces in the third mode are all zero, implying that this mode will not be excited by horizontal ground motion.

Part b

The modal displacements from Eq. (13.1.10) are

$$\mathbf{u}_1(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}_1 = \Gamma_1 \phi_1 D_1(t) = 0.397 \begin{Bmatrix} 1 \\ -1.949 \\ 1.949 \end{Bmatrix} D_1(t) = \begin{Bmatrix} 0.397 \\ -0.774 \\ 0.774 \end{Bmatrix} D_1(t)$$

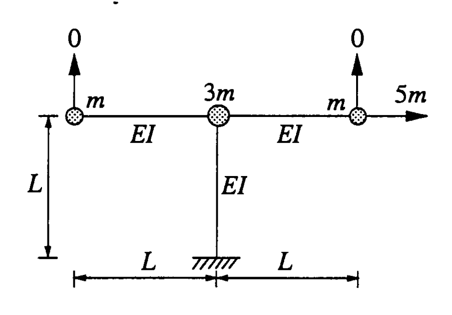
$$\mathbf{u}_2(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}_2 = \Gamma_2 \phi_2 D_2(t) = 0.603 \begin{Bmatrix} 1 \\ 1.283 \\ -1.283 \end{Bmatrix} D_2(t) = \begin{Bmatrix} 0.603 \\ 0.774 \\ -0.774 \end{Bmatrix} D_2(t)$$

$$\mathbf{u}_3(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}_3 = \Gamma_3 \phi_3 D_3(t) = 0 \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} D_3(t) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} D_3(t)$$

Combining the modal displacements gives the total displacements:

$$u_1(t) = 0.397 D_1(t) + 0.603 D_2(t)$$
$$u_2(t) = -0.774 D_1(t) + 0.774 D_2(t)$$
$$u_3(t) = 0.774 D_1(t) - 0.774 D_2(t)$$

Observe that the third mode does not contribute to the total displacements, and the total displacements $u_2(t)$ and $u_3(t)$ are antisymmetric.



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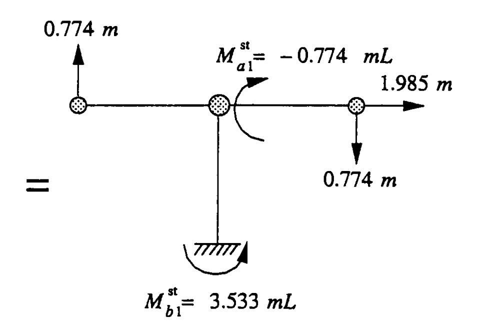

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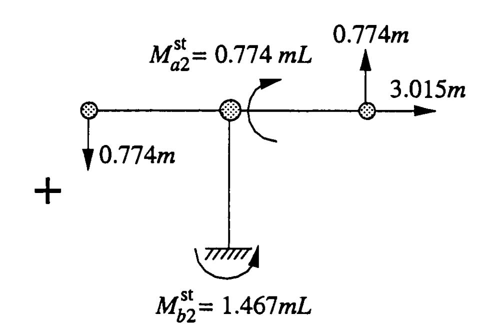

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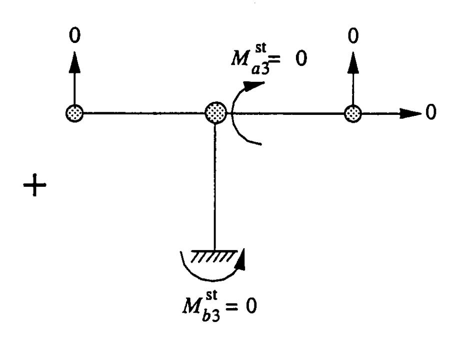


Fig. P13.17b

Part c

The bending moment at the base of the column due to the n^th mode is

$$M_{bn}(t) = M_{bn}^{st} A_n(t)$$

Substituting the modal static responses M_{bn}^{st}, shown in Fig. P13.17b, and combining modal responses gives

$$M_b(t) = \sum_{n=1}^{3} M_{bn}(t) = 3.533\, mL\, A_1(t) + 1.467\, mL\, A_2(t)$$

The bending moment at location a of the beam due to the n^{th} mode is

$$M_{an}(t) = M_{an}^{st}\, A_n(t)$$

Substituting the modal static responses M_{an}^{st}, shown in Fig. P13.17b, and combining modal responses gives

$$M_a(t) = \sum_{n=1}^{3} M_{an}(t) = -0.774\, mL\, A_1(t) + 0.774\, mL\, A_2(t)$$

Problem 13.18

Part a

The properties of the structure, **m** and **k**, ω_n and ϕ_n are given in Problem 13.17. The influence vector is (from Problem 9.13):

$$\boldsymbol{\iota} = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

The modal quantities, given by Eq. (13.1.5), are:

$$L_1 = \phi_1^T \mathbf{m} \boldsymbol{\iota} = 0$$

$$M_1 = \phi_1^T \mathbf{m} \phi_1 = 12.579m$$

$$\Gamma_1 = \frac{L_1}{M_1} = 0$$

$$L_2 = \phi_2^T \mathbf{m} \boldsymbol{\iota} = 0$$

$$M_2 = \phi_2^T \mathbf{m} \phi_2 = 8.292m$$

$$\Gamma_2 = \frac{L_2}{M_2} = 0$$

$$L_3 = \phi_3^T \mathbf{m} \boldsymbol{\iota} = 2m$$

$$M_3 = \phi_3^T \mathbf{m} \phi_3 = 2m$$

$$\Gamma_3 = \frac{L_3}{M_3} = 1$$

The effective forces, Eq. (13.1.4), are

$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m} \boldsymbol{\iota}\, \ddot{u}_g(t) = -\begin{bmatrix} 5m & & \\ & m & \\ & & m \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \ddot{u}_g(t)$$

$$= -\begin{Bmatrix} 0 \\ m \\ m \end{Bmatrix} \ddot{u}_g(t)$$

Substituting Γ_n, **m** and ϕ_n in Eq. (13.1.6) gives

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m} \phi_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{s}_2 = \Gamma_2 \mathbf{m} \phi_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{s}_3 = \Gamma_3 \mathbf{m} \phi_3 = \begin{Bmatrix} 0 \\ m \\ m \end{Bmatrix}$$

The modal expansion of effective forces is shown in Fig. P13.18. The effective forces in the first two modes are zero, implying that these modes will not be excited by vertical ground motion.

Part b

The response is only due to the third mode; from Eq. (13.1.10):

$$\mathbf{u}_3(t) = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} D_3(t)$$

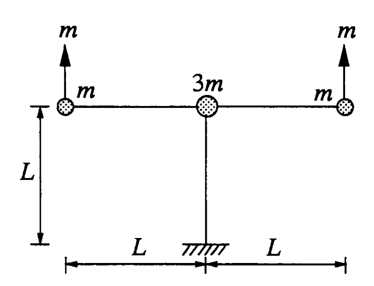
Part c

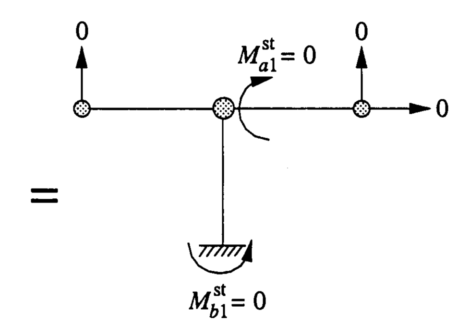
The first two modes are not excited. Due to the third mode, the bending moments at the base of the column and at section a of the beam are

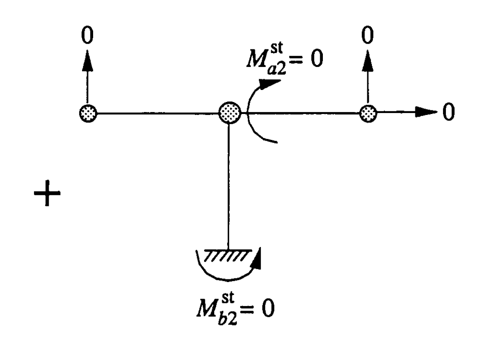
$$M_{b3}(t) = M_{b3}^{\text{st}} A_3(t) \qquad M_{a3}(t) = M_{a3}^{\text{st}} A_3(t)$$

Static analysis of the system in Fig. P13.3 gives $M_{b3}^{\text{st}} = 0$ and $M_{a3}^{\text{st}} = mL$. Thus the total responses are

$$M_b(t) = 0 \qquad M_a(t) = mL\, A_3(t)$$


m
m
m
3m
m
L
L
L


0
0
$M_{a1}^{st} = 0$
0
=
$M_{b1}^{st} = 0$


0
0
$M_{a2}^{st} = 0$
0
+
$M_{b2}^{st} = 0$

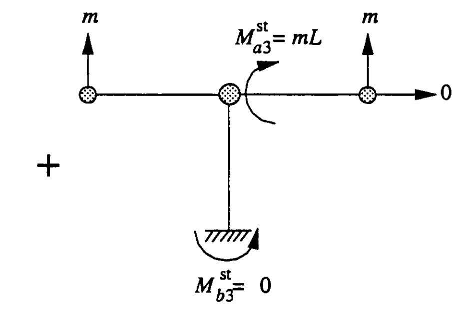

m
m
$M_{a3}^{st} = mL$
0
+
$M_{b3}^{st} = 0$

Fig. P13.18

Problem 13.19

The properties of the structure, $\mathbf{m}$ and $\mathbf{k}$, ω_n and ϕ_n are given in Problem 13.17. The influence vector due to ground motion in the direction *b-d* is (from Problem 9.13, Part c).

$$\boldsymbol{\iota} = \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix}$$

Substituting for $\mathbf{m}$, ι, and ϕ_n in Eq. (13.1.5) gives the modal quantities for the three modes:

$$L_1 = \phi_1^T \mathbf{m}\iota = 3.536m$$
$$M_1 = \phi_1^T \mathbf{m}\phi_1 = 12.597m$$
$$\Gamma_1 = \frac{L_1}{M_1} = 0.281$$

$$L_2 = \phi_2^T \mathbf{m}\iota = 3.536m$$
$$M_2 = \phi_2^T \mathbf{m}\phi_2 = 8.292m$$
$$\Gamma_2 = \frac{L_2}{M_2} = 0.426$$

$$L_3 = \phi_3^T \mathbf{m}\iota = 1.414m$$
$$M_3 = \phi_3^T \mathbf{m}\phi_3 = 2m$$
$$\Gamma_3 = \frac{L_3}{M_3} = 0.707$$

The effective earthquake forces are given by Eq. (13.1.4):

$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\iota\ddot{u}_{gbd}(t) = -\begin{bmatrix} 5m & & \\ & m & \\ & & m \end{bmatrix}\begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix}\ddot{u}_{gbd}(t)$$
$$= -\begin{Bmatrix} 5m/\sqrt{2} \\ m/\sqrt{2} \\ m/\sqrt{2} \end{Bmatrix}\ddot{u}_{gbd}(t)$$

Substituting Γ_n, $\mathbf{m}$ and ϕ_n in Eq. (13.1.6) gives

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m}\phi_1 = \begin{Bmatrix} 1.405m \\ -0.547m \\ 0.547m \end{Bmatrix}$$

$$\mathbf{s}_2 = \Gamma_2 \mathbf{m}\phi_2 = \begin{Bmatrix} 2.130m \\ 0.547m \\ -0.547m \end{Bmatrix}$$

$$\mathbf{s}_3 = \Gamma_3 \mathbf{m}\phi_3 = \begin{Bmatrix} 0 \\ 0.707m \\ 0.707m \end{Bmatrix}$$

The modal expansion of the spatial distribution of effective forces is shown in Fig. P13.19.

Part b

The modal displacements, form Eq. (13.1.10) are

$$\mathbf{u}_1 = \Gamma_1\phi_1 D_1(t) = \begin{Bmatrix} 0.281 \\ -0.547 \\ 0.547 \end{Bmatrix} D_1(t)$$

$$\mathbf{u}_2 = \Gamma_2\phi_2 D_2(t) = \begin{Bmatrix} 0.426 \\ 0.547 \\ -0.547 \end{Bmatrix} D_2(t)$$

$$\mathbf{u}_3 = \Gamma_3\phi_3 D_3(t) = \begin{Bmatrix} 0 \\ 0.707 \\ 0.707 \end{Bmatrix} D_3(t)$$

Combining the modal displacements gives the total displacements:

$$u_1(t) = 0.281D_1(t) + 0.426D_2(t) + 0D_3(t)$$
$$u_2(t) = -0.547D_1(t) + 0.547D_2(t) + 0.707D_3(t)$$
$$u_3(t) = 0.547D_1(t) - 0.547D_2(t) + 0.707D_3(t)$$

Part c

The bending moment at the base of the column due to the n[th] mode is

$$M_{bn} = M_{bn}^{\text{st}} A_n(t)$$

Substituting the modal static responses M_{bn}^{st}, shown in Fig. P13.19, and combining modal responses gives

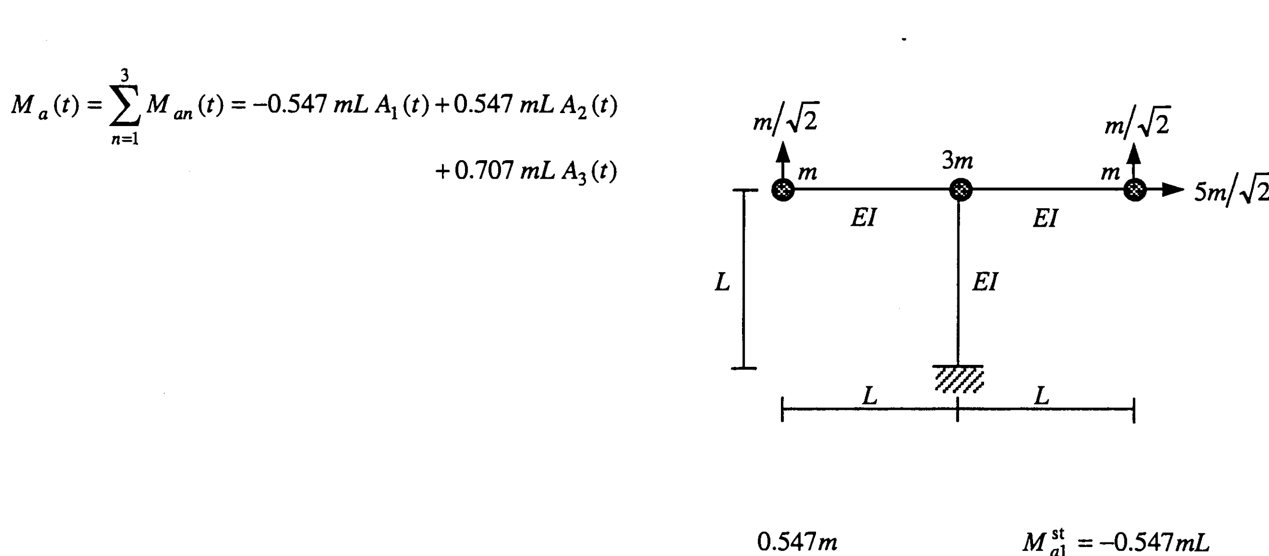
$$M_b(t) = \sum_{n=1}^{3} M_{bn}(t)$$
$$= 2.499\, mL\, A_1(t) + 1.036\, mL\, A_2(t)$$

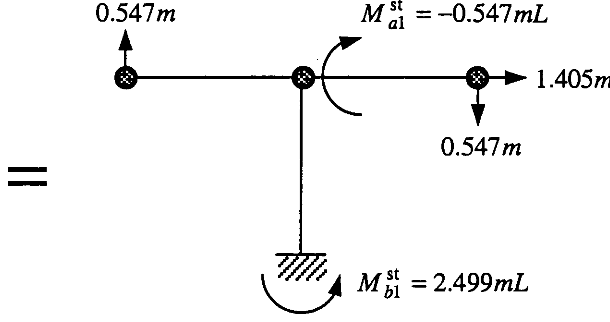
The bending moment at location a of the beam due to the n[th] mode is

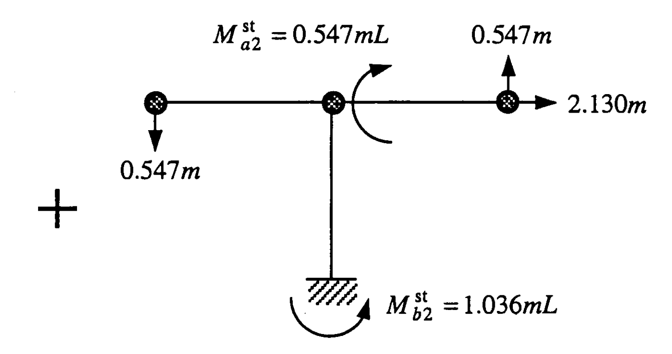
$$M_{an}(t) = M_{an}^{\text{st}} A_n(t)$$

Substituting the modal static responses M_{an}^{st}, shown in Fig. P13.19, and combining modal responses gives

$$M_a(t) = \sum_{n=1}^{3} M_{an}(t) = -0.547\, mL\, A_1(t) + 0.547\, mL\, A_2(t) + 0.707\, mL\, A_3(t)$$







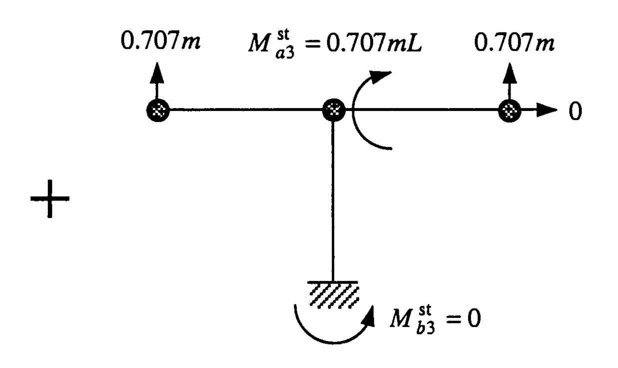


Fig. P13.19

Problem 13.20

Part a

The properties of the structure, $\mathbf{m}$ and $\mathbf{k}$, ω_n and ϕ_n are given in Problem 13.17. The influence vector due to ground motion in the direction *b-c* is (from Problem 9.13, Part c).

$$\boldsymbol{\iota} = \begin{Bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix}$$

Substituting for $\mathbf{m}$, ι and ϕ_n in Eq. (13.1.5) gives the modal quantities for the three modes:

$$L_1 = \phi_1^T \mathbf{m}\boldsymbol{\iota} = -3.536m$$
$$M_1 = \phi_1^T \mathbf{m}\phi_1 = 12.597m$$
$$\Gamma_1 = \frac{L_1}{M_1} = -0.281$$

$$L_2 = \phi_2^T \mathbf{m}\boldsymbol{\iota} = -3.536m$$
$$M_2 = \phi_2^T \mathbf{m}\phi_2 = 8.292m$$
$$\Gamma_2 = \frac{L_2}{M_2} = -0.426$$

$$L_3 = \phi_3^T \mathbf{m}\boldsymbol{\iota} = 1.414m$$
$$M_3 = \phi_3^T \mathbf{m}\phi_3 = 2m$$
$$\Gamma_3 = \frac{L_3}{M_3} = 0.707$$

The effective earthquake forces are given by Eq. (13.1.4):

$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\boldsymbol{\iota}\ddot{u}_{gbc}(t) = -\begin{bmatrix} 5m & & \\ & m & \\ & & m \end{bmatrix} \begin{Bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix} \ddot{u}_{gbc}(t)$$
$$= -\begin{Bmatrix} -5m/\sqrt{2} \\ m/\sqrt{2} \\ m/\sqrt{2} \end{Bmatrix} \ddot{u}_{gbc}(t)$$

Substituting Γ_n, $\mathbf{m}$ and ϕ_n in Eq. (13.1.6) gives

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m}\phi_1 = \begin{Bmatrix} -1.405m \\ 0.547m \\ -0.547m \end{Bmatrix}$$

$$\mathbf{s}_2 = \Gamma_2 \mathbf{m}\phi_2 = \begin{Bmatrix} -2.130m \\ -0.547m \\ 0.547m \end{Bmatrix}$$

$$\mathbf{s}_3 = \Gamma_3 \mathbf{m}\phi_3 = \begin{Bmatrix} 0 \\ 0.707m \\ 0.707m \end{Bmatrix}$$

The modal expansion of the spatial distribution of effective forces is shown in Fig. P13.20.

Part b

The modal displacements, form Eq. (13.1.10). are

$$\mathbf{u}_1 = \Gamma_1 \phi_1 D_1(t) = \begin{Bmatrix} -0.281 \\ 0.547 \\ -0.547 \end{Bmatrix} D_1(t)$$

$$\mathbf{u}_2 = \Gamma_2 \phi_2 D_2(t) = \begin{Bmatrix} -0.426 \\ -0.547 \\ 0.547 \end{Bmatrix} D_2(t)$$

$$\mathbf{u}_3 = \Gamma_3 \phi_3 D_3(t) = \begin{Bmatrix} 0 \\ 0.707 \\ 0.707 \end{Bmatrix} D_3(t)$$

Combining the modal displacements gives the total displacements:

$$u_1(t) = -0.281D_1(t) - 0.426D_2(t) + \quad 0D_3(t)$$
$$u_2(t) = \ \ 0.547D_1(t) - 0.547D_2(t) + 0.707D_3(t)$$
$$u_3(t) = -0.547D_1(t) + 0.547D_2(t) + 0.707D_3(t)$$

Part c

The bending moment at the base of the column due to the n[th] mode is

$$M_{bn} = M_{bn}^{\text{st}} A_n(t)$$

Substituting the modal static responses M_{bn}^{st}, shown in Fig. P13.20, and combining modal responses gives

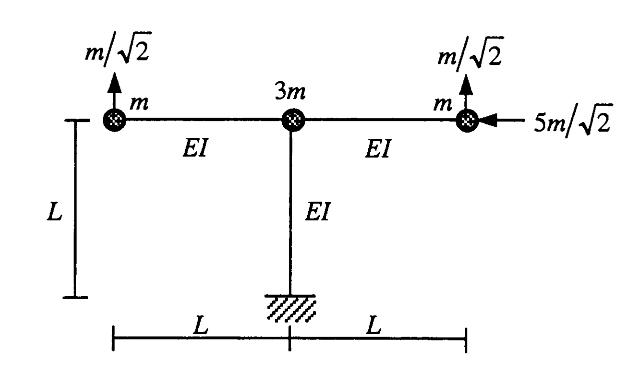
$$M_b(t) = \sum_{n=1}^{3} M_{bn}(t)$$
$$= -2.499\, mL\, A_1(t) - 1.036\, mL\, A_2(t)$$

The bending moment at location a of the beam due to the n[th] mode is

$$M_{an}(t) = M_{an}^{\text{st}} A_n(t)$$

Substituting the modal static responses M_{an}^{st}, shown in Fig. P13.20, and combining modal responses gives

$$M_a(t) = \sum_{n=1}^{3} M_{an}(t) = 0.547\, mL\, A_1(t) - 0.547\, mL\, A_2(t) + 0.707\, mL\, A_3(t)$$



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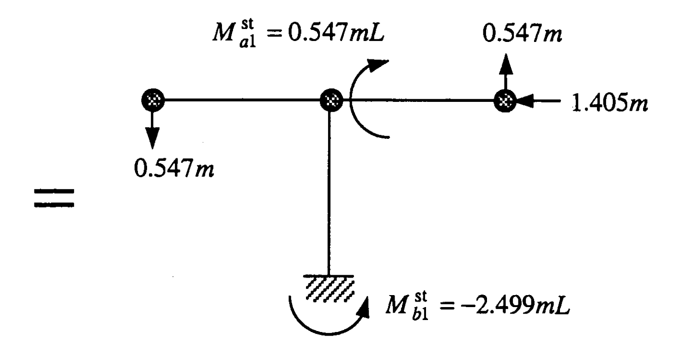

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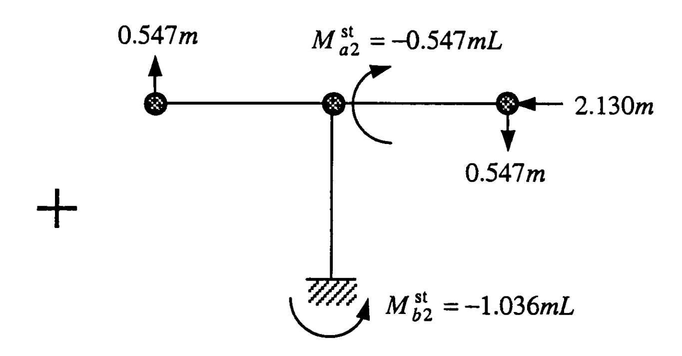

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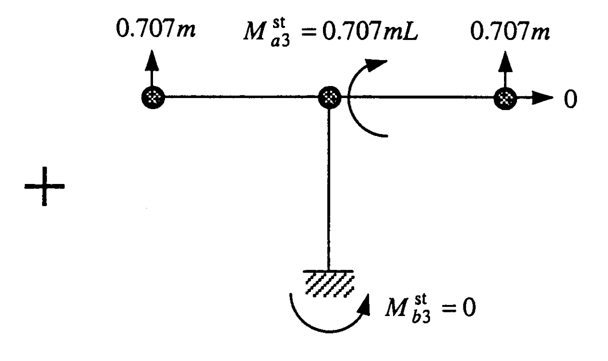


Fig. P13.20

Problem 13.21

Part a

The properties of the structure, $\mathbf{m}$ and $\mathbf{k}$, ω_n and ϕ_n are given in Problem 13.17. The influence vector due to ground rocking $u_{g\theta}$ (in radians) in the counter-clockwise direction is (from Problem 9.13, Part c).

$$\boldsymbol{\iota} = \begin{Bmatrix} -L \\ L \\ -L \end{Bmatrix}$$

Substituting for $\mathbf{m}$, $\boldsymbol{\iota}$ and ϕ_n in Eq. (13.1.5) gives the modal quantities for the three modes:

$$L_1 = \phi_1^T \mathbf{m}\,\boldsymbol{\iota} = -8.898mL$$
$$M_1 = \phi_1^T \mathbf{m}\phi_1 = 12.597m$$
$$\Gamma_1 = \frac{L_1}{M_1} = -0.7064L$$

$$L_2 = \phi_2^T \mathbf{m}\boldsymbol{\iota} = -2.4348mL$$
$$M_2 = \phi_2^T \mathbf{m}\phi_2 = 8.292m$$
$$\Gamma_2 = \frac{L_2}{M_2} = -0.294L$$

$$L_3 = \phi_3^T \mathbf{m}\boldsymbol{\iota} = 0$$
$$M_3 = \phi_3^T \mathbf{m}\phi_3 = 2m$$
$$\Gamma_3 = \frac{L_3}{M_3} = 0$$

The effective earthquake forces are given by Eq. (13.1.4):

$$\mathbf{p}_{\text{eff}}(t) = -\mathbf{m}\,\boldsymbol{\iota}\,\ddot{u}_{g\theta}(t) = -\begin{bmatrix} 5m & & \\ & m & \\ & & m \end{bmatrix}\begin{Bmatrix} -L \\ L \\ -L \end{Bmatrix}\ddot{u}_{g\theta}(t)$$

$$= -\begin{Bmatrix} -5mL \\ mL \\ -mL \end{Bmatrix}\ddot{u}_{g\theta}(t)$$

Substituting Γ_n, $\mathbf{m}$ and ϕ_n in Eq. (13.1.6) gives

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m}\phi_1 = \begin{Bmatrix} -3.532mL \\ 1.377mL \\ -1.377mL \end{Bmatrix}$$

$$\mathbf{s}_2 = \Gamma_2 \mathbf{m}\phi_2 = \begin{Bmatrix} -1.470mL \\ -0.377mL \\ 0.377mL \end{Bmatrix}$$

$$\mathbf{s}_3 = \Gamma_3 \mathbf{m}\phi_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The modal expansion of the spatial distribution of effective forces is shown in Fig. P13.21.

Part b

The modal displacements, from Eq. (13.1.10), are

$$\mathbf{u}_1 = \Gamma_1 \phi_1 D_1(t) = \begin{Bmatrix} -0.706L \\ 1.377L \\ -1.377L \end{Bmatrix} D_1(t)$$

$$\mathbf{u}_2 = \Gamma_2 \phi_2 D_2(t) = \begin{Bmatrix} -0.294L \\ -0.377L \\ 0.377L \end{Bmatrix} D_2(t)$$

$$\mathbf{u}_3 = \Gamma_3 \phi_3 D_3(t) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} D_3(t)$$

Note that, here, $D_1(t)$, $D_2(t)$ and $D_3(t)$ are in radians.

Combining the modal displacements gives the total displacements:

$$u_1(t) = -0.706L\,D_1(t) - 0.294L\,D_2(t)$$
$$u_2(t) = 1.377L\,D_1(t) - 0.377L\,D_2(t)$$
$$u_3(t) = -1.377L\,D_1(t) + 0.377L\,D_2(t)$$

Observe that the third mode does not contribute to the displacement response.

Part c

The bending moment at the base of the column due to the n[th] mode is

$$M_{bn} = M_{bn}^{\text{st}} A_n(t)$$

where $A_n(t)$ is in radians per second per second.

Substituting the modal static responses M_{bn}^{st}, shown in Fig. P13.21, and combining modal responses gives

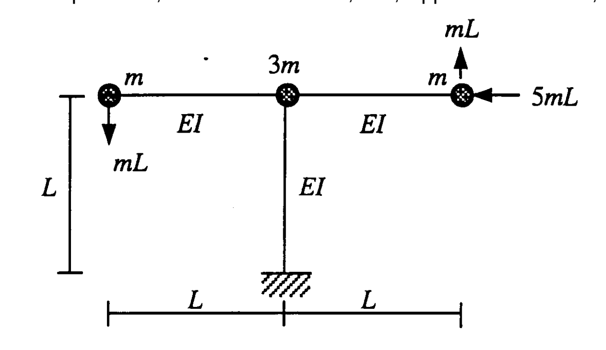
$$M_b(t) = \sum_{n=1}^{3} M_{bn}(t)$$
$$= -6.826\, mL^2\, A_1(t) + 1.470\, mL^2\, A_2(t)$$

The bending moment at location a of the beam due to the n^{th} mode is

$$M_{an}(t) = M_{an}^{\text{st}} A_n(t)$$

Substituting the modal static responses M_{an}^{st}, shown in Fig. P13.21, and combining modal responses gives

$$M_a(t) = \sum_{n=1}^{3} M_{an}(t)$$
$$= 1.377\, mL^2\, A_1(t) - 1.470\, mL^2\, A_2(t)$$



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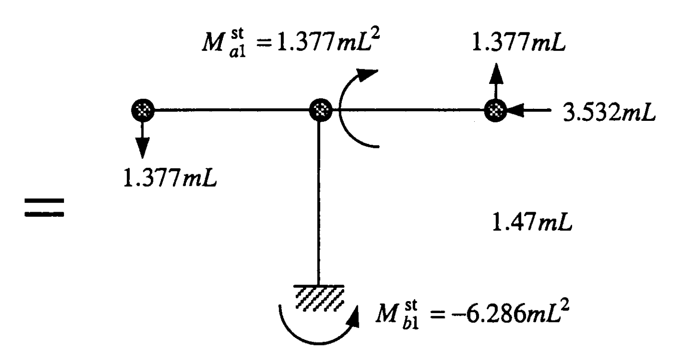

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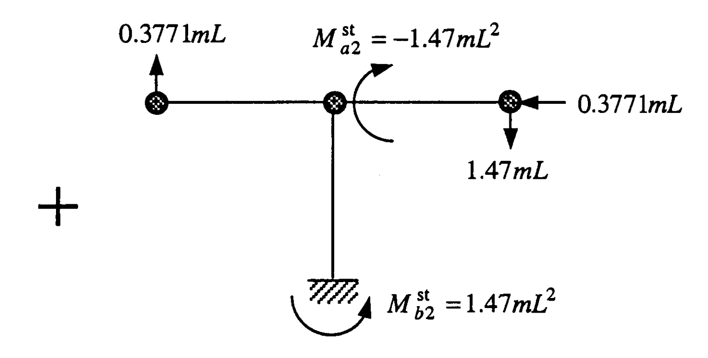

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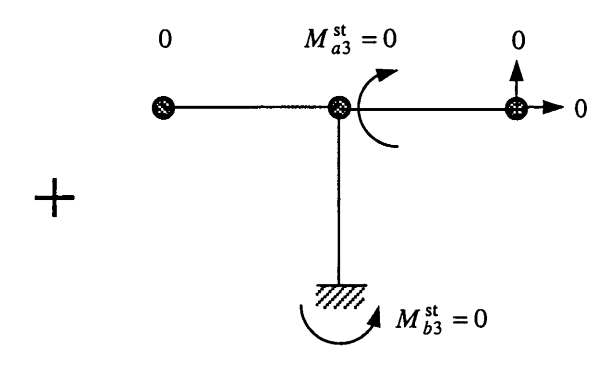
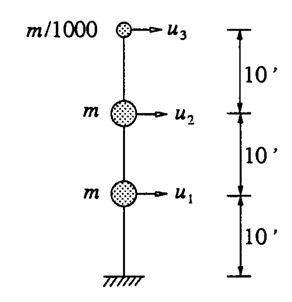


Fig. P13.21

Problem 13.22



Data:

$$m = 0.486 \text{ kip}-\text{sec}^2/\text{in.}$$

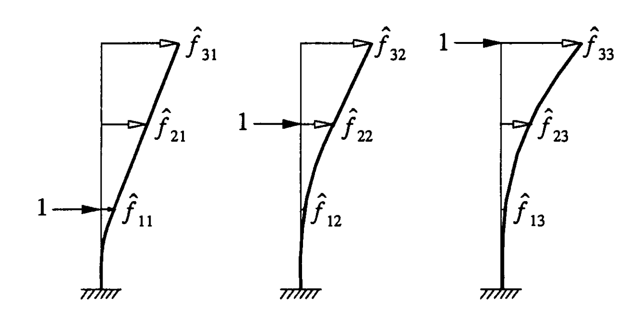
$$\frac{EI}{L^3} = 56.26 \text{ kips/in.}$$

$$\frac{EI'}{L^3} = 0.0064 \text{ kip/in.}$$

$$\zeta_n = 0.05, n = 1, 2, 3$$

Part a

First compute the flexibility matrix:



$$\hat{\mathbf{f}} = \begin{bmatrix} L^3/3EI & 5L^3/6EI & 4L^3/3EI \\ & 8L^3/3EI & 14L^3/3EI \\ (sym) & & 26L^3/3EI + L^3/3EI' \end{bmatrix}$$

Next determine the lateral stiffness matrix $\hat{\mathbf{k}}$:

$$\hat{\mathbf{k}} = \hat{\mathbf{f}}^{-1} = \begin{bmatrix} 771.62 & -241.19 & 0.0329 \\ & 96.55 & -0.0439 \\ (sym) & & 0.0192 \end{bmatrix} \text{kip/in.}$$

The mass matrix is

$$\mathbf{m} = 0.486 \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0.001 \end{bmatrix} \text{kip}-\text{sec}^2/\text{in.}$$

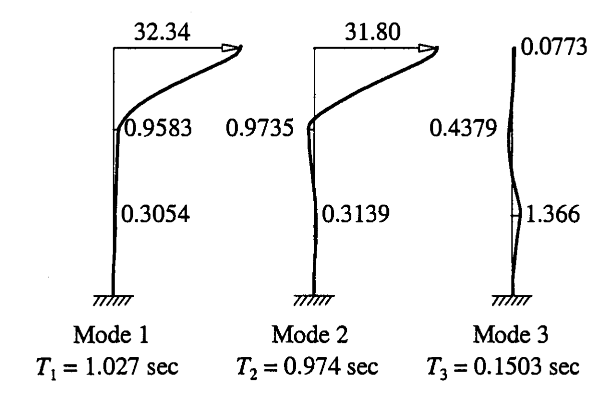
Solution of the eigenvalue problem gives the natural periods and modes:

$$T_1 = 1.027 \text{ sec} \qquad T_2 = 0.974 \text{ sec} \qquad T_3 = 0.1503 \text{ sec}$$

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} -0.3054 \\ -0.9583 \\ -32.34 \end{Bmatrix} \quad \boldsymbol{\phi}_2 = \begin{Bmatrix} 0.3139 \\ 0.9735 \\ -31.80 \end{Bmatrix} \quad \boldsymbol{\phi}_3 = \begin{Bmatrix} 1.366 \\ -0.4379 \\ 0.0773 \end{Bmatrix}$$

The modes have been normalized so that

$$M_n = \boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_n = 1$$



Part b

Computing Eq. (13.2.3) gives the modal properties:

$$M_1 = 1 \qquad M_2 = 1 \qquad M_3 = 1$$

$$L_1^h = -0.6296 \qquad L_2^h = 0.6104 \qquad L_3^h = 0.4511$$

$$\Gamma_1 = -0.6296 \qquad \Gamma_2 = 0.6104 \qquad \Gamma_3 = 0.4511$$

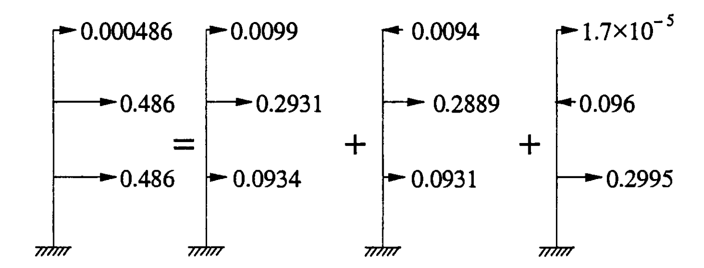
Substituting these data in Eq. (13.2.4) gives

$$\mathbf{s}_1 = \begin{Bmatrix} 0.0934 \\ 0.2931 \\ 0.0099 \end{Bmatrix} \quad \mathbf{s}_2 = \begin{Bmatrix} 0.0931 \\ 0.2889 \\ -0.0094 \end{Bmatrix} \quad \mathbf{s}_3 = \begin{Bmatrix} 0.2995 \\ -0.0960 \\ 1.7 \times 10^{-5} \end{Bmatrix}$$

Thus the modal expansion of $\mathbf{m}\,\mathbf{1}$ is [Eq. (13.2.2)]:

$$\begin{Bmatrix} 0.486 \\ 0.486 \\ 0.000486 \end{Bmatrix} = \begin{Bmatrix} 0.0934 \\ 0.2931 \\ 0.0099 \end{Bmatrix} + \begin{Bmatrix} 0.0931 \\ 0.2889 \\ -0.0094 \end{Bmatrix} + \begin{Bmatrix} 0.2995 \\ -0.0960 \\ 1.7 \times 10^{-5} \end{Bmatrix} \tag{c}$$

This expansion is shown graphically:



Part c

We compute

$$\mathbf{u}^{st} = \hat{\mathbf{k}}^{-1}\,\mathbf{m}\,\mathbf{1} = \begin{bmatrix} 0.0051 & 0.0046 & 0.0004 \\ 0.0161 & 0.0143 & -0.0001 \\ 0.5437 & -0.4664 & 0 \end{bmatrix}$$

The modal static responses of the displacement of the appendage mass are

$$u_{31}^{st} = 0.5437 \qquad u_{32}^{st} = -0.4664 \qquad u_{33}^{st} = 2.0 \times 10^{-5} \tag{a}$$

Static analysis of the system for forces $\mathbf{s}_n$ (n = 1, 2 and 3) gives the shear force in the appendage:

$$V_{a1}^{st} = 0.0099 \qquad V_{a2}^{st} = -0.0094$$
$$V_{a3}^{st} = 1.7 \times 10^{-5} \tag{b}$$

and the shear force at the base of the tower:

$$V_{b1}^{st} = 0.3965 \qquad V_{b2}^{st} = 0.3726 \qquad V_{b3}^{st} = 0.2035 \tag{c}$$

Part d

From Eq. (a) we expect that the first two modes should give the largest contribution to the displacement of the appendage mass. Similarly, these two modes will dominate the shear at the base of the appendage. In addition to important contributions from the first two modes, the third mode should have, according to Eq. (c), significant contribution to the shear at the base of the tower.

Problem 13.23

Part a

The three modal SDF systems have the following properties:

$$T_1 = 1.027\text{ sec} \quad T_2 = 0.974\text{ sec} \quad T_3 = 0.1503\text{ sec}$$

$$\zeta_1 = 0.05 \qquad \zeta_2 = 0.05 \qquad \zeta_3 = 0.05$$

The displacement responses $D_n(t)$ of these three SDF systems to the El Centro ground motion are shown in Fig. P13.23a. The pseudo-acceleration responses are $A_n(t) = \omega_n^2 D_n(t)$ and these are shown in Fig. P13.23b.

Part b

Step 5c of Section 13.2.4 is implemented to determine the contribution of the n$^{\text{th}}$ mode

$$r_n(t) = r_n^{\text{st}} A_n(t)$$

to the desired response quantities: displacement $u_3(t)$ of the appendage mass, shear $V_a(t)$ in the appendage, and shear force $V_b(t)$ at the base of the tower.

The modal static responses u_{3n}^{st} are available in Eq. (a) of Problem 13.22, V_{an}^{st} in Eq. (b), and V_{bn}^{st} in Eq. (c). These modal static responses are multiplied by $A_n(t)$ (Fig. P13.23b) at each time step to obtain the results presented in Figs. P13.23c, P13.23d, and P13.23e. The peak values are noted for each modal response.

Part c

The modal responses are combined at each time instant to determine the total responses shown in Figs. P13.23c, P13.23d, and P13.23e. The peak values are noted:

$u_{3o} = 44.58$ in. $\quad V_{ao} = 0.879$ kips
$V_{bo} = 159.83$ kips

Part d

The seismic coefficient for the appendage is

$$\frac{V_{ao}}{w_a} = \frac{0.879}{w/1000} = \frac{0.879\,(1000)}{0.486\,(386)} = 4.69$$

and that for the tower is

$$\frac{V_{bo}}{w} = \frac{159.83}{2.001\,w} = \frac{159.83}{2.001\,(0.486)\,(386)} = 0.426$$

The seismic coefficient for the appendage is large because its natural frequency is tuned to the fundamental natural frequency of the main tower and the appendage mass is small.

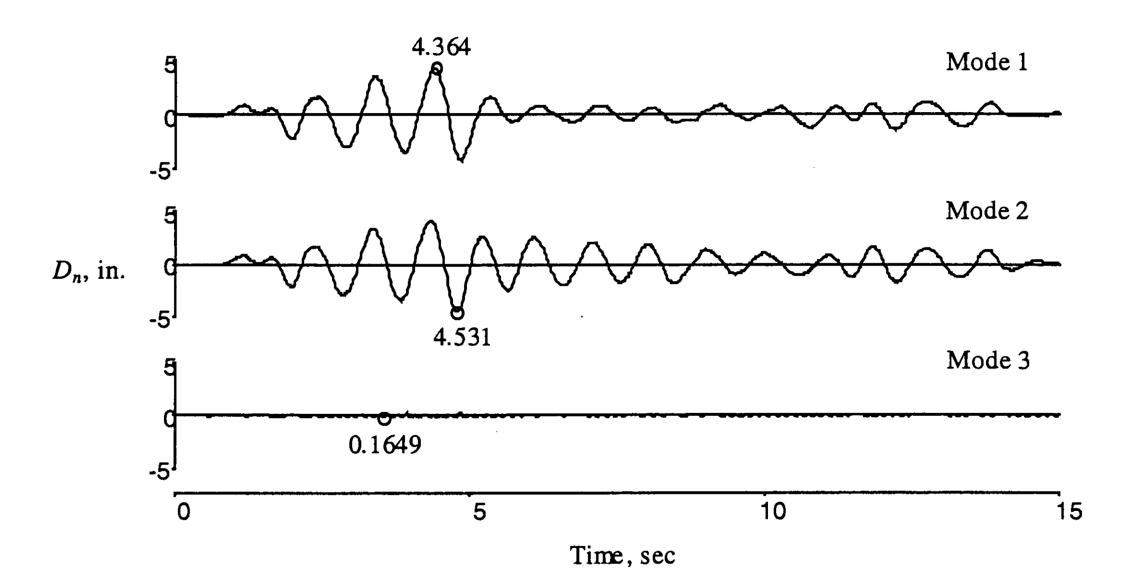


Fig. P13.23a

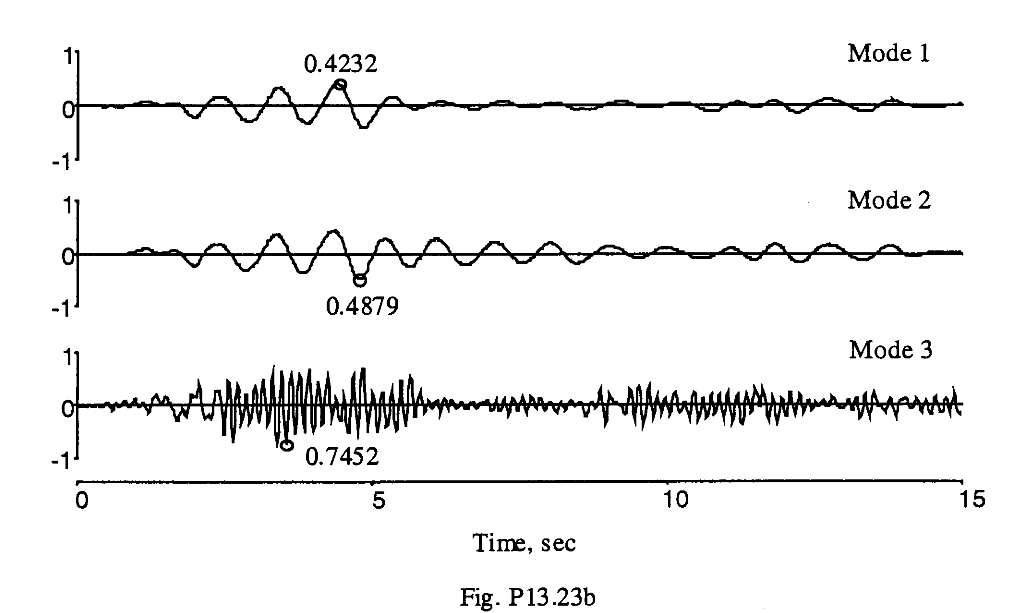


Fig. P13.23b

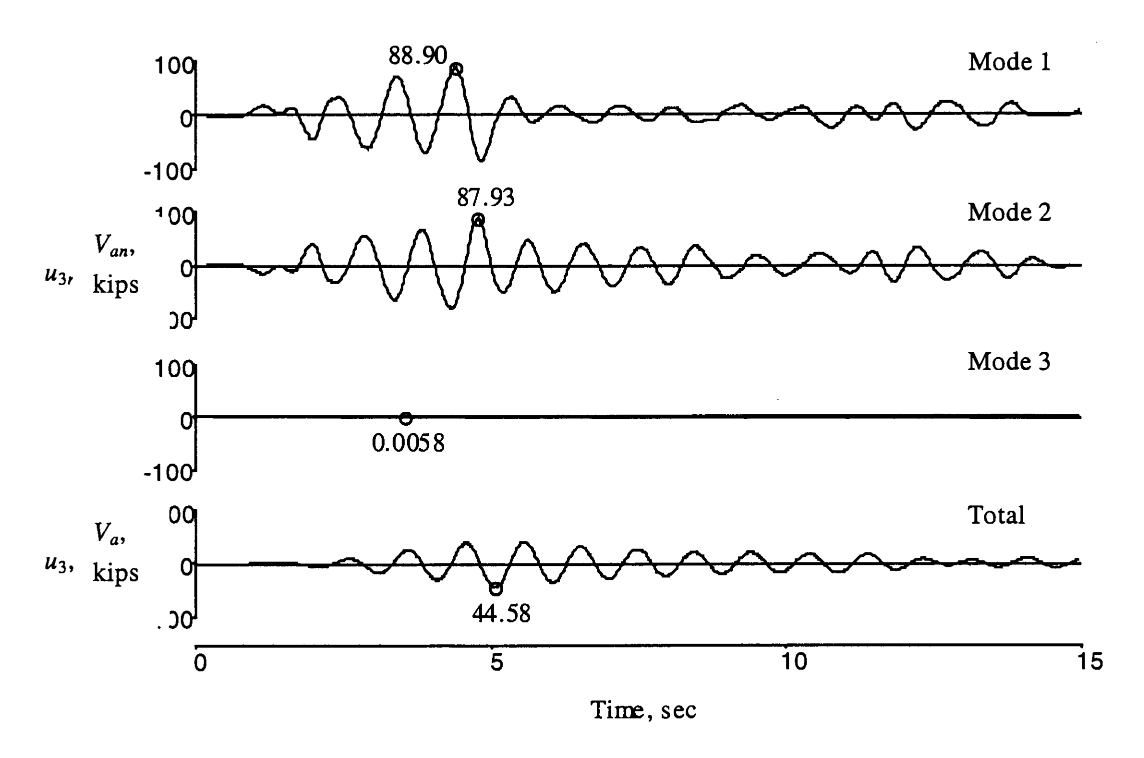


Fig. P13.23c

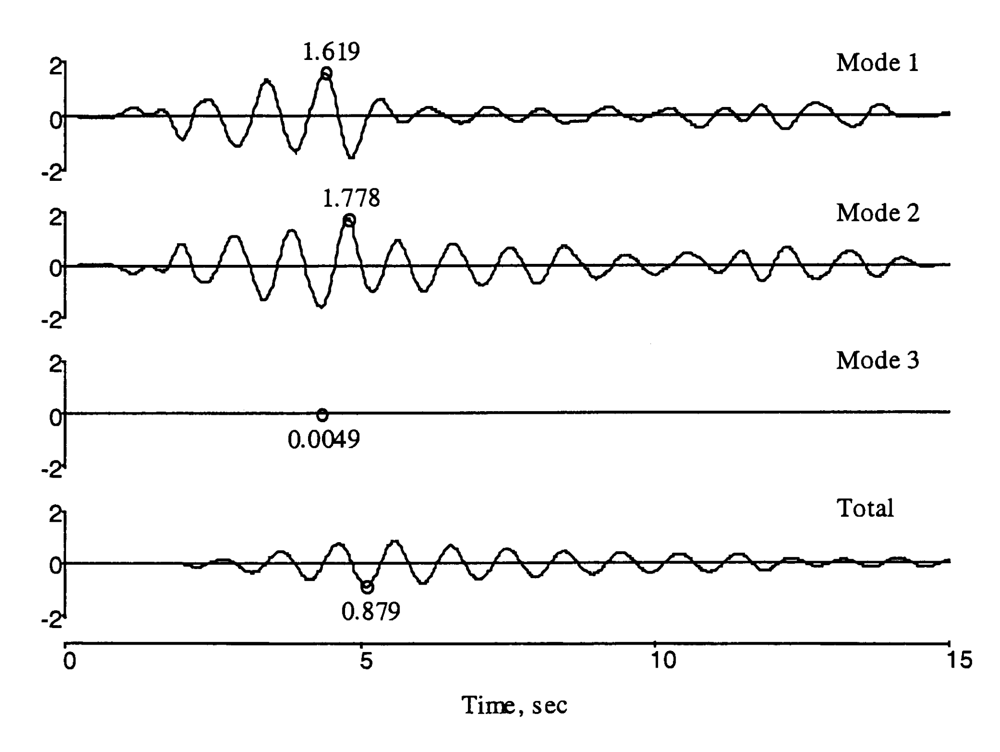


Fig. P13.23d

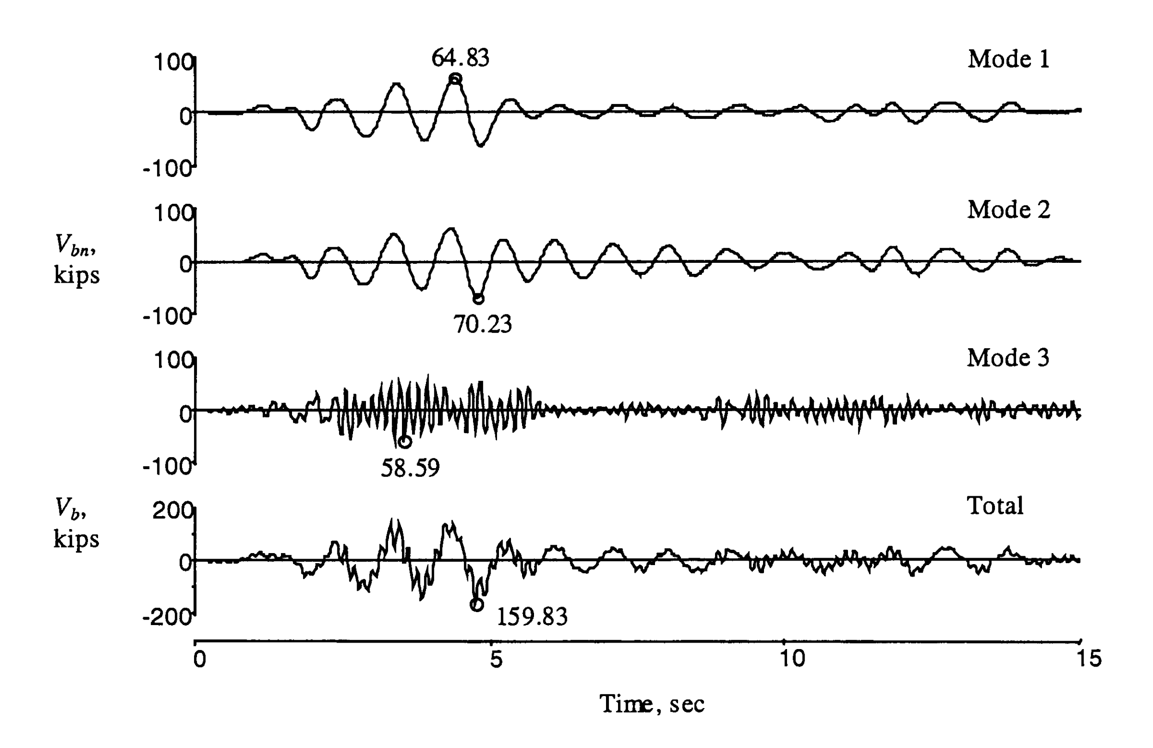

100
0
-100
64.83
Mode 1
V_{bn},
kips
100
0
-100
70.23
Mode 2
100
0
-100
58.59
Mode 3
V_b,
kips
200
0
-200
159.83
Total
0
5
10
15
Time, sec

Fig. P13.23e

Problem 13.24

Data (see Problem 9.14):

$m = 0.2331 \text{ kip}-\text{sec}^2/\text{in.}$

$k = 1.5 \text{ kips/in.}$

$b = 25 \text{ ft}$

$\zeta_n = 0.05, n = 1, 2 \text{ and } 3$

DOFs: $\mathbf{u} = \langle u_x \quad u_y \quad u_\theta \rangle^T$

Equations of motion:

$$m\begin{bmatrix}1 & & \\ & 1 & \\ & & b^2/6\end{bmatrix}\ddot{\mathbf{u}} + k\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & -b\\ 0 & -b & 3b^2\end{bmatrix}\mathbf{u} = \underbrace{-m\begin{Bmatrix}0\\1\\0\end{Bmatrix}}_{\mathbf{m}\iota}\ddot{u}_{gy}(t)$$

Natural frequencies and modes (from Problem 10.17):

$\omega_1 = 5.96 \qquad \omega_2 = 6.21 \qquad \omega_3 = 10.90$

$$\phi_1 = \begin{Bmatrix}0\\2.032\\0.0033\end{Bmatrix} \quad \phi_2 = \begin{Bmatrix}2.071\\0\\0\end{Bmatrix} \quad \phi_3 = \begin{Bmatrix}0\\-0.3988\\0.0166\end{Bmatrix}$$

Part a

$$M_n = \phi_n^T \mathbf{m}\, \phi_n \qquad L_n^h = \phi_n^T \mathbf{m}\, \iota \qquad \Gamma_n = \frac{L_n^h}{M_n} \qquad \text{(a)}$$

Substituting for $\mathbf{m}$, ϕ_n and ι in Eq. (a) gives

$\Gamma_1 = 0.4738 \qquad \Gamma_2 = 0 \qquad \Gamma_3 = -0.0930$

Substituting for $\mathbf{m}$, ϕ_n and Γ_n in Eq. (13.1.6) gives

$$\mathbf{s}_1 = \begin{Bmatrix}0\\0.2245\\5.3966\end{Bmatrix} \qquad \mathbf{s}_2 = \begin{Bmatrix}0\\0\\0\end{Bmatrix} \qquad \mathbf{s}_3 = \begin{Bmatrix}0\\0.0086\\-5.3966\end{Bmatrix}$$

where

$$\mathbf{s}_n = \langle s_{xn} \quad s_{yn} \quad s_{\theta n} \rangle^T$$

The modal expansion of $\mathbf{m}\,\iota$ is shown in the following figure:

0.2331 = 5.3966 / 0.2245 + [ ] + 5.3966 / 0.0086

Part b

From Eq. (13.3.10),

$$M_n^* = \frac{(L_n^h)^2}{M_n} \qquad I_{On}^* = s_{\theta n} \qquad \text{(b)}$$

Substituting numerical data for M_n, L_n^h and $s_{\theta n}$ gives Table P13.24.

Table P13.24

Mode	M_n^*	I_{On}^*
1	0.2245	5.3966
2	0	0
3	0.0086	– 5.3966
$\sum M_n^*$ or $\sum I_{On}^*$	0.2331 = m	0

The last row of the table verifies Eq. (13.3.9).

Part c

The displacements due to the n[th] mode are

$$\mathbf{u}(t) = \sum_{n=1}^{3} \Gamma_n\, \phi_n\, D_n(t)$$

Substituting for Γ_n and ϕ_n gives

$$\begin{Bmatrix}u_x(t)\\u_y(t)\\u_\theta(t)\end{Bmatrix} = 0.4738\begin{Bmatrix}0\\2.032\\0.0033\end{Bmatrix}D_1(t) - 0.0930\begin{Bmatrix}0\\-0.3988\\0.0166\end{Bmatrix}D_3(t)$$

or

$$\begin{Bmatrix}u_x(t)\\u_y(t)\\u_\theta(t)\end{Bmatrix} = \begin{Bmatrix}0\\0.9629\\0.00156\end{Bmatrix}D_1(t) + \begin{Bmatrix}0\\0.0371\\-0.00156\end{Bmatrix}D_3(t)$$

Part d

The modal static responses for base shear in the y-direction and base torque are

$$V_{byn}^{st} = s_{yn} \qquad T_{bn}^{st} = s_{\theta n}$$

Substituting numerical values for s_{yn} and $s_{\theta n}$ gives

$V_{by1}^{st} = 0.2245 \qquad V_{by2}^{st} = 0 \qquad V_{by3}^{st} = 0.0086$

$T_{b1}^{st} = 5.3966 \qquad T_{b2}^{st} = 0 \qquad T_{b3}^{st} = -5.3966$

Substituting V_{byn}^{st} and T_{bn}^{st} in Eq. (13.3.15) gives the modal responses:

$$V_{by1}(t) = 0.2245\, A_1(t)$$

$$V_{by2}(t) = 0$$

$$V_{by3}(t) = 0.0086\, A_3(t)$$

$$T_{b1}(t) = 5.3966\, A_1(t)$$

$$T_{b2}(t) = 0$$

$$T_{b3} = -5.3966\, A_3(t)$$

Combining the modal responses gives the total response:

$$V_{by}(t) = 0.2245\, A_1(t) + 0.0086\, A_3(t)$$

$$T_b(t) = 5.3966\, A_1(t) - 5.3966\, A_3(t)$$

Problem 13.25

Equations of motion:

$$m\begin{bmatrix}1 & & \\ & 1 & \\ & & b^2/6\end{bmatrix}\ddot{\mathbf{u}}+k\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & -b\\ 0 & -b & 3b^2\end{bmatrix}\mathbf{u} = \underbrace{-m\begin{Bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\\ 0\end{Bmatrix}}_{\mathbf{m}\iota}\ddot{u}_g(t)$$

where m, b, k are given in the solution to Problem 13.24. Also the ω_n and ϕ_n are given in the solution to Problem 13.24.

Part a

$$M_n = \phi_n^T \mathbf{m}\,\phi_n \qquad L_n^h = \phi_n^T \mathbf{m}\,\iota \qquad \Gamma_n = \frac{L_n^h}{M_n} \qquad \text{(a)}$$

Substituting for $\mathbf{m}$, ϕ_n and ι in Eq. (a) gives

$$\Gamma_1 = 0.3350 \qquad \Gamma_2 = 0.3414 \qquad \Gamma_3 = -0.0658$$

Substituting for $\mathbf{m}$, ϕ_n and Γ_n in Eq. (13.1.6) gives

$$\mathbf{s}_1 = \begin{Bmatrix}0\\ 0.1588\\ 3.816\end{Bmatrix}; \quad \mathbf{s}_2 = \begin{Bmatrix}0.1649\\ 0\\ 0\end{Bmatrix}; \quad \mathbf{s}_3 = \begin{Bmatrix}0\\ 0.0061\\ -3.816\end{Bmatrix}$$

where

$$\mathbf{s}_n = \langle s_{xn} \quad s_{yn} \quad s_{\theta n}\rangle^T$$

The modal expansion of $\mathbf{m}\,\iota$ is shown in the following figure:

0.1649, 0.1649 = 0.1588, 0, 3.816 + 0.1649 + 0.0061, 0, 3.816

Part b

From Eq. (13.3.10),

$$M_n^* = \frac{(L_n^h)^2}{M_n} \qquad I_{On}^* = s_{\theta n} \qquad \text{(b)}$$

Substituting numerical data for M_n, L_n^h and $s_{\theta n}$ gives Table P13.25.

Table P13.25

Mode	M_n^*	I_{On}^*
1	0.1123	3.816
2	0.1166	0
3	0.0043	− 3.816
$\sum M_n^*$ or $\sum I_{On}^*$	0.2332 = m	0

The last row of the table verifies Eq. (13.3.9).

Part c

The displacement response is

$$\mathbf{u}(t) = \sum_{n=1}^{3} \Gamma_n\, \phi_n\, D_n(t) \qquad \text{(c)}$$

Substituting for Γ_n and ϕ_n gives

$$\begin{Bmatrix}u_x(t)\\ u_y(t)\\ u_\theta(t)\end{Bmatrix} = 0.3350\begin{Bmatrix}0\\ 2.032\\ 0.0033\end{Bmatrix}D_1(t) + 0.3414\begin{Bmatrix}2.071\\ 0\\ 0\end{Bmatrix}D_2(t) - 0.0658\begin{Bmatrix}0\\ -0.3988\\ 0.0166\end{Bmatrix}D_3(t)$$

or

$$\begin{Bmatrix}u_x(t)\\ u_y(t)\\ u_\theta(t)\end{Bmatrix} = \begin{Bmatrix}0\\ 0.6809\\ 0.0011\end{Bmatrix}D_1(t) + \begin{Bmatrix}0.7071\\ 0\\ 0\end{Bmatrix}D_2(t) + \begin{Bmatrix}0\\ 0.0262\\ -0.0011\end{Bmatrix}D_3(t)$$

Part d

The modal static responses for the x-component $V_{bx}(t)$ of base shear, the y-component $V_{by}(t)$ of base shear, and the base torque $T_b(t)$ are

$$V_{bxn}^{st} = s_{xn} \qquad V_{byn}^{st} = s_{yn} \qquad T_{byn}^{st} = s_{\theta n} \qquad \text{(d)}$$

Substituting numerical values for s_{xn}, s_{yn} and $s_{\theta n}$ gives

$$V_{bx1}^{st} = 0 \qquad V_{bx2}^{st} = 0.1649 \qquad V_{bx3}^{st} = 0$$

$$V_{by1}^{st} = 0.1588 \qquad V_{by2}^{st} = 0 \qquad V_{by3}^{st} = 0.0061$$

$$T_{b1}^{st} = 3.816 \qquad T_{b2}^{st} = 0 \qquad T_{b3}^{st} = -3.816$$

Substituting these modal static responses in Eq. (13.3.15) gives the modal responses:

$$V_{bx1}(t) = 0$$

$$V_{bx2}(t) = 0.1649\, A_2(t)$$

$$V_{bx3}(t) = 0$$

$$V_{by1}(t) = 0.1588\, A_1(t)$$

$$V_{by2}(t) = 0$$

$$V_{by3}(t) = 0.0061\, A_3(t)$$

$$T_{b1}(t) = 3.816\, A_1(t)$$

$$T_{b2}(t) = 0$$

$$T_{b3} = -3.816\, A_3(t)$$

Combining the modal responses gives the total response:

$$V_{bx}(t) = 0.1649\, A_2(t)$$

$$V_{by}(t) = 0.1588\, A_1(t) + 0.0061\, A_3(t)$$

$$T_b(t) = 3.816\, A_1(t) - 3.816\, A_3(t)$$

Problem 13.26

Part a

Ground motion in the y-direction excites the first and third natural vibration modes of the system. The corresponding SDF systems have the following properties:

$$T_1 = 1.054 \text{ sec} \qquad T_3 = 0.5764 \text{ sec}$$

$$\zeta_1 = 0.05 \qquad \zeta_3 = 0.05$$

The displacement responses $D_n(t)$ of the two SDF systems are shown in Fig. P13.26a. The pseudo-acceleration responses are $A_n(t) = \omega_n^2 D_n(t)$ and these are shown in Fig. P13.26b.

Parts b and c

From Problem 13.24,

$$u_y(t) = 0.9629 D_1(t) + 0.0371 D_3(t) \qquad \text{(a)}$$

$$(b/2)\, u_\theta(t) = (b/2)\left[0.00156 D_1(t) - 0.00156 D_3(t)\right] \qquad \text{(b)}$$

$$V_{by}(t) = 0.2245 A_1(t) + 0.0086 A_3(t) \qquad \text{(c)}$$

$$T_b(t) = 5.3966 A_1(t) - 5.3966 A_3(t) \qquad \text{(d)}$$

In Eq. (a), the D_n values at each time instant are substituted from the data of Fig. P13.26a to obtain the modal contributions $u_{yn}(t)$ and the total y-displacement $u_y(t)$. The results are shown in Fig. P13.26c. Similar results for $(b/2)\, u_{\theta n}(t)$ and $(b/2)\, u_\theta(t)$ are presented in Fig. P13.26d.

In Eq. (c), the A_n values at each time instant are substituted from the data of Fig. P13.26b to obtain the modal contributions $V_{byn}(t)$ and the total base shear $V_{by}(t)$. The results are shown in Fig. P13.26e. Similar results for $T_{bn}(t)$ and the total base torque $T_b(t)$ are presented in Fig. P13.26f.

For various response quantities, their peak values are

$$u_{yo} = 4.182 \text{ in.} \qquad (b/2)\, u_{\theta o} = 1.22 \text{ in.}$$

$$V_{byo} = 35.6 \text{ kips} \qquad T_{bo} = 1899 \text{ kip-in.}$$

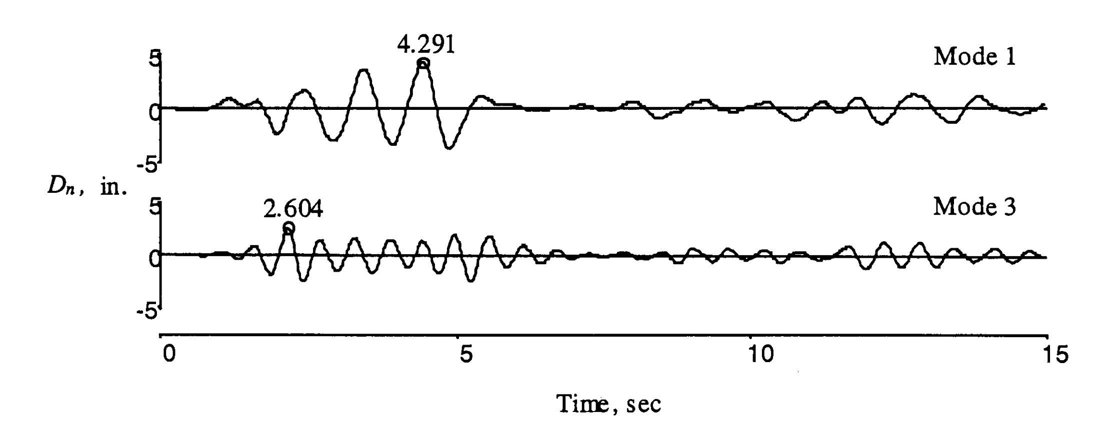


Fig. P13.26a

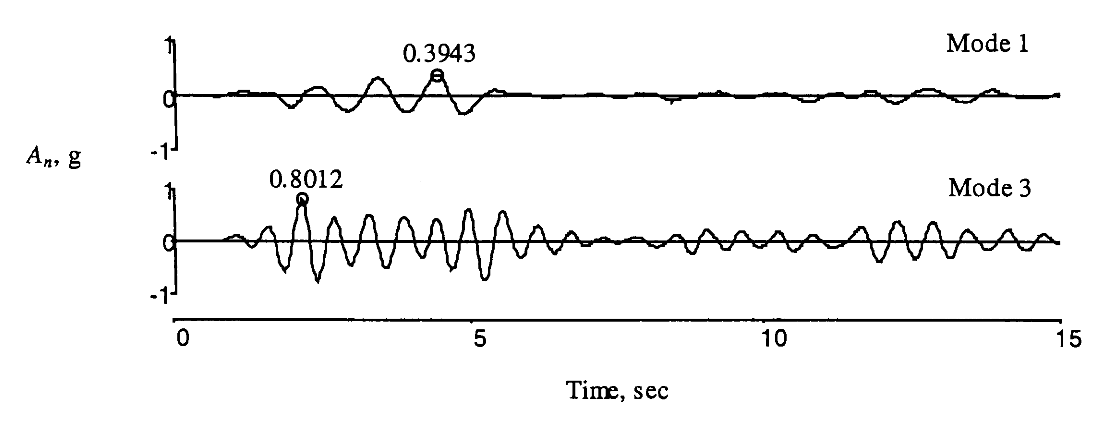


Fig. P13.26b

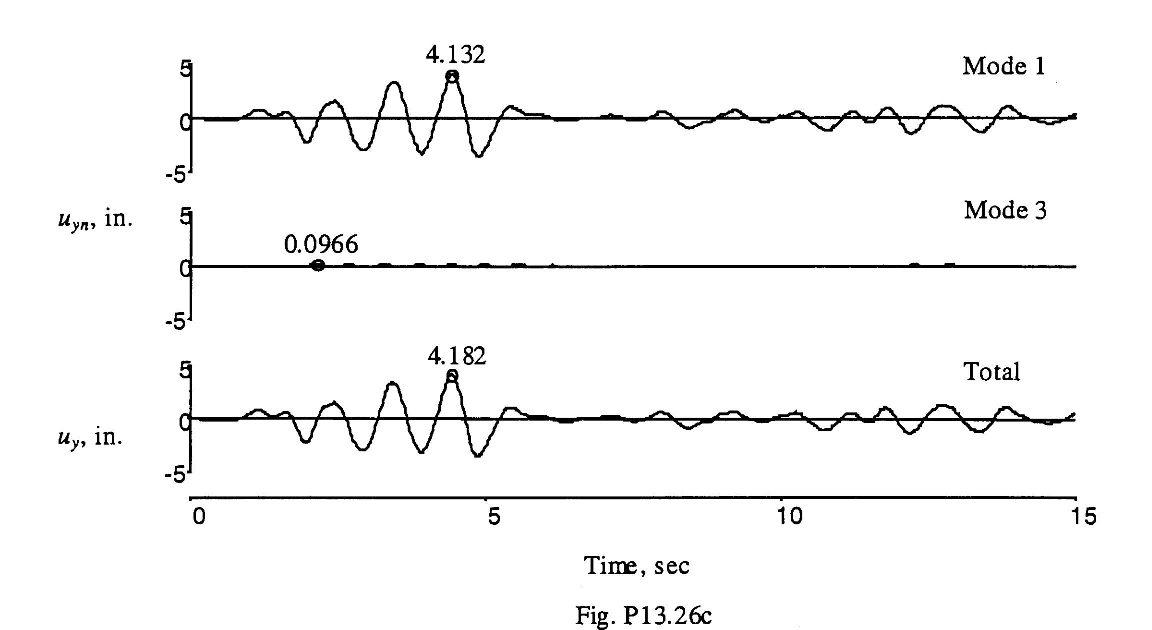


Fig. P13.26c

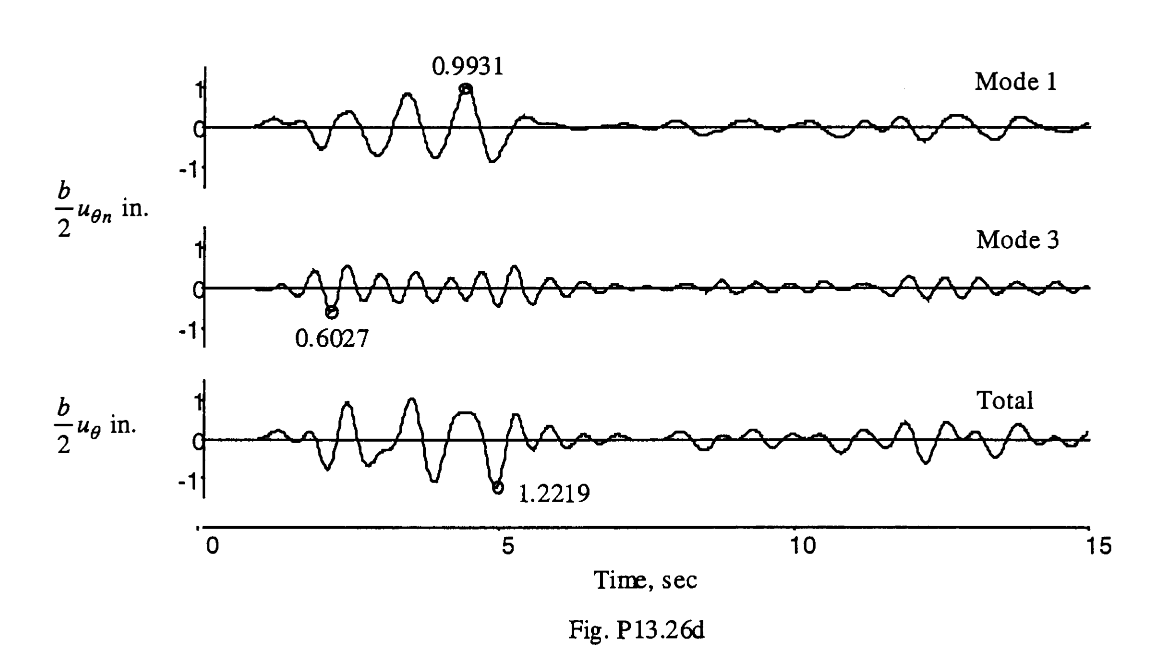


Fig. P13.26d

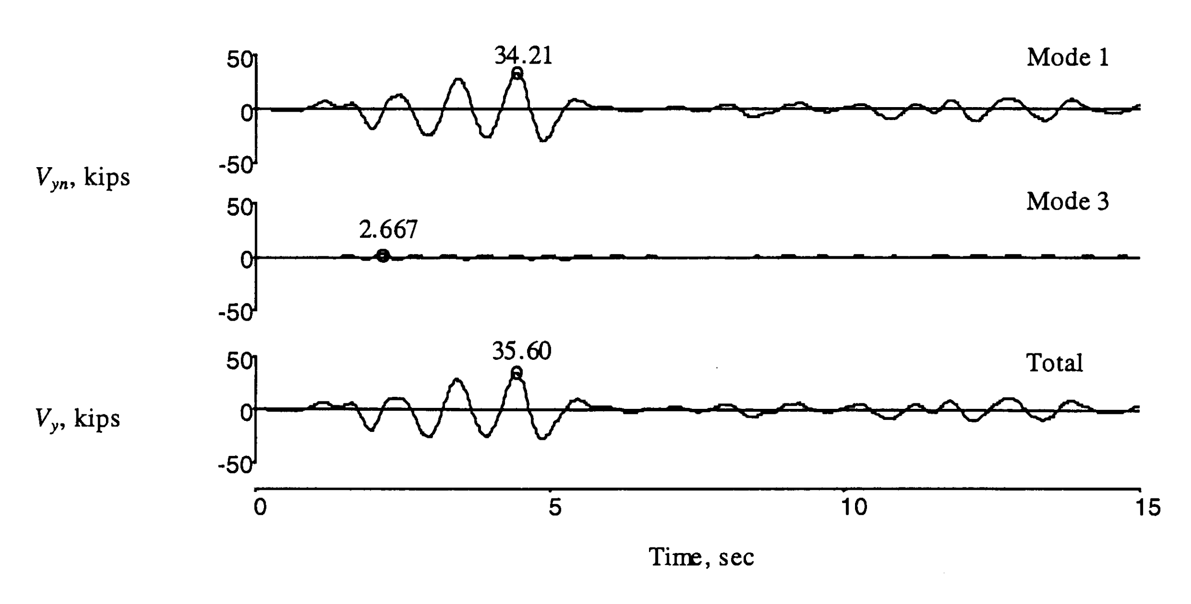


Fig. P13.26e

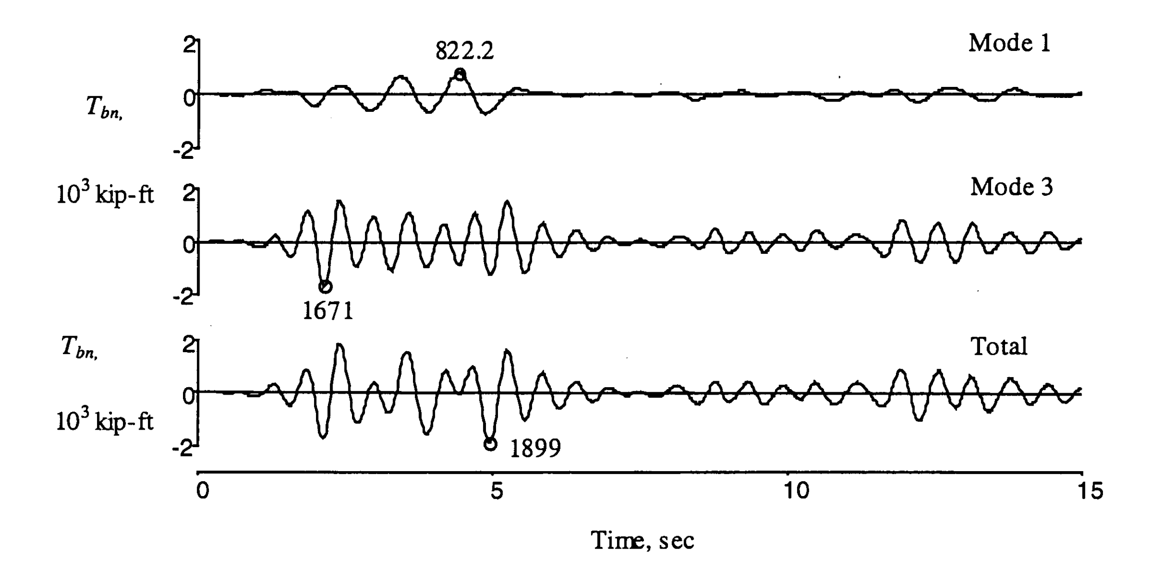
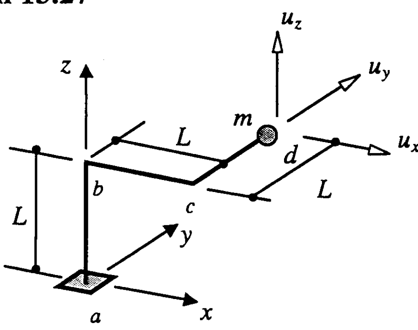


Fig. P13.26f

Problem 13.27



Part a

From Problem 9.18:

$$\mathbf{m} = \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} \qquad \boldsymbol{\iota} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

From Problem 10.28:

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix}$$

Substituting $\mathbf{m}$, $\boldsymbol{\iota}$ and ϕ_n in Eq. (13.1.5) gives:

$$L_1 = \phi_1^T \mathbf{m} \boldsymbol{\iota} = 0.7767m$$
$$M_1 = \phi_1^T \mathbf{m} \phi_1 = m$$
$$\Gamma_1 = \frac{L_1}{M_1} = 0.7767$$

$$L_2 = \phi_2^T \mathbf{m} \boldsymbol{\iota} = -0.2084m$$
$$M_2 = \phi_2^T \mathbf{m} \phi_2 = m$$
$$\Gamma_2 = \frac{L_2}{M_2} = -0.2084$$

$$L_3 = \phi_3^T \mathbf{m} \boldsymbol{\iota} = 0.5943m$$
$$M_3 = \phi_3^T \mathbf{m} \phi_3 = m$$
$$\Gamma_3 = \frac{L_3}{M_3} = 0.5943$$

Substituting , $\mathbf{m}$ and ϕ_n in Eq. (13.1.6) gives the effective earthquake forces:

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m} \phi_1 = m \begin{Bmatrix} 0.6034 \\ -0.3824 \\ -0.3051 \end{Bmatrix}$$

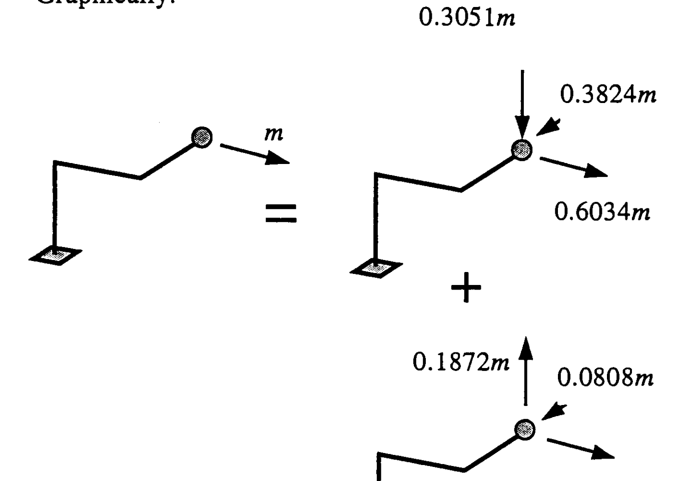
$$\mathbf{s}_2 = \Gamma_2 \mathbf{m} \phi_2 = m \begin{Bmatrix} 0.0434 \\ -0.0808 \\ 0.1872 \end{Bmatrix}$$

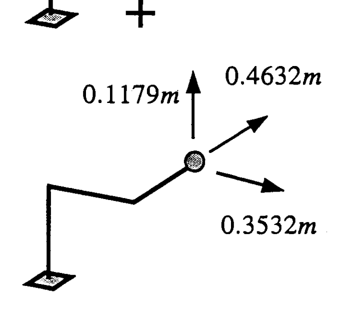
$$\mathbf{s}_3 = \Gamma_3 \mathbf{m} \phi_3 = m \begin{Bmatrix} 0.3532 \\ 0.4632 \\ 0.1179 \end{Bmatrix}$$

Thus, Eq. (13.1.4) specializes to:

$$m \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = m \begin{Bmatrix} 0.6034 \\ -0.3824 \\ -0.3051 \end{Bmatrix} + m \begin{Bmatrix} 0.0434 \\ -0.0808 \\ 0.1872 \end{Bmatrix} + m \begin{Bmatrix} 0.3532 \\ 0.4632 \\ 0.1179 \end{Bmatrix}$$

Graphically:





Part b

Substituting for Γ_n and ϕ_n in Eq. (13.1.10) gives the modal displacements:

$$\mathbf{u}_1(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_1 = \Gamma_1 \phi_1 D_1(t) = \begin{Bmatrix} 0.6034 \\ -0.3824 \\ -0.3051 \end{Bmatrix} D_1(t) \qquad \text{(a)}$$

$$\mathbf{u}_2(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_2 = \Gamma_2\phi_2 D_2(t) = \begin{Bmatrix} 0.0434 \\ -0.0808 \\ 0.1872 \end{Bmatrix} D_2(t) \qquad \text{(b)}$$

$$\mathbf{u}_3(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_3 = \Gamma_3\phi_3 D_3(t) = \begin{Bmatrix} 0.3532 \\ 0.4632 \\ 0.1179 \end{Bmatrix} D_3(t) \qquad \text{(c)}$$

Combining modal responses in Eqs. (a), (b) and (c) gives the total displacement response:

$$\mathbf{u}(t) = \begin{Bmatrix} 0.6034 \\ -0.3824 \\ -0.3051 \end{Bmatrix} D_1(t) + \begin{Bmatrix} 0.0434 \\ -0.0808 \\ 0.1872 \end{Bmatrix} D_2(t) + \begin{Bmatrix} 0.3532 \\ 0.4632 \\ 0.1179 \end{Bmatrix} D_3(t) \qquad \text{(d)}$$

Part c

From Eq. (13.1.13), the earthquake-induced bending moments and torques at the base due to the n^th^ mode are:

$$M_{xn}(t) = M_{xn}^{st} A_n(t)$$

$$M_{yn}(t) = M_{yn}^{st} A_n(t)$$

$$T_n(t) = T_n^{st} A_n(t)$$

where M_{xn}^{st}, M_{yn}^{st} and T_n^{st} are expressed in terms of $\mathbf{s}_n$ as the scalar products:

$$M_{xn}^{st} = \begin{bmatrix} 0 & L & -L \end{bmatrix} \mathbf{s}_n$$

$$M_{yn}^{st} = \begin{bmatrix} -L & 0 & L \end{bmatrix} \mathbf{s}_n$$

$$T_n^{st} = \begin{bmatrix} L & -L & 0 \end{bmatrix} \mathbf{s}_n$$

and $\mathbf{s}_n$ are given in Part a:

$$M_{xn}^{st} = \begin{bmatrix} -0.0773mL, & -0.2679mL, & 0.3453mL \end{bmatrix}$$

$$M_{yn}^{st} = \begin{bmatrix} -0.9084mL, & 0.1437mL, & -0.2353mL \end{bmatrix}$$

$$T_n^{st} = \begin{bmatrix} 0.9858mL, & 0.1242mL, & -0.1100mL \end{bmatrix}$$

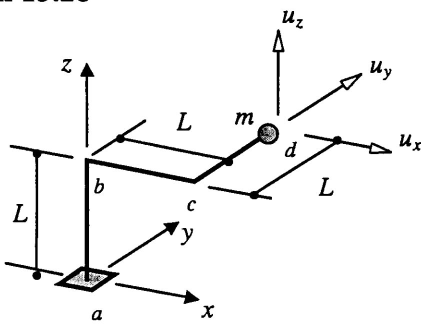
Combining these modal contributions for each response gives the total response in terms of $A_n(t)$:

$$M_x(t) = \sum_{n=1}^{3} M_{xn}(t) = \sum_{n=1}^{3} M_{xn}^{st} A_n(t)$$
$$= -0.0773mLA_1(t) - 0.2679mLA_2(t) + 0.3453mLA_3(t)$$

$$M_y(t) = \sum_{n=1}^{3} M_{yn}(t) = \sum_{n=1}^{3} M_{yn}^{st} A_n(t)$$
$$= -0.9084mLA_1(t) + 0.1437mLA_2(t) - 0.2353mLA_3(t)$$

$$T(t) = \sum_{n=1}^{3} T_n(t) = \sum_{n=1}^{3} T_n^{st} A_n(t)$$
$$= 0.9858mLA_1(t) + 0.1242mLA_2(t) - 0.1100mLA_3(t)$$

Problem 13.28



Part a

From Problem 9.18:

$$\mathbf{m} = \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} \qquad \boldsymbol{\iota} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

From Problem 10.28:

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \boldsymbol{\phi}_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \boldsymbol{\phi}_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix}$$

Substituting $\mathbf{m}$, $\boldsymbol{\iota}$ and $\boldsymbol{\phi}_n$ in Eq. (13.1.5) gives:

$$L_1 = \boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\iota} = -0.4923m$$
$$M_1 = \boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\phi}_1 = m$$
$$\Gamma_1 = \frac{L_1}{M_1} = -0.4923$$

$$L_2 = \boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\iota} = 0.3875\,m$$
$$M_2 = \boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\phi}_2 = m$$
$$\Gamma_2 = \frac{L_2}{M_2} = 0.3875$$

$$L_3 = \boldsymbol{\phi}_3^T \mathbf{m} \boldsymbol{\iota} = 0.7794\,m$$
$$M_3 = \boldsymbol{\phi}_3^T \mathbf{m} \boldsymbol{\phi}_3 = m$$
$$\Gamma_3 = \frac{L_3}{M_3} = 0.7794$$

Substituting Γ_n, $\mathbf{m}$ and $\boldsymbol{\phi}_n$ in Eq. (13.1.6) gives the effective earthquake forces:

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m} \boldsymbol{\phi}_1 = m \begin{Bmatrix} -0.3824 \\ 0.2424 \\ 0.1934 \end{Bmatrix}$$

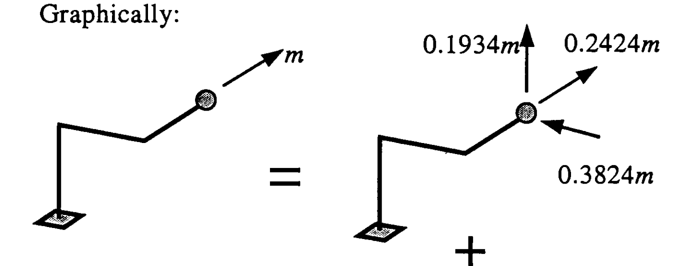
$$\mathbf{s}_2 = \Gamma_2 \mathbf{m} \boldsymbol{\phi}_2 = m \begin{Bmatrix} -0.0808 \\ 0.1502 \\ -0.3480 \end{Bmatrix}$$

$$\mathbf{s}_3 = \Gamma_3 \mathbf{m} \boldsymbol{\phi}_3 = m \begin{Bmatrix} 0.4632 \\ 0.6074 \\ 0.1546 \end{Bmatrix}$$

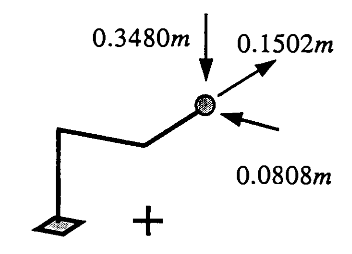
Thus, Eq. (13.1.4) specializes to:

$$m \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = m \begin{Bmatrix} -0.3824 \\ 0.2424 \\ 0.1934 \end{Bmatrix} + m \begin{Bmatrix} -0.0808 \\ 0.1502 \\ -0.3480 \end{Bmatrix} + m \begin{Bmatrix} 0.4632 \\ 0.6074 \\ 0.1546 \end{Bmatrix}$$

Graphically:



+



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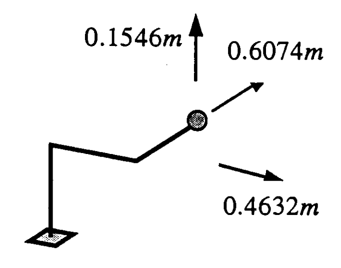


Part b

Substituting for Γ_n and $\boldsymbol{\phi}_n$ in Eq. (13.1.10) gives the modal displacements:

$$\mathbf{u}_1(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_1 = \Gamma_1 \boldsymbol{\phi}_1 D_1(t) = \begin{Bmatrix} -0.3824 \\ 0.2424 \\ 0.1934 \end{Bmatrix} D_1(t) \quad \text{(a)}$$

$$\mathbf{u}_2(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_2 = \Gamma_2 \phi_2 D_2(t) = \begin{Bmatrix} -0.0808 \\ 0.1502 \\ -0.3480 \end{Bmatrix} D_2(t) \qquad \text{(b)}$$

$$\mathbf{u}_3(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_3 = \Gamma_3 \phi_3 D_3(t) = \begin{Bmatrix} 0.4632 \\ 0.6074 \\ 0.1546 \end{Bmatrix} D_3(t) \qquad \text{(c)}$$

Combining modal responses in Eqs. (a), (b) and (c) gives the total displacement response:

$$\mathbf{u}(t) = \begin{Bmatrix} -0.3824 \\ 0.2424 \\ 0.1934 \end{Bmatrix} D_1(t) + \begin{Bmatrix} -0.0808 \\ 0.1502 \\ -0.3480 \end{Bmatrix} D_2(t) + \begin{Bmatrix} 0.4632 \\ 0.6074 \\ 0.1546 \end{Bmatrix} D_3(t) \qquad \text{(d)}$$

Part c

From Eq. (13.1.13), the earthquake-induced bending moments and torques at the base due to the n[th] mode are:

$$M_{xn}(t) = M_{xn}^{st} A_n(t)$$

$$M_{yn}(t) = M_{yn}^{st} A_n(t)$$

$$T_n(t) = T_n^{st} A_n(t)$$

where M_{xn}^{st}, M_{yn}^{st} and T_n^{st} are expressed in terms of $\mathbf{s}_n$ as the scalar products:

$$M_{xn}^{st} = \begin{bmatrix} 0 & L & -L \end{bmatrix} \mathbf{s}_n$$

$$M_{yn}^{st} = \begin{bmatrix} -L & 0 & L \end{bmatrix} \mathbf{s}_n$$

$$T_n^{st} = \begin{bmatrix} L & -L & 0 \end{bmatrix} \mathbf{s}_n$$

and $\mathbf{s}_n$ are given in Part a:

$$M_{xn}^{st} = \begin{bmatrix} 0.0490mL, & 0.4982mL, & 0.4528mL \end{bmatrix}$$

$$M_{yn}^{st} = \begin{bmatrix} 0.5758mL, & -0.2672mL, & -0.3086mL \end{bmatrix}$$

$$T_n^{st} = \begin{bmatrix} -0.6248mL, & -0.2310mL, & -0.1442mL \end{bmatrix}$$

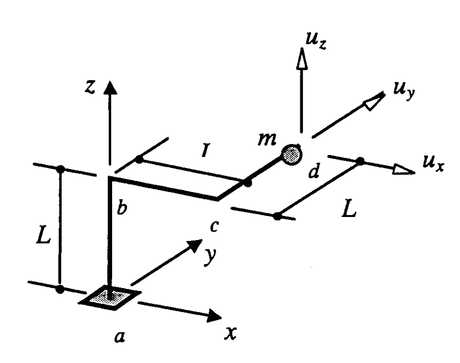
Combining these modal contributions for each response gives the total response in terms of $A_n(t)$:

$$M_x(t) = \sum_{n=1}^{3} M_{xn}(t) = \sum_{n=1}^{3} M_{xn}^{st} A_n(t)$$
$$= 0.0490mLA_1(t) + 0.4982mLA_2(t) + 0.4528mLA_3(t)$$

$$M_y(t) = \sum_{n=1}^{3} M_{yn}(t) = \sum_{n=1}^{3} M_{yn}^{st} A_n(t)$$
$$= 0.5758mLA_1(t) - 0.2672mLA_2(t) - 0.3086mLA_3(t)$$

$$T(t) = \sum_{n=1}^{3} T_n(t) = \sum_{n=1}^{3} T_n^{st} A_n(t)$$
$$= -0.6248mLA_1(t) - 0.2310mLA_2(t) - 0.1442mLA_3(t)$$

Problem 13.29



Part a

From Problem 9.18:

$$\mathbf{m} = \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} \qquad \iota = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

From Problem 10.28:

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix}$$

Substituting $\mathbf{m}$, ι and ϕ_n in Eq. (13.1.5) gives:

$$L_1 = \phi_1^T \mathbf{m} \iota = -0.3928m$$
$$M_1 = \phi_1^T \mathbf{m} \phi_1 = m$$
$$\Gamma_1 = \frac{L_1}{M_1} = -0.3928$$

$$L_2 = \phi_2^T \mathbf{m} \iota = -0.8980m$$
$$M_2 = \phi_2^T \mathbf{m} \phi_2 = m$$
$$\Gamma_2 = \frac{L_2}{M_2} = -0.8980$$

$$L_3 = \phi_3^T \mathbf{m} \iota = 0.1984m$$
$$M_3 = \phi_3^T \mathbf{m} \phi_3 = m$$
$$\Gamma_3 = \frac{L_3}{M_3} = 0.1984$$

Substituting Γ_n, $\mathbf{m}$ and ϕ_n in Eq. (13.1.6) gives the effective earthquake forces:

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m} \phi_1 = m \begin{Bmatrix} -0.3051 \\ 0.1934 \\ 0.1542 \end{Bmatrix}$$

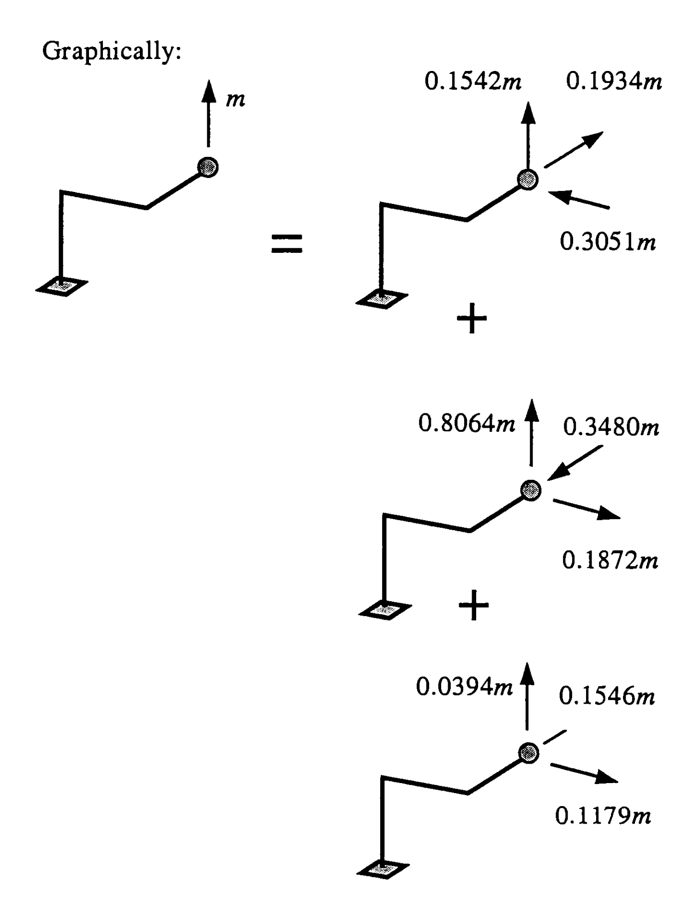
$$\mathbf{s}_2 = \Gamma_2 \mathbf{m} \phi_2 = m \begin{Bmatrix} 0.1872 \\ -0.3480 \\ 0.8064 \end{Bmatrix}$$

$$\mathbf{s}_3 = \Gamma_3 \mathbf{m} \phi_3 = m \begin{Bmatrix} 0.1179 \\ 0.1546 \\ 0.0394 \end{Bmatrix}$$

Thus, Eq. (13.1.4) specializes to:

$$m \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = m \begin{Bmatrix} -0.3051 \\ 0.1934 \\ 0.1542 \end{Bmatrix} + m \begin{Bmatrix} 0.1872 \\ -0.3480 \\ 0.8064 \end{Bmatrix} + m \begin{Bmatrix} 0.1179 \\ 0.1546 \\ 0.0394 \end{Bmatrix}$$

Graphically:



Part b

Substituting for Γ_n and ϕ_n in Eq. (13.1.10) gives the modal displacements:

$$\mathbf{u}_1(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_1 = \Gamma_1 \phi_1 D_1(t) = \begin{Bmatrix} -0.3051 \\ 0.1934 \\ 0.1542 \end{Bmatrix} D_1(t) \qquad \text{(a)}$$

$$\mathbf{u}_2(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_2 = \Gamma_2 \phi_2 D_2(t) = \begin{Bmatrix} 0.1872 \\ -0.3480 \\ 0.8064 \end{Bmatrix} D_2(t) \qquad \text{(b)}$$

$$\mathbf{u}_3(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_3 = \Gamma_3 \phi_3 D_3(t) = \begin{Bmatrix} 0.1179 \\ 0.1546 \\ 0.0394 \end{Bmatrix} D_3(t) \qquad \text{(c)}$$

Combining modal responses in Eqs. (a), (b) and (c) gives the total displacement response:

$$\mathbf{u}(t) = \begin{Bmatrix} -0.3051 \\ 0.1934 \\ 0.1542 \end{Bmatrix} D_1(t) + \begin{Bmatrix} 0.1872 \\ -0.3480 \\ 0.8064 \end{Bmatrix} D_2(t) + \begin{Bmatrix} 0.1179 \\ 0.1546 \\ 0.0394 \end{Bmatrix} D_3(t) \qquad \text{(d)}$$

Part c

From Eq. (13.1.13), the earthquake-induced bending moments and torques at the base due to the n[th] mode are:

$$M_{xn}(t) = M_{xn}^{st} A_n(t)$$

$$M_{yn}(t) = M_{yn}^{st} A_n(t)$$

$$T_n(t) = T_n^{st} A_n(t)$$

where M_{xn}^{st}, M_{yn}^{st} and T_n^{st} are expressed in terms of $\mathbf{s}_n$ as the scalar products:

$$M_{xn}^{st} = \begin{bmatrix} 0 & L & -L \end{bmatrix} \mathbf{s}_n$$

$$M_{yn}^{st} = \begin{bmatrix} -L & 0 & L \end{bmatrix} \mathbf{s}_n$$

$$T_n^{st} = \begin{bmatrix} L & -L & 0 \end{bmatrix} \mathbf{s}_n$$

and $\mathbf{s}_n$ are given in Part a:

$$M_{xn}^{st} = \begin{bmatrix} 0.0391mL, & -1.1544mL, & 0.1153mL \end{bmatrix}$$

$$M_{yn}^{st} = \begin{bmatrix} 0.4594mL, & 0.6192mL, & -0.0786mL \end{bmatrix}$$

$$T_n^{st} = \begin{bmatrix} -0.4985mL, & 0.5352mL, & -0.0367mL \end{bmatrix}$$

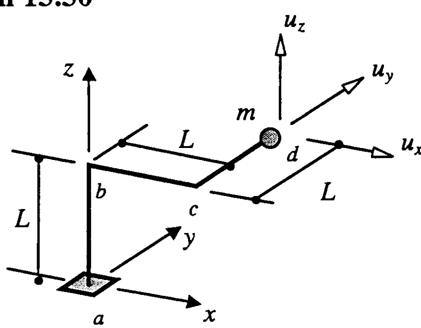
Combining these modal contributions for each response gives the total response in terms of $A_n(t)$:

$$M_x(t) = \sum_{n=1}^{3} M_{xn}(t) = \sum_{n=1}^{3} M_{xn}^{st} A_n(t)$$
$$= 0.0391mLA_1(t) - 1.1544mLA_2(t) + 0.1153mLA_3(t)$$

$$M_y(t) = \sum_{n=1}^{3} M_{yn}(t) = \sum_{n=1}^{3} M_{yn}^{st} A_n(t)$$
$$= 0.4594mLA_1(t) + 0.6192mLA_2(t) - 0.0786mLA_3(t)$$

$$T(t) = \sum_{n=1}^{3} T_n(t) = \sum_{n=1}^{3} T_n^{st} A_n(t)$$
$$= -0.4985mLA_1(t) + 0.5352mLA_2(t) - 0.0367mLA_3(t)$$

Problem 13.30



Part a

From Problem 9.18:

$$\mathbf{m} = \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix} \qquad \iota = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$$

From Problem 10.28:

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix}$$

Substituting $\mathbf{m}$, ι and ϕ_n in Eq. (13.1.5) gives:

$$L_1 = \phi_1^T \mathbf{m} \iota = -0.0626m$$
$$M_1 = \phi_1^T \mathbf{m} \phi_1 = m$$
$$\Gamma_1 = \frac{L_1}{M_1} = -0.0626$$

$$L_2 = \phi_2^T \mathbf{m} \iota = -0.4150m$$
$$M_2 = \phi_2^T \mathbf{m} \phi_2 = m$$
$$\Gamma_2 = \frac{L_2}{M_2} = -0.4150$$

$$L_3 = \phi_3^T \mathbf{m} \iota = 0.9077m$$
$$M_3 = \phi_3^T \mathbf{m} \phi_3 = m$$
$$\Gamma_3 = \frac{L_3}{M_3} = 0.9077$$

Substituting Γ_n, $\mathbf{m}$ and ϕ_n in Eq. (13.1.6) gives the effective earthquake forces:

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m} \phi_1 = m \begin{Bmatrix} -0.0485 \\ 0.0308 \\ 0.0246 \end{Bmatrix}$$

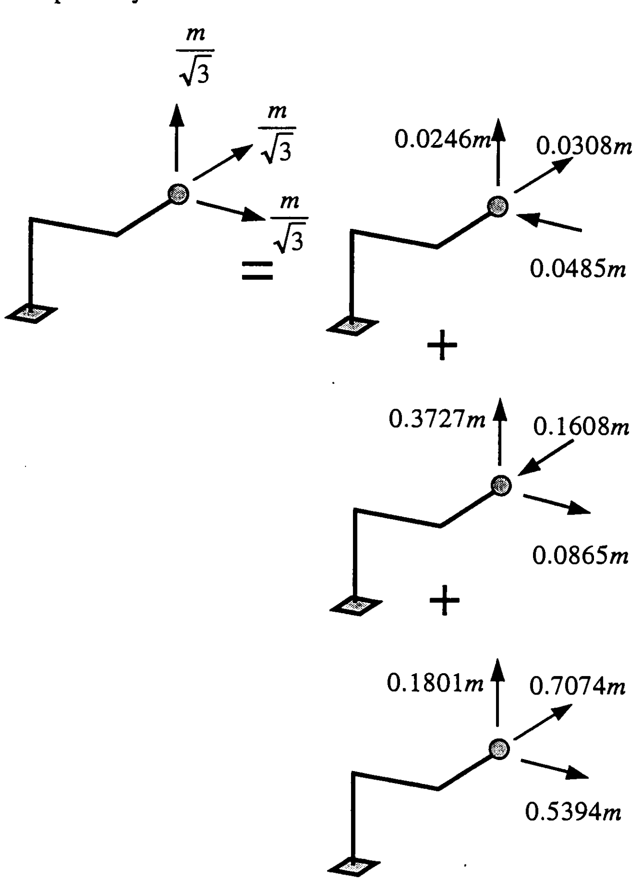
$$\mathbf{s}_2 = \Gamma_2 \mathbf{m} \phi_2 = m \begin{Bmatrix} 0.0865 \\ -0.1608 \\ 0.3727 \end{Bmatrix}$$

$$\mathbf{s}_3 = \Gamma_3 \mathbf{m} \phi_3 = m \begin{Bmatrix} 0.5394 \\ 0.7074 \\ 0.1801 \end{Bmatrix}$$

Thus, Eq. (13.1.4) specializes to:

$$\frac{m}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = m \begin{Bmatrix} -0.0485 \\ 0.0308 \\ 0.0246 \end{Bmatrix} + m \begin{Bmatrix} 0.0865 \\ -0.1608 \\ 0.3727 \end{Bmatrix} + m \begin{Bmatrix} 0.5394 \\ 0.7074 \\ 0.1801 \end{Bmatrix}$$

Graphically:



Part b

Substituting for Γ_n and ϕ_n in Eq. (13.1.10) gives the modal displacements:

$$\mathbf{u}_1(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_1 = \Gamma_1\phi_1 D_1(t) = \begin{Bmatrix} -0.0485 \\ 0.0308 \\ 0.0246 \end{Bmatrix} D_1(t) \quad \text{(a)}$$

$$\mathbf{u}_2(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_2 = \Gamma_2\phi_2 D_2(t) = \begin{Bmatrix} 0.0865 \\ -0.1608 \\ 0.3727 \end{Bmatrix} D_2(t) \quad \text{(b)}$$

$$\mathbf{u}_3(t) = \begin{Bmatrix} u_x(t) \\ u_y(t) \\ u_z(t) \end{Bmatrix}_3 = \Gamma_3\phi_3 D_3(t) = \begin{Bmatrix} 0.5394 \\ 0.7074 \\ 0.1801 \end{Bmatrix} D_3(t) \quad \text{(c)}$$

Combining modal responses in Eqs. (a), (b), and (c) gives the total displacement response:

$$\mathbf{u}(t) = \begin{Bmatrix} -0.0485 \\ 0.0308 \\ 0.0246 \end{Bmatrix} D_1(t) + \begin{Bmatrix} 0.0865 \\ -0.1608 \\ 0.3727 \end{Bmatrix} D_2(t) + \begin{Bmatrix} 0.5394 \\ 0.7074 \\ 0.1801 \end{Bmatrix} D_3(t) \quad \text{(d)}$$

Part c

From Eq. (13.1.13), the earthquake-induced bending moments and torques at the base due to the nth mode are:

$$M_{xn}(t) = M_{xn}^{st} A_n(t)$$

$$M_{yn}(t) = M_{yn}^{st} A_n(t)$$

$$T_n(t) = T_n^{st} A_n(t)$$

where M_{xn}^{st}, M_{yn}^{st} and T_n^{st} are expressed in terms of $\mathbf{s}_n$ as the scalar products:

$$M_{xn}^{st} = \begin{bmatrix} 0 & L & -L \end{bmatrix} \mathbf{s}_n$$

$$M_{yn}^{st} = \begin{bmatrix} -L & 0 & L \end{bmatrix} \mathbf{s}_n$$

$$T_n^{st} = \begin{bmatrix} L & -L & 0 \end{bmatrix} \mathbf{s}_n$$

and $\mathbf{s}_n$ are given in Part a:

$$M_{xn}^{st} = \begin{bmatrix} 0.0062mL, & -0.5335mL, & 0.5273mL \end{bmatrix}$$
$$M_{yn}^{st} = \begin{bmatrix} 0.0732mL, & 0.0262mL, & -0.03594mL \end{bmatrix}$$
$$T_n^{st} = \begin{bmatrix} -0.0794mL, & 0.2474mL, & -0.1680mL \end{bmatrix}$$

Combining these modal contributions for each response gives the total response in terms of $A_n(t)$:

$$M_x(t) = \sum_{n=1}^{3} M_{xn}(t) = \sum_{n=1}^{3} M_{xn}^{st} A_n(t)$$
$$= 0.0062mLA_1(t) - 0.5335mLA_2(t) + 0.5273mLA_3(t)$$

$$M_y(t) = \sum_{n=1}^{3} M_{yn}(t) = \sum_{n=1}^{3} M_{yn}^{st} A_n(t)$$
$$= 0.0732mLA_1(t) + 0.2862mLA_2(t) - 0.3594mLA_3(t)$$

$$T(t) = \sum_{n=1}^{3} T_n(t) = \sum_{n=1}^{3} T_n^{st} A_n(t)$$
$$= -0.0794mLA_1(t) + 0.2474mLA_2(t) - 0.1680mLA_3(t)$$

Problem 13.31

1. *Equations of motion* (from Problem 9.19).

$$m\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{Bmatrix}\ddot{u}_1\\ \ddot{u}_2\end{Bmatrix} + k\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = -m\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{bmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{bmatrix}\begin{Bmatrix}\ddot{u}_{g1}\\ \ddot{u}_{g2}\end{Bmatrix}$$

2. *Natural frequencies and modes.*

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{3k}{m}}$$

$$\phi_1 = \begin{Bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{Bmatrix}$$

3. *Determine* $\Gamma_{nl} = L_{nl}/M_n$ *from Eq. (13.5.3).*

$$\Gamma = \left[\Gamma_{nl}\right] = \begin{bmatrix}0.7071 & 0.7071\\ -0.2357 & 0.2357\end{bmatrix}\begin{matrix}\leftarrow \text{mode 1}\\ \leftarrow \text{mode 2}\end{matrix}$$

$$\begin{matrix}\uparrow & \uparrow\\ \ddot{u}_{g1} & \ddot{u}_{g2}\end{matrix}$$

4. *Dynamic displacements.*

$$\mathbf{u}(t) = \sum_{l=1}^{2}\sum_{n=1}^{2}\Gamma_{nl}\,\phi_n\,D_{nl}(t) \qquad \text{(a)}$$

Substituting for Γ_{nl} and ϕ_n gives

$$\begin{aligned}\mathbf{u}(t) &= 0.7071\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}D_{11}(t) + (-0.2357)\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}D_{21}(t)\\ &+ 0.7071\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}D_{12}(t) + 0.2357\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}D_{22}(t)\\ &= \begin{bmatrix}1/2\\ 1/2\end{bmatrix}D_{11}(t) + \begin{bmatrix}1/6\\ -1/6\end{bmatrix}D_{21}(t) + \begin{bmatrix}1/2\\ 1/2\end{bmatrix}D_{12}(t)\\ &+ \begin{bmatrix}-1/6\\ 1/6\end{bmatrix}D_{22}(t)\end{aligned} \qquad \text{(b)}$$

5. *Quasistatic displacements.*

$$\mathbf{u}^s(t) = \sum_{l=1}^{2}\iota_l\,u_{gl}(t) = \begin{bmatrix}2/3\\ 1/3\end{bmatrix}u_{g1}(t) + \begin{bmatrix}1/3\\ 2/3\end{bmatrix}u_{g2}(t) \qquad \text{(c)}$$

6. *Total displacements.*

$$\mathbf{u}^t(t) = \mathbf{u}^s(t) + \mathbf{u}(t) \qquad \text{(d)}$$

where $\mathbf{u}(t)$ and $\mathbf{u}^s(t)$ are given by Eqs. (b) and (c), respectively.

7. *Excitation a.*

$$u_{g1}(t) = -u_{g2}(t) = u_g(t)$$

For these excitations,

$$D_{n1}(t) = D_n(t) \quad D_{n2}(t) = -D_n(t) \qquad \text{(e)}$$

Substituting Eq. (e) in Eqs. (b), (c) and (d) gives

$$\mathbf{u}(t) = \begin{bmatrix}1/3\\ -1/3\end{bmatrix}D_2(t)$$

$$\mathbf{u}^s(t) = \begin{bmatrix}1/3\\ -1/3\end{bmatrix}u_g(t)$$

$$\mathbf{u}^t(t) = \begin{bmatrix}u_g(t)/3 + D_2(t)/3\\ -u_g(t)/3 - D_2(t)/3\end{bmatrix}$$

Note that for this symmetric excitation only the second mode is excited.

8. *Excitation b.*

$$u_{g1}(t) = u_{g2}(t) = u_g(t)$$

For these excitations,

$$D_{n1}(t) = D_n(t) \quad D_{n2}(t) = D_n(t) \qquad \text{(f)}$$

Substituting Eq. (f) in Eqs. (b), (c) and (d) gives

$$\mathbf{u}(t) = \begin{bmatrix}1\\ 1\end{bmatrix}D_1(t)$$

$$\mathbf{u}^s(t) = \begin{bmatrix}1\\ 1\end{bmatrix}u_g(t)$$

$$\mathbf{u}^t(t) = \begin{bmatrix}u_g(t) + D_1(t)\\ u_g(t) + D_1(t)\end{bmatrix}$$

Note that (1) for this antisymmetric excitation only the first mode is excited;. and (2) because both supports undergo identical motion, the response analysis reduces to the standard case of a single support excitation.

Problem 13.32

From Problem 9.20, the equation of motion is:

$$m\ddot{u} + ku = -m\langle 1/2 \quad 1/2\rangle \begin{Bmatrix} \ddot{u}_{g1}(t) \\ \ddot{u}_{g2}(t) \end{Bmatrix} \qquad \text{(a)}$$

where

$$k = \frac{6EI}{L^3}$$

The natural vibration frequency of the beam is

$$\omega_n = \sqrt{\frac{6EL}{mL^3}} = \sqrt{\frac{6(5 \times 10^8)}{(0.2)(50 \times 12)^3}} = 8.333 \text{ rad/sec}$$

Excitation (i)

$$u_{g1}(t) = u_{go} \sin(0.8\,\omega_n t) \qquad u_{g2}(t) = 0$$

Substituting in Eq. (a) gives the equations of motion:

$$m\ddot{u} + ku = \left(\frac{m}{2}\right)\omega^2 u_{go} \sin \omega t \qquad \text{(b)}$$

where $\omega = 0.8\,\omega_n = 6.667$.

The steady state solution of Eq. (b) is

$$u(t) = \frac{(m/2)\,\omega^2 u_{go}}{k} \frac{1}{1 - (\omega/\omega_n)^2} \sin \omega t$$

$$= \frac{u_{go}}{2} \frac{1}{(\omega_n/\omega)^2 - 1} \sin \omega t$$

$$= \frac{u_{go}}{2} \frac{1}{(1.25)^2 - 1} \sin \omega t$$

$$= 0.8889\, u_{go} \sin 6.667t \qquad \text{(c)}$$

From Problem 9.20, the quasistatic displacement is

$$u^s(t) = \frac{u_{go}}{2} \sin 6.667t \qquad \text{(d)}$$

The total displacement is

$$u^t(t) = u^s(t) + u(t) = 1.3889\, u_{go} \sin 6.667t \quad \text{(e)}$$

To compute the bending moment we determine the equivalent static forces f_s, f_{sg1} and f_{sg2}.

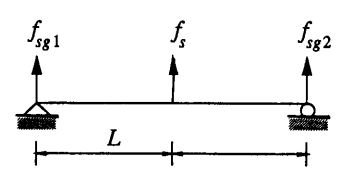


Fig. P13.32a

From Problem 9.20,

$$\mathbf{k} = \frac{6EI}{L^3} \qquad \text{(f.1)}$$

$$\mathbf{k}_g = \frac{6EI}{L^3}\left\langle -\frac{1}{2} \quad -\frac{1}{2} \right\rangle \qquad \text{(f.2)}$$

$$\mathbf{k}_{gg} = \frac{6EI}{L^3}\begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} \qquad \text{(f.3)}$$

From Eq. (13.5.9),

$$f_S(t) = ku(t) = \frac{6EI}{L^3} u(t) \qquad \text{(g)}$$

From Eq. (13.5.12),

$$\mathbf{f}_{S_g} = \mathbf{k}_g^T \mathbf{u}^t + \mathbf{k}_{gg}\mathbf{u}_g$$

$$= \frac{6EI}{L^3}\left\{ \begin{Bmatrix} -1/2 \\ -1/2 \end{Bmatrix}(u^s + u) + \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}\begin{Bmatrix} u_{g1} \\ u_{g2} \end{Bmatrix} \right\}$$

where

$$u^s = \langle 1/2 \quad 1/2 \rangle \begin{Bmatrix} u_{g1} \\ u_{g2} \end{Bmatrix}$$

Therefore,

$$\mathbf{f}_{S_g} = -\begin{Bmatrix} 3EI/L^3 \\ 3EI/L^3 \end{Bmatrix} u(t) \qquad \text{(h)}$$

Static analysis of the beam due to the forces in Fig. P13.32a gives the bending moment at mid-span:

$$M(t) = (f_{S_{g1}})\,L = -\frac{3EI}{L^2} u(t) \qquad \text{(i)}$$

Alternative derivation

Because the quasistatic displacements represent rigid-body motion, only the dynamic displacement u causes bending moment. For a simply supported beam,

$$f_S = \frac{6EI}{L^3} u$$

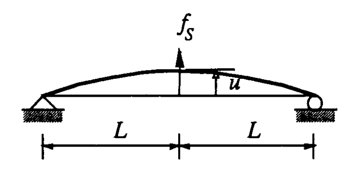


Fig. P13.32b

The bending moment at mid-span is

$$M(t) = -\frac{f_S(t)}{2} L = -\frac{3EI}{L^2} u(t) \qquad \text{(j)}$$

Substituting Eq. (c) in Eq. (i) or (j) gives

$$M(t) = -\frac{3(5 \times 10^8)}{(50 \times 12)^2} 0.8889\, u_{go} \sin 6.667t$$

$$M(t) = -3704.2 \sin 6.667t \qquad \text{(k)}$$

Excitation (ii)

$$u_{g1}(t) = u_{g2}(t) = u_{go} \sin(0.8\,\omega_n t)$$

Substituting in Eq. (a) gives

$$m\ddot{u} + ku = m\,\omega^2 u_{go} \sin \omega t \qquad \text{(l)}$$

where $\omega = 0.8\omega_n = 6.667$. The steady state solution of Eq. (l) is

$$u(t) = 1.7778\, u_{go} \sin 6.667t \qquad \text{(m)}$$

The quasistatic displacement is

$$u^s(t) = \frac{1}{2}\left[u_{g1}(t) + u_{g2}(t)\right] = u_{go} \sin 6.667t \qquad \text{(n)}$$

The total displacement is

$$u^t(t) = u^s(t) + u(t) = 2.7778\, u_{go} \sin 6.667t \qquad \text{(o)}$$

Substituting Eq. (m) in Eq. (i) or (j) gives

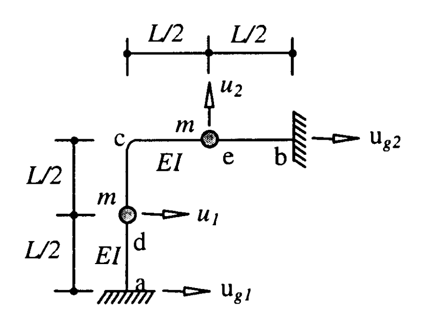
$$M(t) = -7408.3 \sin 6.667t \qquad \text{(p)}$$

Comments

(1) If the beam is subjected to motion at only one support, the dynamic response (displacement and bending moment) and quasistatic displacement at the mid-point is one-half of the corresponding values if both supports are excited.

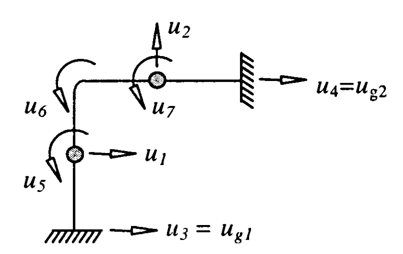
(2) Because the structure is statically determinate, the quasistatic displacements do not cause any bending moment.

Problem 13.33



Preliminaries

From Problem 9.21



$$\begin{array}{c} \\ \left[\begin{array}{c|c} \mathbf{k}_{tt} & \mathbf{k}_{t\theta} \\ \hline \mathbf{k}_{\theta t} & \mathbf{k}_{\theta\theta} \end{array}\right] = \dfrac{4EI}{L^3} \end{array} \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \end{array} \\ \left[\begin{array}{cccc|ccc} 48 & 0 & -24 & -24 & 0 & -6L & 0 \\ 0 & 48 & 0 & 0 & 0 & -6L & 0 \\ -24 & 0 & 24 & 0 & -6L & 0 & 0 \\ -24 & 0 & 0 & 24 & 6L & 6L & 0 \\ \hline 0 & 0 & -6L & 6L & 4L^2 & L^2 & 0 \\ -6L & -6L & 0 & 6L & L^2 & 4L^2 & L^2 \\ 0 & 0 & 0 & 0 & 0 & L^2 & 4L^2 \end{array}\right] \end{array}$$

Condense out the rotational DOF $[u_5, u_6, u_7]^T$

$$\left[\begin{array}{c|c} \mathbf{k} & \mathbf{k}_g \\ \hline \mathbf{k}_g^T & \mathbf{k}_{gg} \end{array}\right] = \mathbf{k}_{tt} - \mathbf{k}_{t\theta}\mathbf{k}_{\theta\theta}^{-1}\mathbf{k}_{\theta t}$$

$$= \frac{6EI}{7L^3}\left[\begin{array}{cc|cc} 176 & -48 & -100 & -76 \\ -48 & 176 & 12 & 36 \\ \hline -100 & 12 & 67 & 33 \\ -76 & 36 & 33 & 43 \end{array}\right]$$

$$\mathbf{m} = \begin{bmatrix} m & \\ & m \end{bmatrix} \quad (u_1 \;\; u_2) \qquad \boldsymbol{\iota} = \frac{1}{448}\begin{bmatrix} 266 & 182 \\ 42 & -42 \end{bmatrix} = [\boldsymbol{\iota}_1 \;\; \boldsymbol{\iota}_2] \quad (u_1 \;\; u_2 \;\; u_{g1} \;\; u_{g2})$$

Solve $\mathbf{k}\boldsymbol{\varphi} = \omega^2 \mathbf{m}\boldsymbol{\varphi}$ for natural frequencies and modes:

$$\omega_1 = 10.4745\sqrt{\frac{EI}{mL^3}} \qquad \omega_2 = 13.8564\sqrt{\frac{EI}{mL^3}}$$

$$\phi_1 = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \phi_2 = \frac{1}{\sqrt{2}}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Determine Γ_{nl}:

$$\Gamma_{nl} = \frac{L_{nl}}{M_n}$$

for mode n and ground motion at support l, where

$$L_{nl} = \phi_n^T \mathrm{m}\boldsymbol{\iota} \text{ and } M_n = \phi_n^T \mathrm{m}\phi_n$$

$$\Gamma = [\Gamma_{nl}] = \begin{bmatrix} 0.4861 & 0.2210 \\ -0.3536 & -0.3536 \end{bmatrix}$$

Part a

Given $\ddot{u}_{g1}(t) = \ddot{u}_g(t)$ and $\ddot{u}_{g2}(t) = \ddot{u}_g(t - t')$.

Then $D_{n1}(t) = D_n(t)$ and $D_{n2}(t) = D_n(t - t')$.

Also, $A_{n1}(t) = A_n(t)$ and $A_{n2}(t) = A_n(t - t')$.

1. *Determine the total displacements, u_1 and u_2 of the valves.*

Substituting for Γ_{nl}, ϕ_n, $D_{nl}(t)$ and $\boldsymbol{\iota}_l$ in Eq. (13.5.6) with $N = 2$ and $N_g = 2$ gives:

$$\begin{Bmatrix} u_1^t(t) \\ u_2^t(t) \end{Bmatrix} = \begin{Bmatrix} 0.5938 \\ 0.0938 \end{Bmatrix} u_g(t) + \begin{Bmatrix} 0.4062 \\ -0.0938 \end{Bmatrix} u_g(t - t')$$
$$+ \begin{Bmatrix} 0.3438 \\ 0.3438 \end{Bmatrix} D_1(t) + \begin{Bmatrix} 0.1563 \\ 0.1563 \end{Bmatrix} D_1(t - t') \qquad \text{(a)}$$
$$+ \begin{Bmatrix} 0.2500 \\ -0.2500 \end{Bmatrix} D_2(t) + \begin{Bmatrix} 0.2500 \\ -0.2500 \end{Bmatrix} D_2(t - t')$$

2. *Determine the bending moments at* a, b, c, d and e.

Recover the rotational DOF $[u_5, u_6, u_7]^T$

$$\begin{Bmatrix} u_5(t) \\ u_6(t) \\ u_7(t) \end{Bmatrix} = -\mathbf{k}_{\theta\theta}^{-1}\mathbf{k}_{\theta t} \begin{Bmatrix} u_1^t(t) \\ u_2^t(t) \\ u_{g1}(t) \\ u_{g2}(t) \end{Bmatrix}$$

$$= -\frac{3}{28L}\begin{bmatrix} 4 & 4 & -15 & 11 \\ -16 & -16 & 4 & 12 \\ 4 & 4 & -1 & -3 \end{bmatrix} \begin{Bmatrix} u_1^t(t) \\ u_2^t(t) \\ u_g(t) \\ u_g(t-t') \end{Bmatrix} \quad \text{(b)}$$

Substitute Eq. (a) into Eq. (b) and collect terms

$$\begin{Bmatrix} u_5(t) \\ u_6(t) \\ u_7(t) \end{Bmatrix} = \frac{1}{L}\left[\begin{Bmatrix} 1.3125 \\ 0.7500 \\ -0.1875 \end{Bmatrix} u_g(t) + \begin{Bmatrix} -1.3125 \\ -0.7500 \\ 0.1875 \end{Bmatrix} u_g(t-t') + \begin{Bmatrix} -0.2946 \\ 1.1786 \\ -0.2946 \end{Bmatrix} D_1(t) + \begin{Bmatrix} -0.1339 \\ 0.5357 \\ -0.1339 \end{Bmatrix} D_1(t-t')\right] \quad \text{(c)}$$

Note that there is no contribution from the second mode to the rotational displacements, an observation that could have been predicted from the second mode shape presented above

Compute bending moments in terms of nodal DOF and support displacements

$$M_a(t) = \frac{4EI}{L^2}\left[6u_1^t(t) - 6u_{g1}(t) + Lu_5(t)\right] \quad \text{(d)}$$

$$M_b(t) = \frac{4EI}{L^2}\left[6u_2^t(t) + Lu_7(t)\right] \quad \text{(e)}$$

$$M_c(t) = \frac{4EI}{L^2}\left[-6u_1^t(t) + 6u_{g2}(t) + Lu_5(t) + 2Lu_6(t)\right] \quad \text{(f)}$$

$$M_d(t) = \frac{4EI}{L^2}\left[6u_1^t(t) - 6u_{g1}(t) + 2Lu_5(t)\right] \quad \text{(g)}$$

$$M_e(t) = \frac{4EI}{L^2}\left[6u_2^t(t) + 2Lu_7(t)\right] \quad \text{(h)}$$

Substitute Eqs. (a) and (c) into Eqs. (d) through (h) and collect terms

$$M_a(t) = \frac{EI}{L^2}\Big[-4.5u_g(t) + 4.5u_g(t-t') + 7.0714D_1(t) + 3.2143D_1(t-t') + 6.0000D_2(t) + 6.0000D_2(t-t')\Big]$$

$$M_b(t) = \frac{EI}{L^2}\Big[1.5u_g(t) - 1.5u_g(t-t') + 7.0714D_1(t) + 3.2143D_1(t-t') - 6.0000D_2(t) - 6.0000D_2(t-t')\Big]$$

$$M_c(t) = \frac{EI}{L^2}\Big[-3.0u_g(t) + 3.0u_g(t-t') - 6.0000D_2(t) - 6.0000D_2(t-t')\Big]$$

$$M_d(t) = \frac{EI}{L^2}\Big[0.75u_g(t) - 0.75u_g(t-t') + 5.8929D_1(t) + 2.6786D_1(t-t') + 6.0000D_2(t) + 6.0000D_2(t-t')\Big]$$

$$M_e(t) = \frac{EI}{L^2}\Big[-0.75u_g(t) + 0.75u_g(t-t') - 5.8929D_1(t) - 2.6786D_1(t-t') + 6.0000D_2(t) + 6.0000D_2(t-t')\Big]$$

Substituting $D_n(t) = A_n(t)/\omega_n^2$:

$$M_a(t) = \frac{EI}{4L^2}\Big[-18u_g(t) + 18u_g(t-t')\Big] + \frac{mL}{4}\Big[0.258A_1(t) + 0.117A_1(t-t') + 0.125A_2(t) + 0.125A_2(t-t')\Big] \quad \text{(i)}$$

$$M_b(t) = \frac{EI}{4L^2}\Big[6.0u_g(t) - 6.0u_g(t-t')\Big] + \frac{mL}{4}\Big[0.258A_1(t) + 0.117A_1(t-t') - 0.125A_2(t) - 0.125A_2(t-t')\Big] \quad \text{(j)}$$

$$M_c(t) = \frac{EI}{4L^2}\Big[-12.0u_g(t) + 12.0u_g(t-t')\Big] + \frac{mL}{4}\Big[-0.125A_2(t) - 0.125A_2(t-t')\Big] \quad \text{(k)}$$

$$M_d(t) = \frac{EI}{4L^2}\Big[3.0u_g(t) - 3.0u_g(t-t')\Big] + \frac{mL}{4}\Big[0.215A_1(t) + 0.098A_1(t-t') + 0.125A_2(t) + 0.125A_2(t-t')\Big] \quad \text{(l)}$$

$$M_e(t) = \frac{EI}{4L^2}\left[-3.0u_g(t) + 3.0u_g(t-t')\right] + \frac{mL}{4}\left[-0.215A_1(t) - 0.215A_1(t-t') + 0.125A_2(t) + 0.125A_2(t-t')\right] \quad \text{(m)}$$

Notes:

(1) When the supports undergo different ground motions, the resulting bending moments in the structure depend, in part, on the relative displacements of the supports.

(2) The first mode of vibration does not contribute to the bending moment at c. This result could have been predicted by plotting the first mode shape.

Part b

For identical support motions,

$$\ddot{u}_{g1}(t) = \ddot{u}_{g2}(t) = \ddot{u}_g(t) \quad \text{(n)}$$

$$D_{n1}(t) = D_{n2}(t) = D_n(t) \quad \text{(o)}$$

$$A_{n1}(t) = A_{n2}(t) = A_n(t) \quad \text{(p)}$$

Substituting Eqs. (n), (o) and (p) into Eqs. (a) and (i) - (m) and collecting terms gives the displacements and bending moments:

1. *Total displacements, u_1 and u_2 of the valves.*

$$\begin{Bmatrix} u_1^t(t) \\ u_2^t(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} u_g(t) + \begin{Bmatrix} 0.5 \\ 0.5 \end{Bmatrix} D_1(t) + \begin{Bmatrix} 0.5 \\ -0.5 \end{Bmatrix} D_2(t)$$

The first term in Eq. (q) corresponds to the rigid body motion of the structure when the supports are subjected to identical ground motions. The displacements of the valves relative to the ground are given by the last two terms in Eq. (q).

2. *Bending moments at* a, b, c, d and e.

$$M_a(t) = (mL/4)\left[0.375A_1(t) + 0.25A_2(t)\right]$$

$$M_b(t) = (mL/4)\left[0.375A_1(t) - 0.25A_2(t)\right]$$

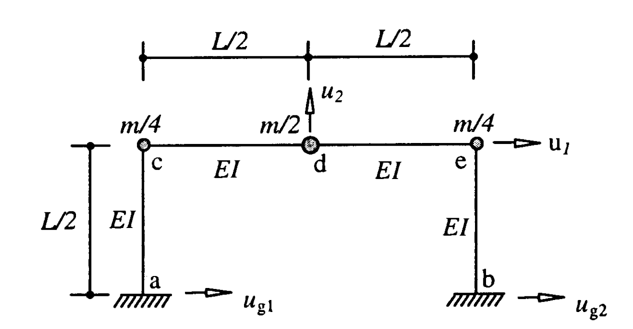
$$M_c(t) = (mL/4)\left[-0.25A_2(t)\right]$$

$$M_d(t) = (mL/4)\left[0.3125A_1(t) + 0.25A_2(t)\right]$$

$$M_e(t) = (mL/4)\left[-0.3125A_1(t) + 0.25A_2(t)\right]$$

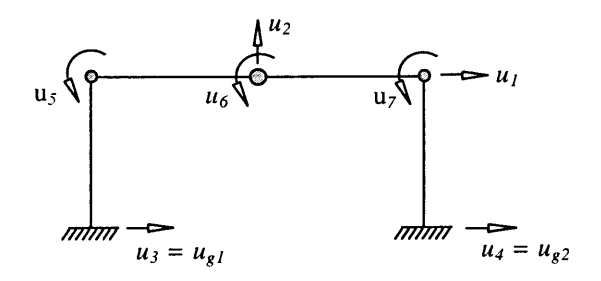
As we might expect, when the supports undergo identical motions, the resulting bending moments in the structure are functions of the modal responses only; they are not affected by the support displacements (c.f. with the results obtained in Part a).

Problem 13.34



Preliminaries

From Problem 9.22:



$$\begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t\theta} \\ \mathbf{k}_{\theta t} & \mathbf{k}_{\theta\theta} \end{bmatrix} = \frac{4EI}{L^3} \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \end{array} \\ \begin{bmatrix} 48 & 0 & -24 & -24 & 6L & 0 & 6L \\ 0 & 48 & 0 & 0 & -6L & 0 & 6L \\ -24 & 0 & 24 & 0 & -6L & 0 & 0 \\ -24 & 0 & 0 & 24 & 0 & 0 & -6L \\ 6L & -6L & -6L & 0 & 4L^2 & L^2 & 0 \\ 0 & 0 & 0 & 0 & L^2 & 4L^2 & L^2 \\ 6L & 6L & 0 & -6L & 0 & L^2 & 4L^2 \end{bmatrix} \end{array}$$

Condense out the rotational DOF $[u_5, u_6, u_7]^T$

$$\begin{bmatrix} \mathbf{k} & \mathbf{k}_g \\ \mathbf{k}_g^T & \mathbf{k}_{gg} \end{bmatrix} = \mathbf{k}_{tt} - \mathbf{k}_{t\theta}\mathbf{k}_{\theta\theta}^{-1}\mathbf{k}_{\theta t}$$

$$= \frac{6EI}{7L^3} \begin{array}{c} \begin{bmatrix} 128 & 0 & -64 & -64 \\ 0 & 140 & -42 & 42 \\ -64 & -42 & 67 & -3 \\ -64 & 42 & -3 & 67 \end{bmatrix} \\ \begin{array}{cccc} u_1 & u_2 & u_{g1} & u_{g2} \end{array} \end{array}$$

$$\mathbf{m} = \begin{array}{c} \begin{bmatrix} m & \\ & m/2 \end{bmatrix} \\ \begin{array}{cc} u_1 & u_2 \end{array} \end{array} \qquad \boldsymbol{\iota} = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & -0.3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\iota}_1 & \boldsymbol{\iota}_2 \end{bmatrix}$$

Solve $\mathbf{k}\boldsymbol{\varphi} = \omega^2 \mathbf{m}\boldsymbol{\varphi}$ for natural frequencies and modes:

$$\omega_1 = 10.4745\sqrt{\frac{EI}{mL^3}} \qquad \omega_2 = 15.4919\sqrt{\frac{EI}{mL^3}}$$

$$\phi_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Determine Γ_{nl}:

$$\Gamma_{nl} = \frac{L_{nl}}{M_n}$$

for support mode n and ground motion at support l, where

$$L_{nl} = \phi_n^T \mathbf{m}\, \boldsymbol{\iota}_l \text{ and } M_n = \phi_n^T \mathbf{m} \phi_n$$

$$\boldsymbol{\Gamma} = [\Gamma_{nl}] = \begin{bmatrix} 0.50 & 0.50 \\ 0.30 & -0.30 \end{bmatrix}$$

Part a

Given $\ddot{u}_{g1}(t) = \ddot{u}_g(t)$ and $\ddot{u}_{g2}(t) = \ddot{u}_g(t-t')$. Then $D_{n1}(t) = D_n(t)$ and $D_{n2}(t) = D_n(t-t')$. Also, $A_{n1}(t) = A_n(t)$ and $A_{n2}(t) = A_n(t-t')$.

1. *Determine the total displacements, u_1 and u_2.*

Substituting for Γ_{nl}, ϕ_n, $D_{nl}(t)$ and $\boldsymbol{\iota}_l$ in Eq. (13.5.6) with N = 2 and N_g = 2 gives:

$$\begin{Bmatrix} u_1^t(t) \\ u_2^t(t) \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0.3 \end{Bmatrix} u_g(t) + \begin{Bmatrix} 0.5 \\ -0.3 \end{Bmatrix} u_g(t-t')$$
$$+ \begin{Bmatrix} 0.5 \\ 0 \end{Bmatrix} D_1(t) + \begin{Bmatrix} 0.5 \\ 0 \end{Bmatrix} D_1(t-t') \qquad \text{(a)}$$
$$+ \begin{Bmatrix} 0 \\ 0.30 \end{Bmatrix} D_2(t) + \begin{Bmatrix} 0 \\ -0.30 \end{Bmatrix} D_2(t-t')$$

2. *Determine the bending moments at* a, b, c, d and e.

Recover the rotational DOF $[u_5, u_6, u_7]^T$

$$\begin{Bmatrix} u_5(t) \\ u_6(t) \\ u_7(t) \end{Bmatrix} = -\mathbf{k}_{\theta\theta}^{-1}\mathbf{k}_{\theta t} \begin{Bmatrix} u_1^t(t) \\ u_2^t(t) \\ u_{g1}(t) \\ u_{g2}(t) \end{Bmatrix}$$

$$= -\frac{3}{28L}\begin{bmatrix} 16 & -14 & -15 & -1 \\ -8 & 0 & 4 & 4 \\ 16 & 14 & -1 & -15 \end{bmatrix} \begin{Bmatrix} u_1^t(t) \\ u_{21}^t(t) \\ u_g(t) \\ u_g(t-t') \end{Bmatrix} \qquad \text{(b)}$$

Substitute Eq. (a) into Eq. (b) and collect terms

$$\begin{Bmatrix} u_5(t) \\ u_6(t) \\ u_7(t) \end{Bmatrix} = \frac{1}{L}\left[\begin{Bmatrix} 1.2 \\ 0 \\ -1.2 \end{Bmatrix} u_g(t) + \begin{Bmatrix} -1.2 \\ 0 \\ 1.2 \end{Bmatrix} u_g(t-t') + \begin{Bmatrix} -0.8571 \\ 0.4286 \\ -0.8571 \end{Bmatrix} D_1(t) + \begin{Bmatrix} -0.8571 \\ 0.4286 \\ -0.8571 \end{Bmatrix} D_1(t-t') + \begin{Bmatrix} 0.45 \\ 0 \\ -0.45 \end{Bmatrix} D_2(t) + \begin{Bmatrix} -0.45 \\ 0 \\ 0.45 \end{Bmatrix} D_2(t-t') \right] \qquad \text{(c)}$$

Compute bending moments in terms of nodal DOF and support displacements

$$M_a(t) = \frac{4EI}{L^2}\left[6u_1^t(t) - 6u_{g1}(t) + Lu_5(t)\right] \qquad \text{(d)}$$

$$M_b(t) = \frac{4EI}{L^2}\left[6u_1^t(t) - 6u_{g2}(t) + Lu_7(t)\right] \qquad \text{(e)}$$

$$M_c(t) = \frac{4EI}{L^2}\left[-6u_2^t(t) + 2Lu_5(t) + Lu_6(t)\right] \qquad \text{(f)}$$

$$M_d(t) = \frac{4EI}{L^2}\left[-6u_2^t(t) + Lu_5(t) + 2Lu_6(t)\right] \qquad \text{(g)}$$

$$M_e(t) = \frac{4EI}{L^2}\left[6u_2^t(t) + Lu_6(t) + 2Lu_7(t)\right] \qquad \text{(h)}$$

Substitute Eqs. (a) and (c) into Eqs. (d) through (h) and collect terms

$$M_a(t) = \frac{EI}{L^2}\Big[-7.2u_g(t) + 7.2u_g(t-t') + 8.5714D_1(t) + 8.5714D_1(t-t') + 1.8D_2(t) - 1.8D_2(t-t')\Big]$$

$$M_b(t) = \frac{EI}{L^2}\Big[7.2u_g(t) - 7.2u_g(t-t') + 8.5714D_1(t) + 8.5714D_1(t-t') - 1.8D_2(t) + 1.8D_2(t-t')\Big]$$

$$M_c(t) = \frac{EI}{L^2}\Big[2.4u_g(t) - 2.4u_g(t-t') - 5.1429D_1(t) - 5.1429D_1(t-t') - 3.6D_2(t) + 3.6D_2(t-t')\Big]$$

$$M_d(t) = \frac{EI}{L^2}\Big[-2.4u_g(t) + 2.4u_g(t-t') - 5.4D_2(t) + 5.4D_2(t-t')\Big]$$

$$M_e(t) = \frac{EI}{L^2}\Big[-2.4u_g(t) + 2.4u_g(t-t') - 5.1429D_1(t) - 5.1429D_1(t-t') + 3.6D_2(t) - 3.6D_2(t-t')\Big]$$

Substituting $D_n(t) = A_n(t)/\omega_n^2$:

$$M_a(t) = \frac{EI}{L^2}\Big[-7.2u_g(t) + 7.2u_g(t-t')\Big] + mL\big[0.0781A_1(t) + 0.0781A_1(t-t') + 0.0075A_2(t) - 0.0075A_2(t-t')\big] \qquad \text{(i)}$$

$$M_b(t) = \frac{EI}{L^2}\Big[7.2u_g(t) - 7.2u_g(t-t')\Big] + mL\big[0.0781A_1(t) + 0.0781A_1(t-t') - 0.0075A_2(t) + 0.0075A_2(t-t')\big] \qquad \text{(j)}$$

$$M_c(t) = \frac{EI}{L^2}\Big[2.4u_g(t) - 2.4u_g(t-t')\Big] + mL\big[-0.0469A_1(t) - 0.0469A_1(t-t') - 0.015A_2(t) + 0.015A_2(t-t')\big] \qquad \text{(k)}$$

$$M_d(t) = \frac{EI}{L^2}\Big[-2.4u_g(t) + 2.4u_g(t-t')\Big] + mL\big[-0.0225A_2(t) + 0.0225A_2(t-t')\big] \qquad \text{(l)}$$

$$\begin{aligned} M_e(t) = \frac{EI}{L^2}&\left[-2.4u_g(t) + 2.4u_g(t-t')\right] \\ &+mL[-0.0469A_1(t) - 0.0469A_1(t-t') \\ &+0.015A_2(t) - 0.015A_2(t-t')] \end{aligned} \qquad \text{(m)}$$

Part b

For identical support motions,

$$\ddot{u}_{g1}(t) = \ddot{u}_{g2}(t) = \ddot{u}_g(t) \qquad \text{(n)}$$

$$D_{n1}(t) = D_{n2}(t) = D_n(t) \qquad \text{(o)}$$

$$A_{n1}(t) = A_{n2}(t) = A_n(t) \qquad \text{(p)}$$

Substituting Eqs. (n), (o), and (p) into Eqs. (a) and (i) - (m) and collecting terms:

1. *Total displacements, u_1 and u_2.*

$$\begin{Bmatrix} u_1^t(t) \\ u_2^t(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} u_g(t) + \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} D_1(t)$$

2. *Bending moments at* a, b, c, d, and e.

$$M_a(t) = 0.1563mLA_1(t)$$

$$M_b(t) = 0.1563mLA_1(t)$$

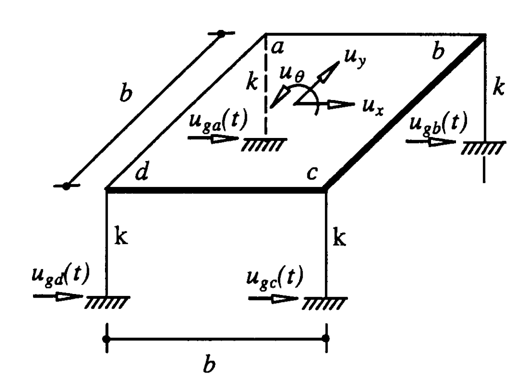
$$M_c(t) = -0.0938mLA_1(t)$$

$$M_d(t) = 0$$

$$M_e(t) = -0.0938mLA_1(t)$$

Observe that when the supports undergo identical ground motions, there is no contribution from the second mode to the response of the bridge. Because the masses do not displace horizontally in this mode, it is not excited by identical horizontal motion of the supports.

Problem 13.35



Preliminaries

From Problem 9.23:

$$\begin{bmatrix} \mathbf{k} & \mathbf{k}_g \\ \mathbf{k}_g^T & \mathbf{k}_{gg} \end{bmatrix} = k \left[\begin{array}{ccc|cccc} 4 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2b^2 & b/2 & b/2 & -b/2 & -b/2 \\ \hline -1 & 0 & b/2 & 1 & 0 & 0 & 0 \\ -1 & 0 & b/2 & 0 & 1 & 0 & 0 \\ -1 & 0 & -b/2 & 0 & 0 & 1 & 0 \\ -1 & 0 & -b/2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{m} = m \begin{bmatrix} 1 & & \\ & 1 & \\ & & \dfrac{b^2}{6} \end{bmatrix}$$

$$\iota = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1/b & -1/b & -1/b & -1/b \end{bmatrix} = \begin{bmatrix} \iota_a & \iota_b & \iota_c & \iota_d \end{bmatrix}$$

Solve $k\phi = \omega^2 \mathbf{m}\,\varphi$ for natural frequencies and mode:

$$\omega_1 = \omega_2 = 2\sqrt{\frac{k}{m}} \qquad \omega_3 = 2\sqrt{\frac{3k}{m}}$$

$$\phi_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \qquad \phi_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

Determine Γ_{nl}:

$$\Gamma_{nl} = \frac{L_{nl}}{M_n}$$

for support mode n and ground motion at support l, where

$$L_{nl} = \phi_n^T \mathbf{m}\,\iota_l \text{ and } M_n = \phi_n^T \mathbf{m} \phi_n$$

$$\Gamma = [\Gamma_{nl}] = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1/b & -1/b & -1/b & -1/b \end{bmatrix}$$

Part a

Given:

$$\ddot{u}_{ga}(t) = \ddot{u}_{gb}(t) = \ddot{u}_g(t)$$

and

$$\ddot{u}_{gc}(t) = \ddot{u}_{gd}(t) = \ddot{u}_g(t - t'),$$

then

$$D_{na}(t) = D_{nb}(t) = D_n(t)$$

and

$$D_{nc}(t) = D_{nd}(t) = D_n(t - t'),$$

also,

$$A_{na}(t) = A_{nb}(t) = A_n(t)$$

and

$$A_{nc}(t) = A_{nd}(t) = A_n(t - t').$$

1. *Determine the total displacements, u, u_y and u_θ*

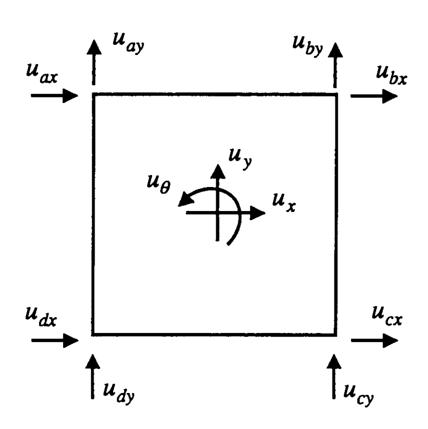
Substituting for Γ_{nl}, ϕ_n, $D_{nl}(t)$ and ι_l in Eq. (13.5.6) with $N = 3$ and $N_g = 4$ gives:

$$\begin{Bmatrix} u_x^t(t) \\ u_y^t(t) \\ u_\theta^t(t) \end{Bmatrix} = \frac{1}{2}\left[\begin{Bmatrix} 1 \\ 0 \\ -1/b \end{Bmatrix} u_g(t) + \begin{Bmatrix} 1 \\ 0 \\ 1/b \end{Bmatrix} u_g(t - t') + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} D_1(t) + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} D_1(t - t') + \begin{Bmatrix} 0 \\ 0 \\ -1/b \end{Bmatrix} D_3(t) + \begin{Bmatrix} 0 \\ 0 \\ 1/b \end{Bmatrix} D_3(t - t') \right] \quad \text{(a)}$$

Note that there is no response in the y-direction and no contribution to the response from the second mode.

2. *Determine the shear forces in the columns.*

Define nodal displacements at the top of the columns as shown:



Relate these displacements to u_x^t, u_y^t and u_θ^t:

$$u_{ax} = u_x^t - \frac{b}{2}u_\theta^t \qquad u_{ay} = u_y^t - \frac{b}{2}u_\theta^t$$

$$u_{bx} = u_x^t - \frac{b}{2}u_\theta^t \qquad u_{by} = u_y^t + \frac{b}{2}u_\theta^t$$

$$u_{cx} = u_x^t + \frac{b}{2}u_\theta^t \qquad u_{cy} = u_y^t + \frac{b}{2}u_\theta^t$$

$$u_{dx} = u_x^t + \frac{b}{2}u_\theta^t \qquad u_{dy} = u_y^t - \frac{b}{2}u_\theta^t$$

The x and y components of the column drifts are:

$$\Delta_{ax}(t) = u_{ax}^t(t) - u_g(t) = u_x^t(t) - \frac{b}{2}u_\theta^t(t) - u_g(t)$$

$$\Delta_{ay}(t) = u_{ay}^t(t) = u_y(t) - \frac{b}{2}u_\theta^t(t)$$

$$\Delta_{bx}(t) = u_{bx}^t(t) - u_g(t) = u_x(t) - \frac{b}{2}u_\theta^t(t) - u_g(t)$$

$$\Delta_{by}(t) = u_{by}^t(t) = u_y(t) + \frac{b}{2}u_\theta^t(t)$$

$$\Delta_{cx}(t) = u_{cx}^t(t) - u_g(t-t') = u_x(t) + \frac{b}{2}u_\theta^t(t) - u_g(t-t')$$

$$\Delta_{cy}(t) = u_{cy}^t(t) = u_y(t) + \frac{b}{2}u_\theta^t(t)$$

$$\Delta_{dx}(t) = u_{dy}^t(t) - u_g(t-t') = u_x(t) + \frac{b}{2}u_\theta(t) - u_g(t-t')$$

$$\Delta_{dy}(t) = u_{dx}^t(t) = u_y(t) - \frac{b}{2}u_\theta(t)$$

The x and y components of the column shears are:

$$V_{ax}(t) = k\Delta_{ax}(t) = k\left(u_x^t(t) - \frac{b}{2}u_\theta^t(t) - u_g(t)\right)$$

$$V_{ay}(t) = k\Delta_{ay}(t) = k\left(u_y^t(t) - \frac{b}{2}u_\theta^t(t)\right)$$

$$V_{bx}(t) = k\Delta_{bx}(t) = k\left(u_x^t(t) - \frac{b}{2}u_\theta^t(t) - u_g(t)\right)$$

$$V_{by}(t) = k\Delta_{by}(t) = k\left(u_y^t(t) + \frac{b}{2}u_\theta^t(t)\right)$$

$$V_{cx}(t) = k\Delta_{cx}(t) = k\left(u_x^t(t) + \frac{b}{2}u_\theta^t(t) - u_g(t-t')\right) \qquad \text{(b)}$$

$$V_{cy}(t) = k\Delta_{cy}(t) = k\left(u_y^t(t) + \frac{b}{2}u_\theta^t(t)\right)$$

$$V_{dx}(t) = k\Delta_{dx}(t) = k\left(u_x^t(t) + \frac{b}{2}u_\theta^t(t) - u_g(t-t')\right)$$

$$V_{dy}(t) = k\Delta_{dy}(t) = k\left(u_y^t(t) - \frac{b}{2}u_\theta^t(t)\right)$$

Substitute Eq. (a) into Eq. (b) and collect terms

$$V_{ax}(t) = k\left[-\frac{1}{4}u_g(t) + \frac{1}{4}u_g(t-t') + \frac{1}{2}D_1(t) + \frac{1}{2}D_1(t-t') + \frac{1}{4}D_3(t) - \frac{1}{4}D_3(t-t')\right]$$

$$V_{ay}(t) = k\left[\frac{1}{4}u_g(t) - \frac{1}{4}u_g(t-t') + \frac{1}{4}D_3(t) - \frac{1}{4}D_3(t-t')\right]$$

$$V_{bx}(t) = k\left[-\frac{1}{4}u_g(t) + \frac{1}{4}u_g(t-t') + \frac{1}{2}D_1(t) + \frac{1}{2}D_1(t-t') + \frac{1}{4}D_3(t) - \frac{1}{4}D_3(t-t')\right]$$

$$V_{by}(t) = k\left[-\frac{1}{4}u_g(t) + \frac{1}{4}u_g(t-t') - \frac{1}{4}D_3(t) + \frac{1}{4}D_3(t-t')\right]$$

$$V_{cx}(t) = k\left[\frac{1}{4}u_g(t) - \frac{1}{4}u_g(t-t') + \frac{1}{2}D_1(t) + \frac{1}{2}D_1(t-t') - \frac{1}{4}D_3(t) + \frac{1}{4}D_3(t-t')\right]$$

$$V_{cy}(t) = k\left[-\frac{1}{4}u_g(t) + \frac{1}{4}u_g(t-t') - \frac{1}{4}D_3(t) + \frac{1}{4}D_3(t-t')\right]$$

$$V_{dx}(t) = k\left[\frac{1}{4}u_g(t) - \frac{1}{4}u_g(t-t') + \frac{1}{2}D_1(t) + \frac{1}{2}D_1(t-t') - \frac{1}{4}D_3(t) + \frac{1}{4}D_3(t-t')\right]$$

$$V_{dy}(t) = k\left[\frac{1}{4}u_g(t) - \frac{1}{4}u_g(t-t') + \frac{1}{4}D_3(t) - \frac{1}{4}D_3(t-t')\right]$$

Substituting $D_n(t) = A_n(t)/\omega_n^2$:

$$V_{ax}(t) = \frac{k}{4}\left[-u_g(t) + u_g(t-t')\right]$$
$$+\frac{m}{8}\left[A_1(t) + A_1(t-t') + \frac{1}{6}A_3(t) - \frac{1}{6}A_3(t-t')\right]$$

$$V_{ay}(t) = \frac{k}{4}\left[u_g(t) - u_g(t-t')\right] + \frac{m}{48}\left[A_3(t) - A_3(t-t')\right]$$

$$V_{bx}(t) = \frac{k}{4}\left[-u_g(t) + u_g(t-t')\right]$$
$$+\frac{m}{8}\left[A_1(t) + A_1(t-t') + \frac{1}{6}A_3(t) - \frac{1}{6}A_3(t-t')\right]$$

$$V_{by}(t) = \frac{k}{4}\left[-u_g(t) + u_g(t-t')\right] + \frac{m}{48}\left[-A_3(t) + A_3(t-t')\right]$$

$$V_{cx}(t) = \frac{k}{4}\left[u_g(t) - u_g(t-t')\right]$$
$$+\frac{m}{8}\left[A_1(t) + A_1(t-t') - \frac{1}{6}A_3(t) + \frac{1}{6}A_3(t-t')\right]$$

$$V_{cy}(t) = \frac{k}{4}\left[-u_g(t) + u_g(t-t')\right] + \frac{m}{48}\left[-A_3(t) + A_3(t-t')\right]$$

$$V_{dx}(t) = \frac{k}{4}\left[u_g(t) - u_g(t-t')\right]$$
$$+\frac{m}{8}\left[A_1(t) + A_1(t-t') - \frac{1}{6}A_3(t) + \frac{1}{6}A_3(t-t')\right]$$

$$V_{dy}(t) = \frac{k}{4}\left[u_g(t) - u_g(t-t')\right] + \frac{m}{48}\left[A_3(t) - A_3(t-t')\right] \quad \text{(c)}$$

Part b

For identical support motions,

$$\ddot{u}_{ga}(t) = \ddot{u}_{gb}(t) = \ddot{u}_{gc}(t) = \ddot{u}_{gd}(t) = \ddot{u}_g(t)$$

$$D_{na}(t) = D_{nb}(t) = D_{nc}(t) = D_{nd}(t) = D_n(t)$$

$$A_{na}(t) = A_{nb}(t) = A_{nc}(t) = A_{nd}(t) = A_n(t) \quad \text{(d)}$$

Substituting Eqs. (d) into Eqs. (a) and (c) and collecting terms:

1. *The total displacements, u_x, u_y and u_θ.*

$$\begin{Bmatrix} u_x^t(t) \\ u_y^t(t) \\ u_\theta^t(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} u_g(t) + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} D_1(t)$$

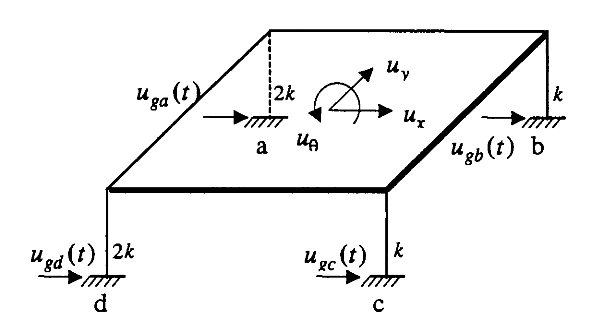
2. *The column shear forces.*

$$V_{ax}(t) = V_{bx}(t) = V_{cx}(t) = V_{dx}(t) = \frac{m}{4}A_1(t)$$

$$V_{ay}(t) = V_{by}(t) = V_{cy}(t) = V_{dy}(t) = 0$$

If the support motions in the x-direction are identical, only the first mode of vibration is excited, causing slab displacements and column shears in the x-direction only. However, if the supports are subjected to spatially varying ground motions, the third mode of vibration (a purely torsional mode) is also excited and contributes to the response of the structure.; i.e., the structure undergoes coupled lateral-torsional motion. Note however that due to the symmetry of the structure and the direction of the applied ground motions, the second mode of vibration is not excited for either excitation case considered. Consequently, as evidenced by the above results, there is no contribution to the response of the structure from the second mode.

Problem 13.36



1. *Establish data* (from Problem 9.14).

$m = 0.2331$ kip - sec^2 / in

$k = 1.5$ kip / in

$b = 25$ ft

$\zeta_n = 0.05, \quad n = 1, 2$ and 3

DOFs: $\mathbf{u}^t = \langle u_x^t \quad u_y^t \quad u_\theta^t \rangle^T$

2. *Set up mass and stiffness matrices.*

The mass matrix (from Problem 9.14) is

$$\mathbf{m} = m\begin{bmatrix} 1 & & \\ & 1 & \\ & & \dfrac{b^2}{6} \end{bmatrix} \tag{a}$$

The structural stiffness matrix (from Problem 9.14) is

$$\mathbf{k} = k\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & -b \\ 0 & -b & 3b^2 \end{bmatrix} \tag{b}$$

The stiffness matrix associated with support DOFs, and the coupling submatrix between structural DOFs and support DOFs (from Problem 9.24).are

$$\mathbf{k}_{gg} = k\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{k}_g = k\begin{bmatrix} -2 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ b & 0.5b & -0.5b & -b \end{bmatrix}$$

and the influence matrix:

$$\iota = -\mathbf{k}^{-1}\mathbf{k}_g = \begin{bmatrix} 0.3333 & 0.1667 & 0.1667 & 0.3333 \\ -0.0588 & -0.0294 & 0.0294 & 0.0588 \\ -\dfrac{0.3529}{b} & -\dfrac{0.1765}{b} & \dfrac{0.1765}{b} & \dfrac{0.3529}{b} \end{bmatrix} \tag{c}$$

3. *Determine the natural vibration frequencies and modes.*

From Problem 10.24 ω_n in rad/sec are:

$$\omega_1 = 5.96 \qquad \omega_2 = 6.21 \qquad \omega_3 = 10.90 \tag{d}$$

$$\phi_1 = \begin{Bmatrix} 0 \\ 2.032 \\ 0.0033 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} 2.071 \\ 0 \\ 0 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0 \\ -0.3988 \\ 0.0166 \end{Bmatrix} \tag{e}$$

4. *Determine* $\Gamma_{nl} = L_{nl}/M_n$.

$$L_{nl} = \phi_n^T \mathbf{m}\,\iota \qquad \text{and} \qquad M_n = \phi_n^T \mathbf{m} \phi_n = 1 \tag{f}$$

$n = 1, 2, 3 \qquad l = 1, 2$

$$\Gamma = [\Gamma_{nl}] = \begin{bmatrix} -0.0413 & -0.0206 & 0.0206 & 0.0413 \\ 0.1609 & 0.0805 & 0.0805 & 0.1609 \\ -0.0628 & -0.0314 & 0.0314 & 0.0628 \end{bmatrix}$$

5. *Determine the response of the* n^{th} *-mode SDF system to* $\ddot{u}_{gl}(t)$.

Given:

$\ddot{u}_{ga}(t) = \ddot{u}_g(t)$, $\ddot{u}_{gb}(t) = \ddot{u}_g(t)$, $\ddot{u}_{gc}(t) = \ddot{u}_g(t - t')$

and

$\ddot{u}_{gd}(t) = \ddot{u}_g(t - t')$,

then

$$D_{na}(t) = D_{nb}(t) = D_n(t) \tag{h}$$

$$D_{nc}(t) = D_{nd}(t) = D_n(t - t')$$

6. *Determine the displacement response.*

In Eq. (13.5.6) with $N = 3$ and $N_g = 4$, substituting for Γ_{nl}, ϕ_n, and $D_{nl}(t)$ from Eqs. (g), (e), and (h), respectively, and for ι_l from Eq. (e) of Problem 9.31 give

$$
\begin{Bmatrix} u_x^t(t) \\ u_y^t(t) \\ u_\theta^t(t) \end{Bmatrix} = \begin{Bmatrix} 0.3333 \\ -0.0588 \\ -0.0012 \end{Bmatrix} u_g(t) + \begin{Bmatrix} 0.1667 \\ -0.0294 \\ -0.0006 \end{Bmatrix} u_g(t) + \begin{Bmatrix} 0.1667 \\ 0.0294 \\ 0.0006 \end{Bmatrix} u_g(t-t') + \begin{Bmatrix} 0.3333 \\ 0.0588 \\ 0.0012 \end{Bmatrix} u_g(t-t')
$$

$$
-0.0413\begin{Bmatrix} 0 \\ 2.032 \\ 0.0033 \end{Bmatrix} D_1(t) \quad -0.0206\begin{Bmatrix} 0 \\ 2.032 \\ 0.0033 \end{Bmatrix} D_1(t) \quad +0.0206\begin{Bmatrix} 0 \\ 2.032 \\ 0.0033 \end{Bmatrix} D_1(t-t') \quad +0.0413\begin{Bmatrix} 0 \\ 2.032 \\ 0.0033 \end{Bmatrix} D_1(t-t')
$$

$$
+0.1609\begin{Bmatrix} 2.071 \\ 0 \\ 0 \end{Bmatrix} D_2(t) \quad +0.0805\begin{Bmatrix} 2.071 \\ 0 \\ 0 \end{Bmatrix} D_2(t) \quad +0.0805\begin{Bmatrix} 2.071 \\ 0 \\ 0 \end{Bmatrix} D_2(t-t') \quad +0.1609\begin{Bmatrix} 2.071 \\ 0 \\ 0 \end{Bmatrix} D_2(t-t')
$$

$$
-0.0628\begin{Bmatrix} 0 \\ -0.3988 \\ 0.0166 \end{Bmatrix} D_3(t) \quad -0.0314\begin{Bmatrix} 0 \\ -0.3988 \\ 0.0166 \end{Bmatrix} D_3(t) \quad +0.0314\begin{Bmatrix} 0 \\ -0.3988 \\ 0.0166 \end{Bmatrix} D_3(t-t') \quad +0.0628\begin{Bmatrix} 0 \\ -0.3988 \\ 0.0166 \end{Bmatrix} D_3(t-t')
$$

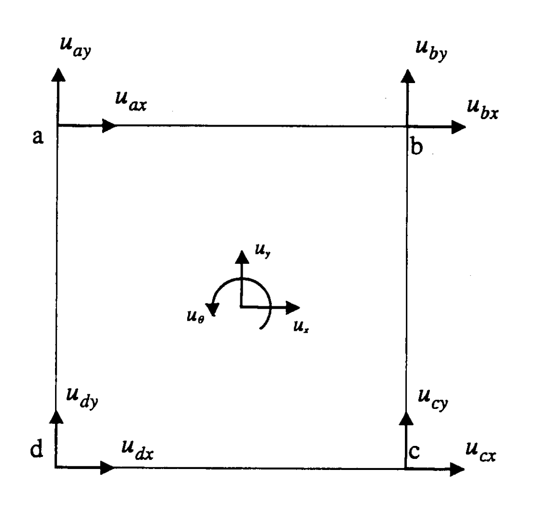
or

$$
\begin{Bmatrix} u_x^t(t) \\ u_y^t(t) \\ u_\theta^t(t) \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ -0.0882 \\ -0.0018 \end{Bmatrix} u_g(t) + \begin{Bmatrix} 0.5 \\ 0.0882 \\ 0.0018 \end{Bmatrix} u_g(t-t') + \begin{Bmatrix} 0 \\ -0.1258 \\ -0.0002 \end{Bmatrix} D_1(t) + \begin{Bmatrix} 0 \\ 0.1258 \\ 0.0002 \end{Bmatrix} D_1(t-t') + \begin{Bmatrix} 0.5 \\ 0 \\ 0 \end{Bmatrix} D_2(t)
$$

$$
+\begin{Bmatrix} 0.5 \\ 0 \\ 0 \end{Bmatrix} D_2(t-t') + \begin{Bmatrix} 0 \\ 0.0376 \\ -0.0015 \end{Bmatrix} D_3(t) + \begin{Bmatrix} 0 \\ -0.0376 \\ 0.0015 \end{Bmatrix} D_3(t-t') \tag{i}
$$

7. *Determine shears in columns.*

Define nodal displacements at the top of the columns as shown in the accompanying figure



Relate these displacements to u_x, u_y and u_θ:

$$
u_{ax}^t = u_x^t - \frac{b}{2}u_\theta^t \quad u_{ay}^t = u_y^t - \frac{b}{2}u_\theta^t
$$

$$
u_{bx}^t = u_x^t - \frac{b}{2}u_\theta^t \quad u_{by}^t = u_y^t + \frac{b}{2}u_\theta^t
$$

$$
u_{cx}^t = u_x^t + \frac{b}{2}u_\theta^t \quad u_{cy}^t = u_y^t + \frac{b}{2}u_\theta^t
$$

$$
u_{dx}^t = u_x^t + \frac{b}{2}u_\theta^t \quad u_{dy}^t = u_y^t - \frac{b}{2}u_\theta^t \tag{h}
$$

The x- and y- components of deformations in columns a, b, c, and d are:

$$\Delta_{ax}(t) = u_{ax}^t(t) - u_g(t) = u_x^t(t) - \frac{b}{2}u_\theta^t(t) - u_g(t), \qquad \Delta_{ay}(t) = u_{ay}^t(t) = u_y^t(t) - \frac{b}{2}u_\theta^t(t)$$

$$\Delta_{bx}(t) = u_{bx}^t(t) - u_g(t) = u_x^t(t) - \frac{b}{2}u_\theta^t(t) - u_g(t), \qquad \Delta_{by}(t) = u_{by}^t(t) = u_y^t(t) + \frac{b}{2}u_\theta^t(t) \tag{k}$$

$$\Delta_{cx}(t) = u_{cx}^t(t) - u_g(t-t') = u_x^t(t) + \frac{b}{2}u_\theta^t(t) - u_g(t-t'), \qquad \Delta_{cy}(t) = u_{cy}^t(t) = u_y^t(t) + \frac{b}{2}u_\theta^t(t)$$

$$\Delta_{dx}(t) = u_{dx}^t(t) - u_g(t-t') = u_x^t(t) + \frac{b}{2}u_\theta^t(t) - u_g(t-t'), \qquad \Delta_{dy}(t) = u_{dy}^t(t) = u_y^t(t) - \frac{b}{2}u_\theta^t(t)$$

The x and y components of shear in columns a, b, c and d are:

$$\begin{aligned} V_{ax} &= 2k\,\Delta_{ax} & V_{ay} &= 2k\,\Delta_{ay} \\ V_{bx} &= k\,\Delta_{bx} & V_{by} &= k\,\Delta_{by} \\ V_{cx} &= k\,\Delta_{cx} & V_{cy} &= k\,\Delta_{cy} \\ V_{dx} &= 2k\,\Delta_{dx} & V_{dy} &= 2k\,\Delta_{dy} \end{aligned} \tag{l}$$

Substituting Eqs (i) and (k) into (l) we obtain:

$$\begin{aligned} V_{ax} &= -0.69\,u_g(t) + 0.69\,u_g(t-t') \\ &\quad + 0.09\,D_1(t) + 1.5\,D_2(t) + 0.675\,D_3(t) \\ &\quad - 0.09\,D_1(t-t') + 1.5\,D_2(t-t') - 0.675\,D_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{ay} &= 0.5454\,u_g(t) - 0.5454\,u_g(t-t') \\ &\quad - 0.2874\,D_1(t) + 0.7878\,D_3(t) \\ &\quad + 0.2874\,D_1(t-t') - 0.7878\,D_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{bx} &= -0.345\,u_g(t) + 0.345\,u_g(t-t') \\ &\quad + 0.045\,D_1(t) + 0.75\,D_2(t) + 0.3375\,D_3(t) \\ &\quad - 0.045\,D_1(t-t') + 0.75\,D_2(t-t') - 0.3375\,D_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{by} &= -0.5373\,u_g(t) + 0.5373\,u_g(t-t') \\ &\quad - 0.2337\,D_1(t) - 0.2811\,D_3(t) \\ &\quad + 0.2337\,D_1(t-t') + 0.2811\,D_3(t-t') \end{aligned} \tag{m}$$

$$\begin{aligned} V_{cx} &= 0.345\,u_g(t) - 0.345\,u_g(t-t') \\ &\quad - 0.045\,D_1(t) + 0.75\,D_2(t) - 0.3375\,D_3(t) \\ &\quad + 0.045\,D_1(t-t') + 0.75\,D_2(t-t') + 0.3375\,D_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{cy} &= -0.5373\,u_g(t) + 0.5373\,u_g(t-t') \\ &\quad - 0.2337\,D_1(t) - 0.2811\,D_3(t) \\ &\quad + 0.2337\,D_1(t-t') + 0.2811\,D_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{dx} &= 0.69\,u_g(t) - 0.69\,u_g(t-t') \\ &\quad - 0.09\,D_1(t) + 1.5\,D_2(t) - 0.675\,D_3(t) \\ &\quad + 0.09\,D_1(t-t') + 1.5\,D_2(t-t') + 0.675\,D_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{dy} &= 0.5454\,u_g(t) - 0.5454\,u_g(t-t') \\ &\quad - 0.2874\,D_1(t) + 0.7878\,D_3(t) \\ &\quad + 0.2874\,D_1(t-t') - 0.7878\,D_3(t-t') \end{aligned}$$

or, using the fact that $D_n(t) = A_n(t)/\omega_n^2$, we can write:

$$\begin{aligned} V_{ax} &= -0.69\,u_g(t) + 0.69\,u_g(t-t') \\ &\quad + 0.0025\,A_1(t) + 0.0389\,A_2(t) + 0.0057\,A_3(t) \\ &\quad - 0.0025\,A_1(t-t') + 0.0389\,A_2(t-t') - 0.0057\,A_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{ay} &= 0.5454\,u_g(t) - 0.5454\,u_g(t-t') \\ &\quad - 0.0081\,A_1(t) + 0.0066\,A_3(t) \\ &\quad + 0.0081\,A_1(t-t') - 0.0066\,A_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{bx} &= -0.345\,u_g(t) + 0.345\,u_g(t-t') \\ &\quad + 0.0013\,A_1(t) + 0.0194\,A_2(t) + 0.0028\,A_3(t) \\ &\quad - 0.0013\,A_1(t-t') + 0.0194\,A_2(t-t') - 0.0028\,A_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{by} &= -0.5373\,u_g(t) + 0.5373\,u_g(t-t') \\ &\quad - 0.0066\,A_1(t) - 0.0024\,A_3(t) \\ &\quad + 0.0066\,A_1(t-t') + 0.0024\,A_3(t-t') \end{aligned} \qquad \text{(n)}$$

$$\begin{aligned} V_{cx} &= 0.345\,u_g(t) - 0.345\,u_g(t-t') \\ &\quad - 0.0013\,A_1(t) + 0.0194\,A_2(t) - 0.0028\,A_3(t) \\ &\quad + 0.0013\,A_1(t-t') + 0.0194\,A_2(t-t') + 0.0028\,A_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{cy} &= -0.5373\,u_g(t) + 0.5373\,u_g(t-t') \\ &\quad - 0.0066\,A_1(t) - 0.0024\,A_3(t) \\ &\quad + 0.0066\,A_1(t-t') + 0.0024\,A_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{dx} &= 0.69\,u_g(t) - 0.69\,u_g(t-t') \\ &\quad - 0.0025\,A_1(t) + 0.0389\,A_2(t) - 0.0057\,A_3(t) \\ &\quad + 0.0025\,A_1(t-t') + 0.0389\,A_2(t-t') + 0.0057\,A_3(t-t') \end{aligned}$$

$$\begin{aligned} V_{dy} &= 0.5454\,u_g(t) - 0.5454\,u_g(t-t') \\ &\quad - 0.0081\,A_1(t) + 0.0066\,A_3(t) \\ &\quad + 0.0081\,A_1(t-t') - 0.0066\,A_3(t-t') \end{aligned}$$

8. *Identical support motions.*

If all the supports undergo identical motions, the motion at the structure is given by Eq.(13.1.15), where Γ_n is defined by Eq.(13.1.5) with

$$\iota = \langle 1 \quad 0 \quad 0 \rangle^T$$

For this case:

$$\Gamma_1 = 0\,,\ \Gamma_2 = 0.4828\,,\ \Gamma_3 = 0$$

Then Eq.(13.1.15) gives

$$\mathbf{u}(t) = \Gamma_2\,\phi_2\,D_2(t) = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} D_2(t) \qquad \text{(o)}$$

Observe that the first and third modes are not excited by identical support motions in the x-direction; all the response is due to the second mode.

Because $u_y = u_\theta = 0$, column shears in the y-direction are zero:

$$V_{ay} = V_{by} = V_{cy} = V_{dy} = 0$$

and the shears in the x-direction in the four columns are:

$$V_{ax}(t) = V_{dx}(t) = 2k\,D_2(t) = 3.0D_2(t)$$

$$V_{bx}(t) = V_{cx}(t) = k\,D_2(t) = 1.5D_2(t)$$

9. *Comparison*

If the support motions in the x-direction are identical, only the second mode is excited, causing floor displacements and column shears in the x-direction only. However, this symmetric plan structure responds to spatially varying ground motion in a much more complicated manner: (1) the three vibration modes are excited; (2) the roof slab displaces in x and y directions and rotates about the vertical axis (implying that the building undergoes coupled lateral-torsional motion); and (3) shears in columns develop in both x and y directions.

Problem 13.37

In the solution of Problem 13.36 the displacements u_x^t, u_y^t and u_θ^t were expressed in terms of $u_g(t)$ and $D_n(t)$, Eq. (i); and column shears in terms of $u_g(t)$ and $A_n(t)$, Eq. (n).

1. *Determine ground displacement.*

Double integration of $\ddot{u}_g(t)$ gives the ground displacement $u_g(t)$ shown in Fig. P13.37a.

2. *Determine $D_n(t)$.*

The properties of the three, nth-mode SDF systems are ($n = 1, 2, 3$)

$\omega_1 = 5.96$ rad/sec $\quad \zeta_1 = 5\,\%$

$\omega_2 = 6.21$ rad/sec $\quad \zeta_2 = 5\,\%$

$\omega_3 = 10.90$ rad/sec $\quad \zeta_3 = 5\,\%$

The deformation responses $D_n(t)$ of these systems are shown in Fig. P13.37b.

3. *Determine displacement response.*

Substituting $u_g(t)$, $D_n(t)$ and t' in Eq. (i) of Problem 13.36 gives the displacements $u_x^t(t)$, $u_y^t(t)$ and $u_\theta^t(t)$ shown in Fig. P13.37c.

4. *Determine column shears.*

From known $u_g(t)$ and $D_n(t)$, the x- and y-components of shear in each column are determined from Eq. (m). The results are shown in Fig. P13.37d.

5. *Identical support motions.*

Substituting $D_2(t)$ from Fig. P13.37b into Eq. (o) gives the displacement response of the structure relative to the ground (Fig. P13.37e), and Eq. (p) gives the column shears (Fig. P13.37g). The total displacements (relative displacement plus the ground displacement) is shown in Fig. P13.37f.

6. *Comparison.*

Identical support motions in the x-direction cause floor displacements and column shears in the x-direction only. However, this symmetric plan structure responds to spatially varying ground motion in a much more complicated manner. The roof slab displaces in both x and y directions and also rotates about the vertical axis (implying that the building undergoes coupled lateral-torsional motion). The peak floor displacements and the peak column shear forces are given in Tables P13.37a and P13.37b, respectively.

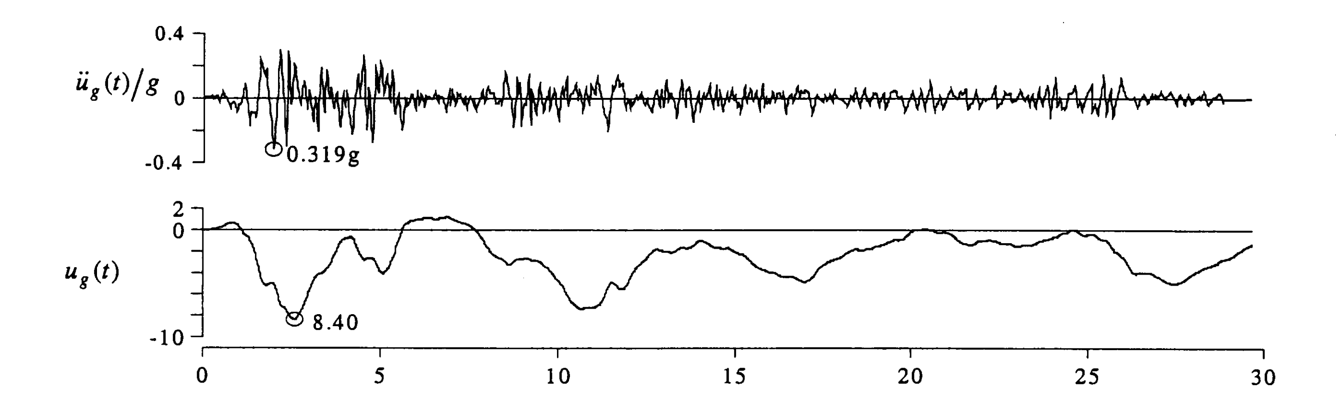
Table P13.37a

Displacement	Non-uniform excitation	Uniform excitation
u_{x0}^t, in	9.941	
u_{y0}^t, in	0.324	0
$b/2\, u_{\theta 0}^t$, in	0.379	0

Table P13.37b

column	column shear, kips			
	non-uniform excitation		uniform excitation	
	x	y	x	y
a	13.684	2.661	13.078	0
b	6.842	1.214	6.539	0
c	6.892	1,214	6.539	0
d	13.783	2.661	13.078	0

Observe that, in the case of different support motions the shear forces in the y-direction are about 20% of those in the x-direction, although the total displacement in the y-direction is negligible compared to the one in x-direction. This is due to the coupled lateral- torsional response of the structure.



Time, sec

Fig. P13.37a

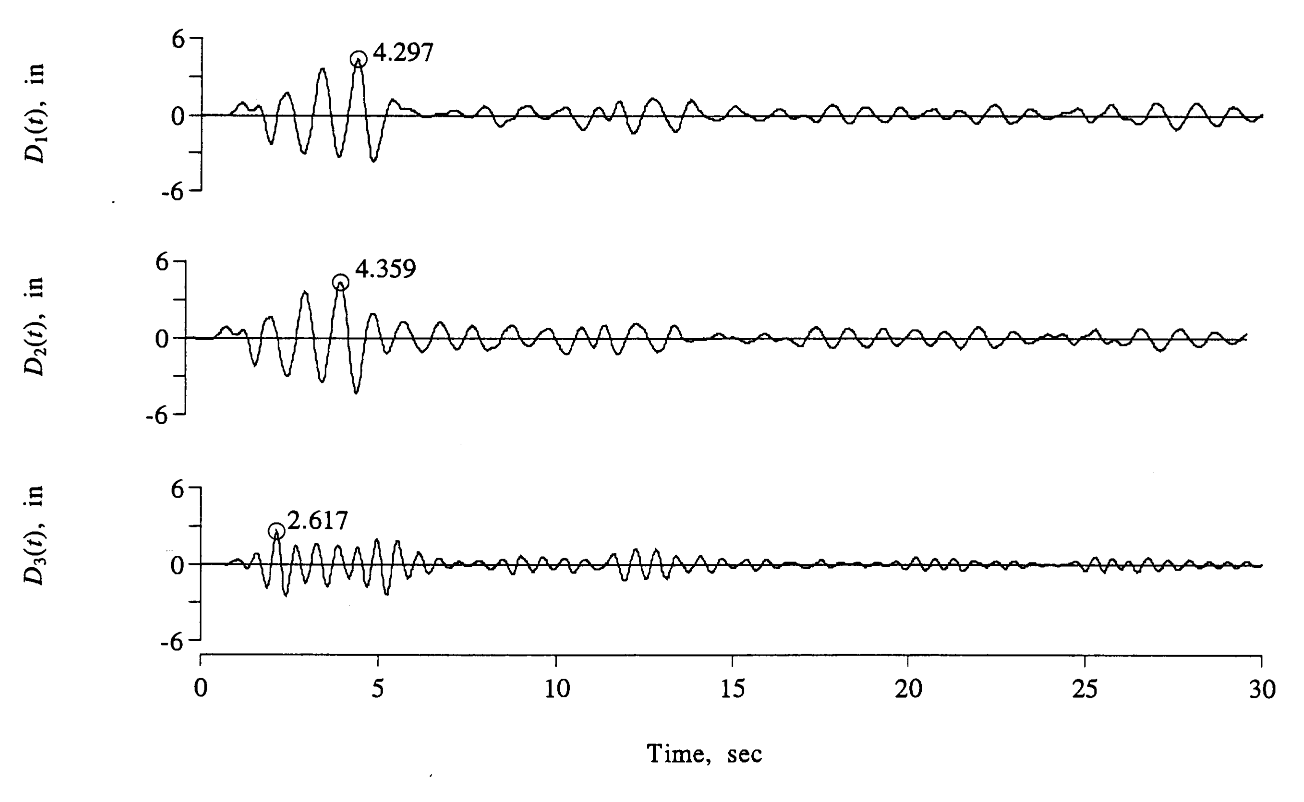


Fig. P13.37b

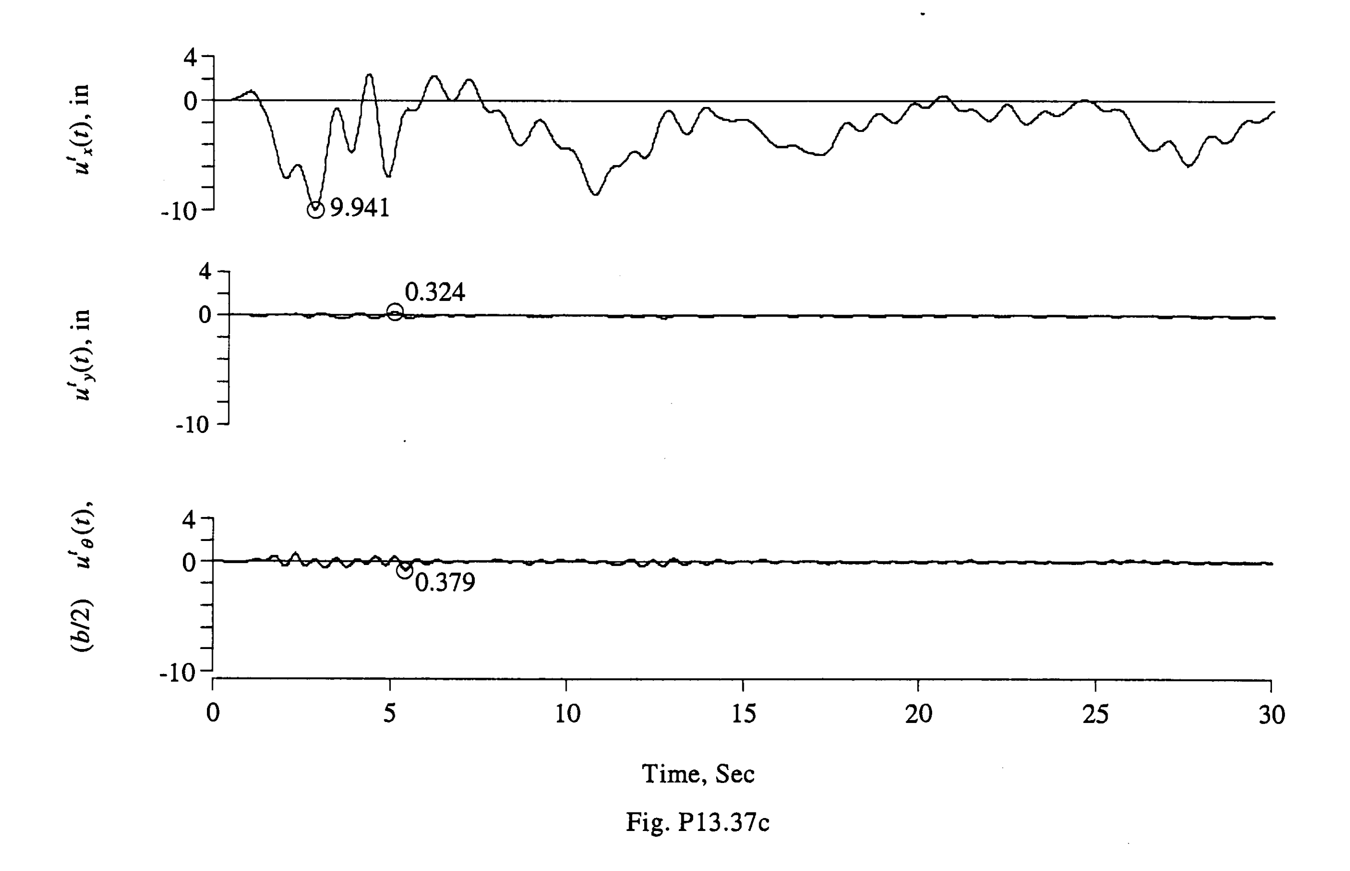

$u^t_x(t)$, in
4
0
-10
9.941
$u^t_y(t)$, in
0.324
$(b/2)$ $u^t_\theta(t)$,
0.379
0
5
10
15
20
25
30
Time, Sec

Fig. P13.37c

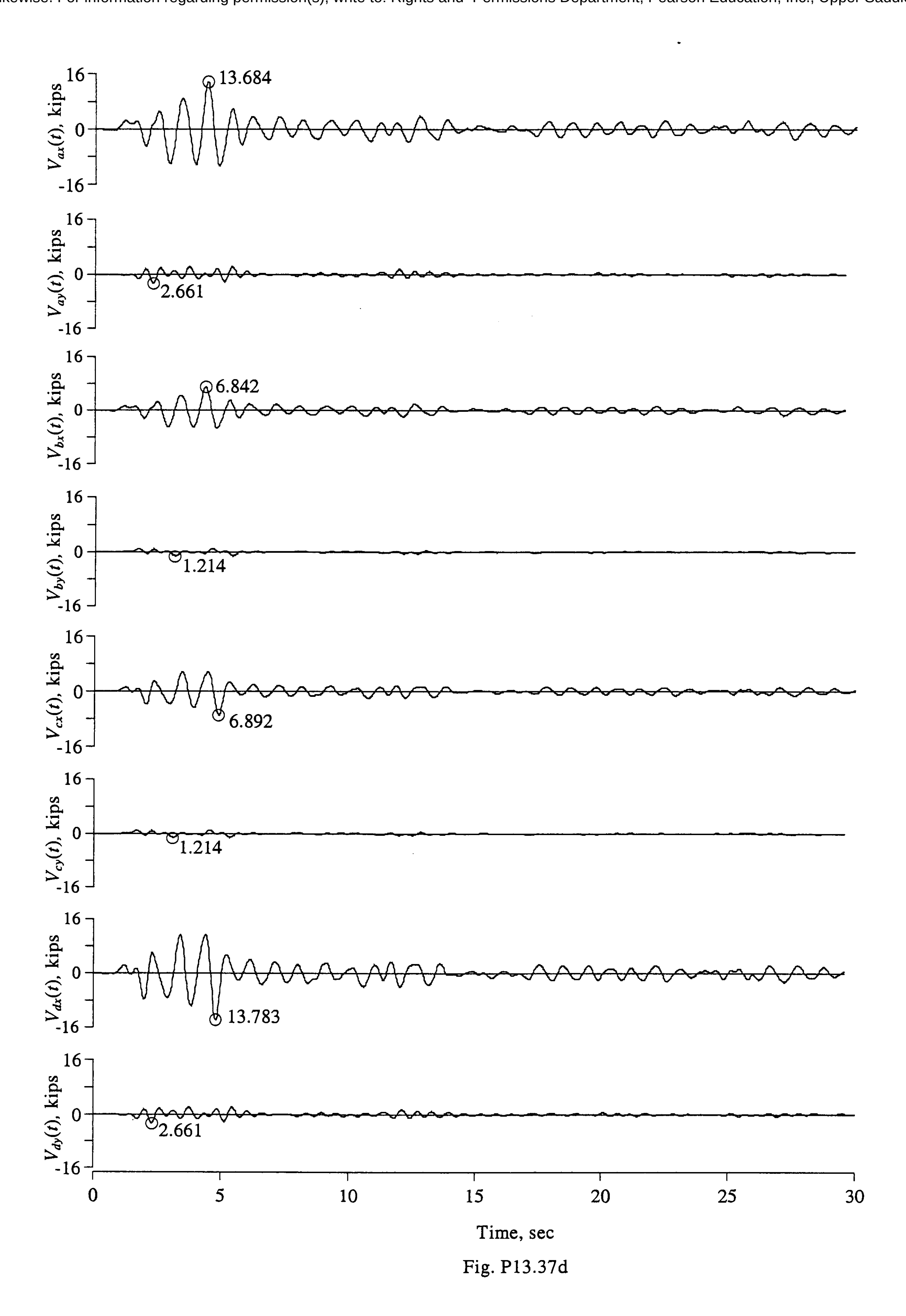

$V_{ax}(t)$, kips
13.684
$V_{ay}(t)$, kips
2.661
$V_{bx}(t)$, kips
6.842
$V_{by}(t)$, kips
1.214
$V_{cx}(t)$, kips
6.892
$V_{cy}(t)$, kips
1.214
$V_{dx}(t)$, kips
13.783
$V_{dy}(t)$, kips
2.661
16
0
-16
0
5
10
15
20
25
30
Time, sec

Fig. P13.37d

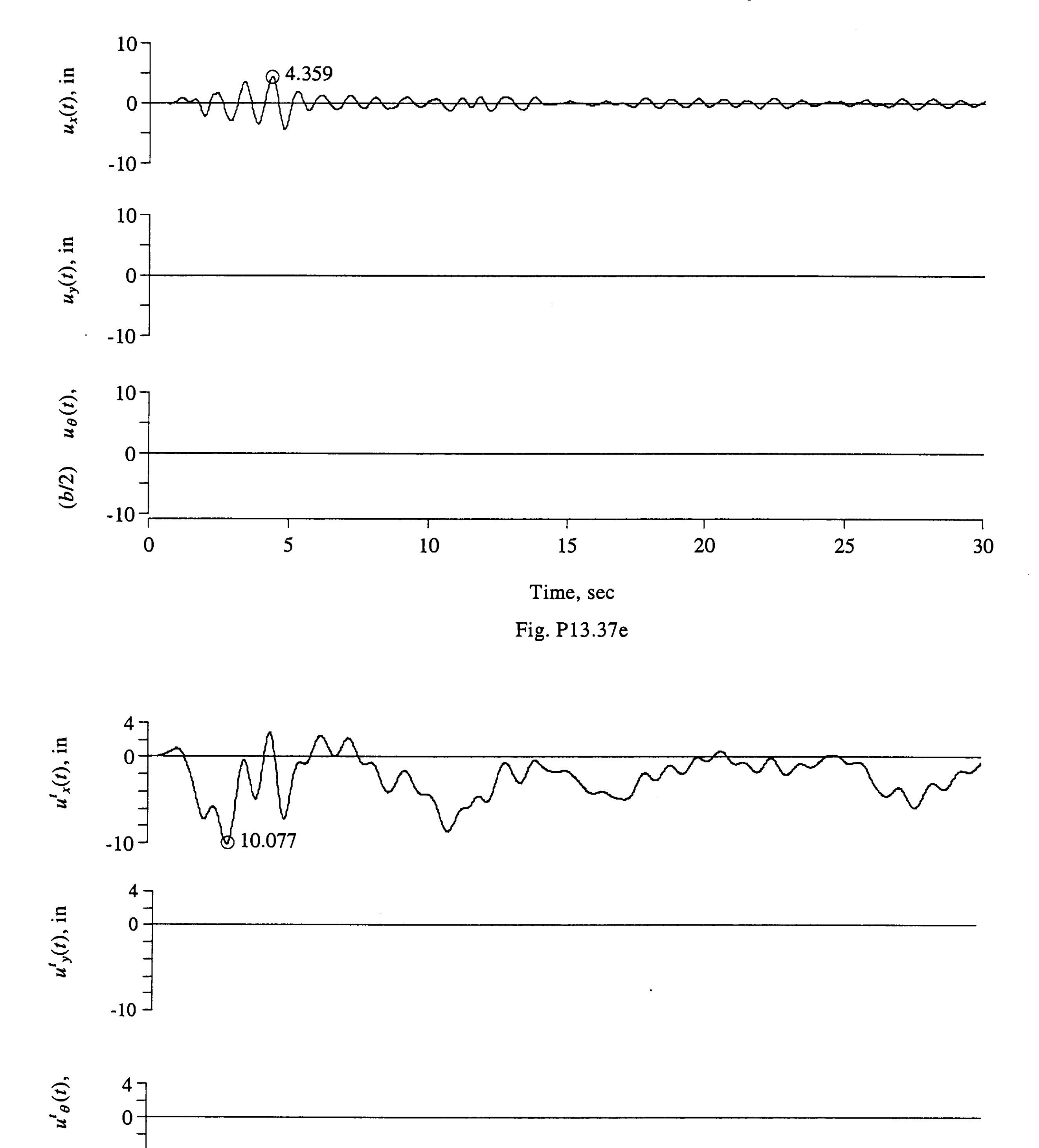
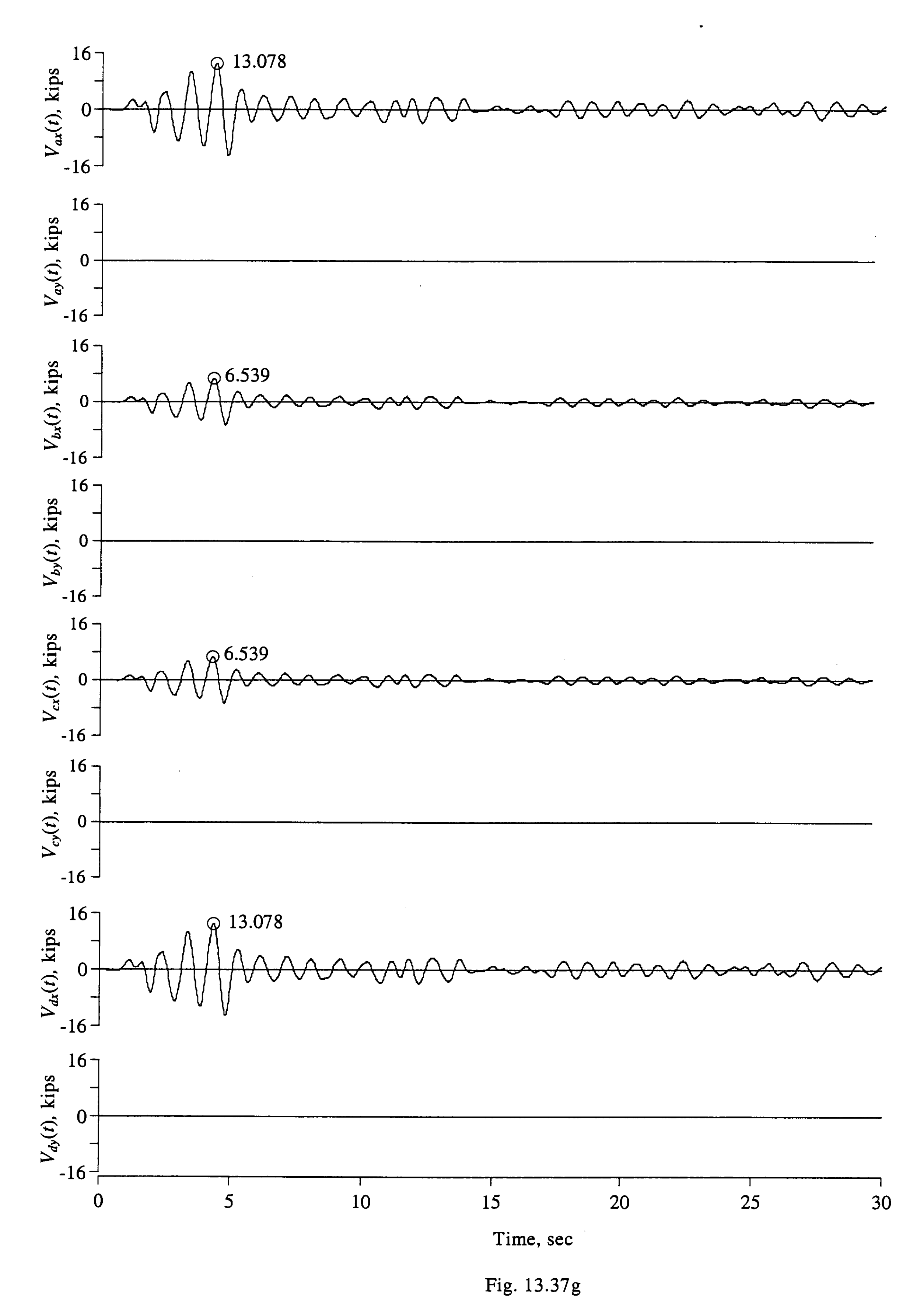


Fig. P13.37e

Fig. P13.37f



16
0
-16
13.078
$V_{ax}(t)$, kips
$V_{ay}(t)$, kips
$V_{bx}(t)$, kips
6.539
$V_{by}(t)$, kips
$V_{cx}(t)$, kips
6.539
$V_{cy}(t)$, kips
$V_{dx}(t)$, kips
13.078
$V_{dy}(t)$, kips
0
5
10
15
20
25
30
Time, sec

Fig. 13.37g

Problem 13.38

Part a

From Problem 9.25,

$$\mathbf{k} = 10^4 \begin{bmatrix} 0.9359 & 0.7701 \\ 0.7701 & 1.5088 \end{bmatrix} \text{kips/in.}$$

$$\mathbf{m} = \begin{bmatrix} 3.624 & \\ & 1.812 \end{bmatrix} \text{kip} - \text{sec}^2/\text{in.}$$

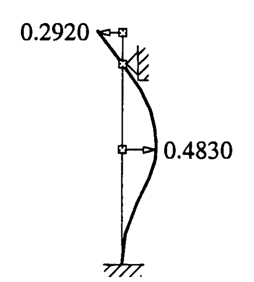
$$\iota_1 = \langle 0.6035 \quad -0.2143 \rangle^T$$

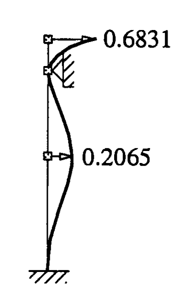
$$\iota_2 = \langle 0.3965 \quad 1.2143 \rangle^T$$

1. Evaluate *natural frequencies and modes.*

$$\omega_1 = 36.02 \qquad \omega_2 = 98.04$$

$$\phi_1 = \begin{Bmatrix} 0.4830 \\ -0.2920 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 0.2065 \\ 0.6831 \end{Bmatrix}$$





2. *Determine* $\Gamma_{nl} = L_{nl}/M_n$.

$$L_{nl} = \phi_n^T \mathbf{m}\, \iota_l$$

$$M_n = 1$$

$$\iota_1 = \langle 0.6035 \quad -0.2143 \rangle^T \qquad \text{(a)}$$
$$\iota_2 = \langle 0.3965 \quad 1.2143 \rangle^T$$

$$\Gamma = \left[\Gamma_{nl}\right] = \begin{bmatrix} 1.1698 & 0.0515 \\ 0.1864 & 1.7997 \end{bmatrix} \begin{matrix} \leftarrow \textbf{mode 1} \\ \leftarrow \textbf{mode 2} \end{matrix} \qquad \text{(b)}$$

$$\uparrow \ddot{u}_{g1} \qquad \uparrow \ddot{u}_{g2}$$

3. *Determine response of the* n^{th}*-mode SDF system to* $\ddot{u}_{gl}(t)$.

$$u_{g1}(t) = u_g(t) \qquad u_{g2}(t) = u_g(t - t')$$

$$D_{n1}(t) = D_n(t) \qquad D_{n2}(t) = D_n(t - t') \quad \text{(c)}$$

$$A_{n1}(t) = A_n(t) \qquad A_{n2}(t) = A_n(t - t')$$

4. *Determine displacements.*

$$\mathbf{u}^t(t) = \sum_{l=1}^{2} \iota_l\, u_{gl}(t) + \sum_{l=1}^{2} \sum_{n=1}^{2} \Gamma_{nl}\, \phi_n\, D_{nl}(t) \qquad \text{(d)}$$

Substituting Eqs. (a), (b) and (c) in Eq. (d) gives

$$\begin{Bmatrix} u_1^t(t) \\ u_2^t(t) \end{Bmatrix} = \begin{Bmatrix} 0.6035 \\ -0.2143 \end{Bmatrix} u_g(t) + \begin{Bmatrix} 0.3965 \\ 1.2143 \end{Bmatrix} u_g(t - t')$$
$$+ 1.1698 \begin{Bmatrix} 0.4830 \\ -0.2920 \end{Bmatrix} D_1(t)$$
$$+0.0515 \begin{Bmatrix} 0.4830 \\ -0.2920 \end{Bmatrix} D_1(t - t')$$
$$+ 0.1864 \begin{Bmatrix} 0.2065 \\ 0.6831 \end{Bmatrix} D_2(t)$$
$$+ 1.7997 \begin{Bmatrix} 0.2065 \\ 0.6831 \end{Bmatrix} D_2(t - t') \qquad \text{(e)}$$

The displacement at the top of the tower is

$$u_2^t(t) = -0.2143\, u_g(t) + 1.2143\, u_g(t - t')$$
$$-0.3416\, D_1(t) - 0.0150\, D_1(t - t')$$
$$+ 0.1273\, D_2(t) + 1.2294\, D_2(t - t') \qquad \text{(f)}$$

5. *Compute equivalent static forces.*

$$\mathbf{f}_S(t) = \sum_{l=1}^{2} \sum_{n=1}^{2} \Gamma_{nl}\, \mathbf{m}\, \phi_n\, A_{nl}(t) \qquad \text{(g)}$$

Substituting for $\mathbf{m}$ and ϕ_n, Eq. (b) for Γ_{nl}, and Eq. (c) for $A_{nl}(t)$ gives

$$\mathbf{f}_S(t) = -\begin{Bmatrix} 2.0476 \\ -0.6189 \end{Bmatrix} A_1(t) + \begin{Bmatrix} 0.0901 \\ -0.0272 \end{Bmatrix} A_1(t - t')$$
$$+ \begin{Bmatrix} 0.1395 \\ 0.2307 \end{Bmatrix} A_2(t) + \begin{Bmatrix} 1.3469 \\ 2.2277 \end{Bmatrix} A_2(t - t')$$
$$\text{(h)}$$

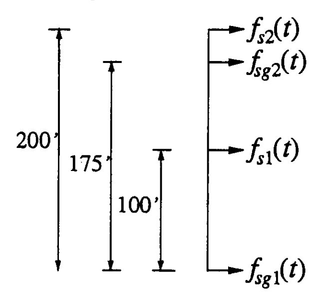
6. *Compute equivalent static support forces.*

$$\mathbf{f}_{S_g}(t) = \mathbf{k}_g^T\, \mathbf{u}(t) + \left(\mathbf{k}_{gg} + \mathbf{k}_g^T\, \iota\right) \mathbf{u}_g(t) \qquad \text{(i)}$$

Substituting for $\mathbf{k}_g$ and $\mathbf{k}_{gg}$ from Problem 9.25 and $\mathbf{u}(t)$ from the second half of Eq. (d), and ι from Eq. (a) gives

$$\mathbf{f}_{S_g}(t) = -\begin{Bmatrix} 1.3686 \\ 0.0603 \end{Bmatrix} A_1(t) - \begin{Bmatrix} 0.0603 \\ 0.0027 \end{Bmatrix} A_1(t-t') - \begin{Bmatrix} 0.0348 \\ 0.3355 \end{Bmatrix} A_2(t) - \begin{Bmatrix} 0.3355 \\ 3.2388 \end{Bmatrix} A_2(t-t') + 159.46 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_g(t) \\ u_g(t-t') \end{Bmatrix} \tag{j}$$

7. *Compute internal forces.*



Observe that at each time instant, the equivalent static forces shown in the accompanying figure and defined by Eqs. (h) and (j) are in equilibrium. By statics, the shear at the base is

$$V_b(t) = f_{S_{g1}}(t) \text{ kips} \tag{k.1}$$

The moment at the base is

$$M_b(t) = \left[200\, f_{S2}(t) + 175\, f_{S_{g2}}(t) + 100\, f_{S1}(t)\right] \text{kip-ft} \tag{k.2}$$

and the

$$\text{axial force in the bridge} = f_{S_{g2}}(t) \text{ kips} \tag{k.3}$$

Part b

If both supports undergo identical motion $u_g(t)$, the relative motion of the structure is

$$\mathbf{u}(t) = \sum_{n=1}^{2} \Gamma_n \phi_n D_n(t) \tag{l}$$

where

$$\Gamma_n = \phi_n^T \mathbf{m} \mathbf{1}$$

Substituting ϕ_n and $\mathbf{m}$ gives

$$\Gamma_1 = 1.2212 \qquad \Gamma_2 = 1.9861$$

Substituting ϕ_n and Γ_n in Eq. (l) gives

$$\mathbf{u}(t) = 1.2212 \begin{Bmatrix} 0.4830 \\ -0.2920 \end{Bmatrix} D_1(t) + 1.9861 \begin{Bmatrix} 0.2065 \\ 0.6831 \end{Bmatrix} D_2(t) \tag{m}$$

The total displacements are

$$\mathbf{u}^t(t) = u_g(t) \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \mathbf{u}(t) \tag{n}$$

In particular, the displacement at top of the tower is

$$u_2^t(t) = u_g(t) - 0.3567\, D_1(t) + 1.3566\, D_2(t) \tag{o}$$

The equivalent static forces are

$$\mathbf{f}_S(t) = \sum_{n=1}^{2} \Gamma_n \mathbf{m} \phi_n A_n(t) \tag{p}$$

Substituting for Γ_n, $\mathbf{m}$, and ϕ_n gives

$$\mathbf{f}_S(t) = \begin{Bmatrix} 2.1375 \\ -0.6461 \end{Bmatrix} A_1(t) + \begin{Bmatrix} 1.4863 \\ 2.4583 \end{Bmatrix} A_2(t) \tag{q}$$

The equivalent static support forces are obtained by substituting $A_n(t) = A_n(t-t')$ and $u_g(t) = u_g(t-t')$ in Eq. (j):

$$\mathbf{f}_{S_g}(t) = -\begin{Bmatrix} 1.4289 \\ 0.0630 \end{Bmatrix} A_1(t) - \begin{Bmatrix} 0.3703 \\ 35743 \end{Bmatrix} A_2(t) \tag{r}$$

These support forces could also be obtained by static analysis of the structure subjected to $\mathbf{f}_S(t)$.

With $\mathbf{f}_S$ and $\mathbf{f}_{S_g}$ given by Eqs. (q) and (r), the internal forces are given by Eqs. (k).

Comments

If the support motions are different, quasistatic support forces are developed. These disappear if the support motions are identical.

Problem 13.39

Initial calculations

Substituting values of $E = 29{,}000$ ksi, $I = 727$ in.^{4}, $m = 100$ kips/g, and $h = 12$ ft in the solution for Problem 13.4 gives:

$$\omega_1 = 2.407\sqrt{\frac{EI}{mh^3}} = 12.57 \text{ rad/sec}$$

$$\omega_2 = 7.193\sqrt{\frac{EI}{mh^3}} = 37.55 \text{ rad/sec}$$

$$T_1 = \frac{2\pi}{12.57} = 0.5 \text{ sec} \qquad T_2 = \frac{2\pi}{37.55} = 0.167 \text{ sec}$$

Corresponding to these periods, the spectral ordinates are

$$A_1 = \frac{1}{3}(2.71\text{g}) = 348.7 \text{ in./sec}^2$$

$$A_2 = \frac{1}{3}(2.71\text{g}) = 348.7 \text{ in./sec}^2$$

$$D_1 = \frac{A_1}{\omega_1^2} = 2.208 \text{ in.} \qquad D_2 = \frac{A_2}{\omega_2^2} = 0.247 \text{ in.}$$

Part a

Substituting the above D_1 and D_2 for $D_1(t)$ and $D_2(t)$ in Eqs. (a) and (c) of the solution to Problem 13.4 gives the peak values of floor displacements due to each of the two modes:

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.647 \\ 1.341 \end{Bmatrix}(2.208) = \begin{Bmatrix} 1.429 \\ 2.961 \end{Bmatrix} \text{in.}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_2 = \begin{Bmatrix} 0.353 \\ -0.341 \end{Bmatrix}(0.247) = \begin{Bmatrix} 0.087 \\ -0.084 \end{Bmatrix} \text{in.}$$

Using the SRSS rule gives an estimate of the total displacements:

$$u_1 \approx \sqrt{(1.429)^2 + (0.087)^2} = 1.43 \text{ in.}$$

$$u_2 \approx \sqrt{(2.961)^2 + (-0.084)^2} = 2.96 \text{ in.}$$

Part b

Bending moments in a first-story column [from Eqs. (k) and (l) in the solution to Problem 13.4] are determined as follows.

First mode:

$$\begin{aligned} M_{a1} &= 0.216\, mhA_1 \\ &= 0.216\,(100/386)\,(12 \times 12)\, 348.7 \\ &= 2810 \text{ kip-in.} = 234.1 \text{ kip-ft} \end{aligned}$$

$$M_{b1} = 0.443\, mhA_1 = 480.2 \text{ kip-ft}$$

Second mode:

$$M_{a2} = 0.0473\, mhA_2 = 51.27 \text{ kip-ft}$$

$$M_{b2} = 0.0441\, mhA_2 = 47.80 \text{ kip-ft}$$

Using the SRSS rule gives an estimate of the total bending moments:

$$M_a \approx \sqrt{(234.1)^2 + (51.27)^2} = 239.69 \text{ kip-ft}$$

$$M_b \approx \sqrt{(480.2)^2 + (47.80)^2} = 482.58 \text{ kip-ft}$$

Bending moments in a second-floor beam [from Eqs. (m) and (n) in the solution to Problem 13.4] are determined as follows.

First mode:

$$M_{a1} = -0.211\, mhA_1 = -228.7 \text{ kip-ft}$$

$$M_{b1} = -0.211\, mhA_1 = -228.7 \text{ kip-ft}$$

Second mode:

$$M_{a2} = 0.0332\, mhA_2 = 35.99 \text{ kip-ft}$$

$$M_{b2} = 0.0332\, mhA_2 = 35.99 \text{ kip-ft}$$

Using the SRSS rule gives the total values:

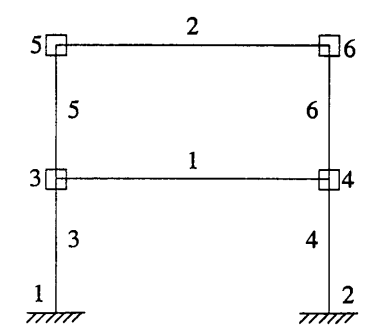
$$M_a \approx \sqrt{(-228.7)^2 + (35.99)^2} = 231.53 \text{ kip-ft}$$

$$M_b \approx \sqrt{(-228.7)^2 + (35.99)^2} = 231.53 \text{ kip-ft}$$

Table P13.39 summarizes the bending moments in all elements which were calculated similarly.

Table P13.39: Peak bending moments, kip-ft

Element	Node	Mode 1	Mode 2	Total
Beam 1	3	– 369	5	369
	4	– 369	5	369
Beam 2	5	229	36	232
	6	229	36	232
Column 3	3	234	51	240
	1	480	48	483
Column 5	5	229	– 36	232
	3	136	– 56	147


2
5
6
5
6
1
3
4
3
4
1
2

Problem 13.57

From Problem 13.24, **m**, ω_n, ϕ_n, and Γ_n are available.

1. *Determine spectral ordinates.*

The spectral ordinates are determined from Fig. 6.9.4 scaled by 0.5:

$$\omega_1 = 5.96,\ T_1 = 1.054 \text{ sec} \Rightarrow$$
$$D_1 = 0.5 \times 110.4/5.96 = 9.26 \text{ in.}$$
$$\omega_2 = 6.21,\ T_2 = 1.012 \text{ sec} \Rightarrow$$
$$D_2 = 0.5 \times 110.4/6.21 = 8.89 \text{ in.}$$
$$\omega_3 = 10.90,\ T_3 = 0.576 \text{ sec} \Rightarrow$$
$$D_3 = 0.5 \times 2.71g/(10.90)^2 = 4.40 \text{ in.}$$

2. *Determine peak modal responses.*

Due to mode n, the peak displacements in the x, y, and θ directions are

$$u_{xn} = 0 \qquad u_{yn} = \Gamma_n \phi_{yn} D_n \qquad u_{\theta n} = \Gamma_n \phi_{\theta n} D_n \tag{a}$$

The base shear (x and y components) and base torque due to mode n are

$$V_{bxn} = 0$$
$$V_{byn} = f_{yn} = \Gamma_n m \phi_{yn} A_n \tag{b}$$
$$T_{bn} = f_{\theta n} = \Gamma_n (mr^2) \phi_{\theta n} A_n$$

In Eqs. (a) and (b), we substitute for m, r, ϕ_{yn}, $\phi_{\theta n}$ and Γ_n from Problem 13.24; D_n from Part a; and $A_n = \omega_n^2 D_n$ to obtain the peak modal responses in Table P13.57a.

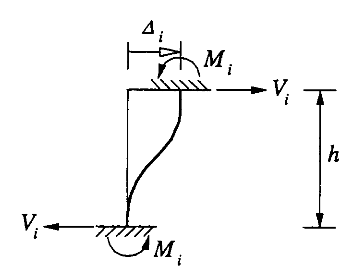
Table P13.57a

Response	Peak modal responses			Peak total response	
	$n=1$	$n=2$	$n=3$	CQC	SRSS
u_x, in.	0	0	0	0	0
u_y, in.	8.917	0	0.163	8.922	8.918
$(b/2)\,u_\theta$, in.	2.143	0	-1.018	0.0157	0.0158
V_{bx}, kips	0	0	0	0	0
V_{by}, kips	73.8	0	4.5	74.1	74.0
T_b, kip-in.	1774.7	0	-2823.1	3297.1	3334.4

3. *Combine peak modal responses.*

The modal peaks (from Part b) and correlation coefficients ρ_{in} from Table P13.56b are substituted in Eqs. (13.7.3) and (13.7.4) for each response quantity to obtain estimates of the peak value of the total response. The results using SRSS and CQC methods are presented in Table P13.57a.

4. *Determine bending moments in columns.*



The bending moment M_i at the ends of a clamped-clamped column i is related to the lateral displacement Δ_i of the column as follows:

$$M_i = V_i \frac{h}{2} = \frac{kh}{2} \Delta_i \tag{c}$$

where the two components of Δ_i — Δ_{ix} and Δ_{iy} — cause moments M_y and M_x about the y and x axes, respectively.

We illustrate the procedure for column a in Fig. P13.24. The peak modal values of Δ_{ax} and Δ_{ay} are

$$\begin{aligned} \Delta_{axn} &= \Gamma_n (-y_a \phi_{\theta n}) D_n = -y_a u_{\theta n} \\ \Delta_{ayn} &= \Gamma_n (\phi_{yn} + x_a \phi_{\theta n}) D_n = u_{yn} + x_a u_{\theta n} \end{aligned} \tag{d}$$

For mode 1 ($n = 1$), substituting the peak values of u_x and u_θ from Table P13.57a in Eqs. (d) and (c) gives

$$\Delta_{ax1} = -(b/2)\,u_{\theta 1} = -2.143 \text{ in.}$$
$$\Delta_{ay1} = u_{y1} - (b/2)\,u_{\theta 1} = 8.917 - 2.143 = 6.774 \text{ in.}$$
$$M_{ay1} = \frac{(2 \times 1.5)(12 \times 12)}{2} \Delta_{ax1} = 216\,\Delta_{ax1} = 216(-2.143) = -462.9 \text{ kip-in.}$$
$$M_{ax1} = 216\,\Delta_{ay1} = 216(6.774) = 1463 \text{ kip-in.}$$

The computations for mode 3 proceed similarly:

$$\Delta_{ax3} = -(b/2)\,u_{\theta 3} = 1.018 \text{ in.}$$
$$\Delta_{ay3} = u_{y3} - (b/2)\,u_{\theta 3} = 0.163 - (-1.018) = 1.181 \text{ in.}$$
$$M_{ay3} = 216\,\Delta_{ax3} = 216(1.018) = 220 \text{ kip-in.}$$
$$M_{ax3} = 216\,\Delta_{ay3} = 216(1.181) = 255.2 \text{ kip-in.}$$

Specializing Eqs. (13.7.3) and (13.7.4) for the SRSS and CQC methods to response M_{ay}, and substituting for M_{ay1} and M_{ay3} (from above) and $\rho_{13} = 0.0247$ (from Problem 13.56) gives

$$M_{ay} = \sqrt{M_{ay1}^2 + M_{ay3}^2} = 512.5 \text{ kip-in.}$$

$$M_{ay} = \sqrt{M_{ay1}^2 + M_{ay3}^2 + 2\rho_{13} M_{ay1} M_{ay3}} = 507.6 \text{ kip-in.}$$

Similar computations lead to Table P13.57b for M_{ax} and bending moments in other columns.

Table P13.57b: Column Moments in kip-in.

Response	Peak modal responses			Peak total response	
	$n=1$	$n=2$	$n=3$	CQC	SRSS
M_{ay}	-462.9	0	220	507.6	512.6
M_{ax}	1463	0	255.2	1491.3	1485.1
M_{by}	-231.5	0	110.0	253.8	256.3
M_{bx}	1194.5	0	-92.4	1195.7	1198.0
M_{cy}	231.5	0	-110.0	253.8	256.3
M_{cx}	1194.5	0	-92.4	1195.7	1198.0
M_{dy}	462.9	0	-220	507.6	512.6
M_{dx}	1463	0	255.2	1491.3	1485.1

In this case both CQC and SRSS methods render similar results because modes 1 and 3, which are the only contributions to the response, have well separated frequencies.

Problem 13.40

Initial calculations

From Problem 13.2, the modal properties are

$$\phi_1 = \begin{Bmatrix} 1.389 \\ 1.965 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1.389 \\ -1.965 \end{Bmatrix}$$

$$\Gamma_1 = 0.614 \qquad \Gamma_2 = 0.105$$

Part a: Spectral ordinates

From Problem 13.2, the peak values of $D_n(t)$ and $A_n(t)$ are

$$D_1 = 0.797 \text{ in.} \qquad D_2 = 0.118 \text{ in.}$$

$$A_1 = 0.791\text{g} \qquad A_2 = 0.684\text{g}$$

Part b

1. *Peak responses due to mode 1*

Displacements:

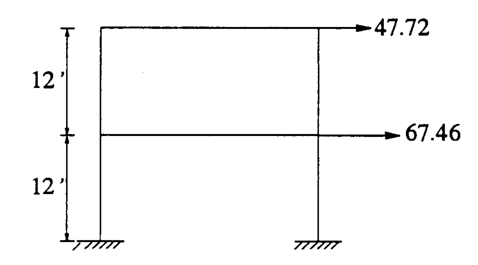
$$\mathbf{u}_1 = \Gamma_1 \phi_1 D_1 = 0.614 \begin{Bmatrix} 1.389 \\ 1.965 \end{Bmatrix} (0.797) = \begin{Bmatrix} 0.680 \\ 0.962 \end{Bmatrix} \text{in.}$$

Equivalent static forces:

$$\mathbf{f}_1 = \Gamma_1 \begin{Bmatrix} m_1\phi_{11} \\ m_2\phi_{21} \end{Bmatrix} A_1 = 0.614 \frac{100}{\text{g}} \begin{Bmatrix} 1\,(1.389) \\ 0.5\,(1.965) \end{Bmatrix} (0.791\text{g})$$

$$= \begin{Bmatrix} 67.46 \\ 47.72 \end{Bmatrix} \text{kips}$$

Story shears:



$$V_{21} = 47.72 \text{ kips}$$

$$V_{11} = 67.46 + 47.72 = 115.18 \text{ kips}$$

Overturning moments: These moments are denoted by M_1 at the first floor level and M_b at the base.

$$M_{11} = 47.72\,(12) = 572.64 \text{ kip-ft}$$

$$M_{b1} = 67.46\,(12) + 47.72\,(24) = 1954.8 \text{ kip-ft}$$

2. *Peak responses due to mode 2.*

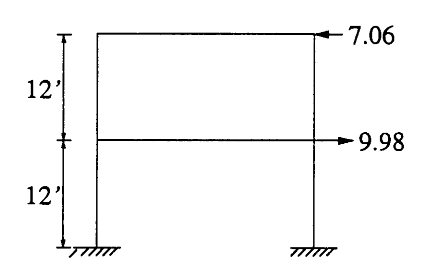
$$\mathbf{u}_2 = \Gamma_2 \phi_2 D_2 = 0.105 \begin{Bmatrix} 1.389 \\ -1.965 \end{Bmatrix} (0.118) = \begin{Bmatrix} 0.0172 \\ -0.0243 \end{Bmatrix} \text{in.}$$

Equivalent static forces:

$$\mathbf{f}_2 = \Gamma_2 \begin{Bmatrix} m_1\phi_{12} \\ m_2\phi_{22} \end{Bmatrix} A_2 = 0.105 \frac{100}{\text{g}} \begin{Bmatrix} 1\,(1.389) \\ 0.5\,(-1.965) \end{Bmatrix} (0.684\text{g})$$

$$= \begin{Bmatrix} 9.98 \\ -7.06 \end{Bmatrix} \text{kips}$$

Story shears:



$$V_{22} = -7.06 \text{ kips}$$

$$V_{12} = 9.98 - 7.06 = 2.92 \text{ kips}$$

Overturning moments:

$$M_{12} = -7.06\,(12) = -84.72 \text{ kip-ft}$$

$$M_{b2} = 9.98\,(12) + (-7.06)\,(24) = -49.68 \text{ kip-ft}$$

The above-determined responses due to each mode should be identical to those determined in Problem 13.2 by RHA; the slight differences are due to numerical truncation errors in the computations for this problem whereas more significant digits were used in the computer work for Problem 13.2.

Part c

The peak modal responses are combined by the SRSS rule:

$$r = \sqrt{r_1^2 + r_2^2}$$

The resulting RSA estimate and the RHA results (from Problem 13.2) are presented in Table P13.40.

Table P13.40

Response	RSA	RHA	Error, %
u_1, in.	0.680	0.679	0.15
u_2, in.	0.962	0.964	0.21
V_b, kips	115.22	115.11	0.10
V_2, kips	48.24	49.56	2.66
M_b, kip-ft	1955.43	1959.25	0.20
M_1, kip-ft	578.87	594.65	2.65

***Part d**: Comments*

For this problem, the RSA method gives results that are very close to the RHA results. The errors are small for all response quantities because the first mode response is dominant.

Problem 13.41

Initial calculations

Substituting values of $E = 29{,}000$ ksi, $I = 1400$ in.^{4}, $m = 100$ kips/g and $h = 12$ ft in the solution for Problem 13.11 gives:

$$\omega_1 = 10.57 \qquad \omega_2 = 34.56 \qquad \omega_3 = 59.42$$

$$T_1 = 0.595 \text{ sec} \quad T_2 = 0.182 \text{ sec} \quad T_3 = 0.106 \text{ sec}$$

From the design spectrum of Fig. 6.9.5, scaled to $u_{go} = (1/3)\,\text{g}$, the spectral ordinates are

$$D_1 = 3.13 \text{ in.} \quad D_2 = 0.292 \text{ in.} \quad D_3 = 0.088 \text{ in.}$$

$$\frac{A_1}{\text{g}} = 0.903 \qquad \frac{A_2}{\text{g}} = 0.903 \qquad \frac{A_3}{\text{g}} = 0.803$$

***Part a**: Floor displacements*

Substituting numerical values for Γ_n and ϕ_n from Problem 13.11 and D_n values above in Eq. (13.8.1a) gives the peak displacements due to each of the three modes:

$$\mathbf{u}_1 = \Gamma_1\,\phi_1\,D_1 = 1.3513\begin{Bmatrix}0.3156\\0.7451\\1\end{Bmatrix}(3.13) = \begin{Bmatrix}1.335\\3.152\\4.230\end{Bmatrix}\text{ in.}$$

$$\mathbf{u}_2 = \Gamma_2\,\phi_2\,D_2 = -0.5083\begin{Bmatrix}-0.7409\\-0.3572\\1\end{Bmatrix}(0.292) = \begin{Bmatrix}0.110\\0.053\\-0.148\end{Bmatrix}\text{ in.}$$

$$\mathbf{u}_3 = \Gamma_3\,\phi_3\,D_3 = 0.1569\begin{Bmatrix}1.2546\\-1.2024\\1\end{Bmatrix}(0.088) = \begin{Bmatrix}0.017\\-0.017\\0.014\end{Bmatrix}\text{ in.}$$

Combining modal displacements by the SRSS rule gives

$$u_1 \approx \sqrt{(1.335)^2 + (0.110)^2 + (0.017)^2} = 1.340 \text{ in.}$$

$$u_2 \approx \sqrt{(3.152)^2 + (0.053)^2 + (-0.017)^2} = 3.152 \text{ in.}$$

$$u_3 \approx \sqrt{(4.230)^2 + (-0.148)^2 + (0.014)^2} = 4.233 \text{ in.}$$

Observe that the first mode contributes essentially the entire floor displacements.

***Part b**: Element forces*

To compute the element forces use the results from Problem 13.11 and replace $A_n(t)$ by the spectral values A_n.

The bending moments in a first-story column due to the first mode are

$$M_{a1} = 0.7526\,mh\,A_1 = 0.7526\,(100/\text{g})\,(12)\,0.903\text{g} = 81552 \text{ kip-ft}$$

$$M_{b1} = 0.3014\,mh\,A_1 = 0.3014\,(100/\text{g})\,(12)\,0.903\text{g} = 326.60 \text{ kip-ft}$$

These bending moments due to the second mode are

$$M_{a2} = 0.08381\,mh\,A_2 = 0.08381\,(100/\text{g})\,(12)\,0.903\text{g} = 90.82 \text{ kip-ft}$$

$$M_{b2} = 0.06823\,mh\,A_2 = 0.06823\,(100/\text{g})\,(12)\,0.903\text{g} = 73.93 \text{ kip-ft}$$

Due to the third mode the bending moments are

$$M_{a3} = 0.02030\,mh\,A_3 = 0.02030\,(100/\text{g})\,(12)\,0.803\text{g} = 19.56 \text{ kip-ft}$$

$$M_{b3} = 0.02302\,mh\,A_3 = 0.02302\,(100/\text{g})\,(12)\,0.803\text{g} = 22.18 \text{ kip-ft}$$

Combining modal responses by the SRSS rule gives

$$M_a \approx \sqrt{(815.52)^2 + (90.82)^2 + (19.56)^2} = 820.80 \text{ kip-ft}$$

$$M_b \approx \sqrt{(326.60)^2 + (73.93)^2 + (22.18)^2} = 335.60 \text{ kip-ft}$$

The bending moments in the second-story beam due to each of the three modes are computed similarly to obtain

$$M_{a1} = M_{b1} = -586.66 \text{ kip-ft}$$
$$M_{a2} = M_{b2} = 61.44 \text{ kip-ft}$$
$$M_{a3} = M_{b3} = 2.68 \text{ kip-ft}$$

Combining modal responses by the SRSS rule gives

$$M_a = M_b \approx \sqrt{(-586.66)^2 + (61.44)^2 + (2.68)^2} = 589.87 \text{ kip-ft}$$

Problem 13.42

Initial calculations

Substituting values of E = 29,000 ksi, I = 1400 in^4, m = 100 kips/g and h = 12 ft in the solution for Problem 13.12 gives

$$\omega_1 = 8.67 \text{ rad/sec} \quad \omega_2 = 30.27 \text{ rad/sec} \quad \omega_3 = 57.25 \text{ rad/sec}$$

$$T_1 = 0.725 \text{ sec} \quad T_2 = 0.207 \text{ sec} \quad T_3 = 0.109 \text{ sec}$$

From the design spectrum of Fig. 6.9.5, scaled to $\ddot{u}_{g0} = (1/3)g$, the spectral ordinates are

$$D_1 = 4.25 \text{ in.} \quad D_2 = 0.381 \text{ in.} \quad D_3 = 0.096 \text{ in.}$$

$$\frac{A_1}{g} = 0.828 \quad \frac{A_2}{g} = 0.903 \quad \frac{A_3}{g} = 0.819$$

Part a: *Floor displacements*

Substituting numerical values for Γ_n and ϕ_n from Problem 13.12 and D_n values above in Eq. (13.8.1a) gives the peak displacements due to each of the three modes:

$$\mathbf{u}_1 = \Gamma_1\phi_1 D_1 = 1.386\begin{Bmatrix} 0.273 \\ 0.698 \\ 1 \end{Bmatrix}(4.25) = \begin{Bmatrix} 1.608 \\ 4.112 \\ 5.89 \end{Bmatrix} \text{ in.}$$

$$\mathbf{u}_2 = \Gamma_2\phi_2 D_2 = -0.542\begin{Bmatrix} -0.706 \\ -0.441 \\ 1 \end{Bmatrix}(0.381) = \begin{Bmatrix} 0.146 \\ 0.091 \\ -0.206 \end{Bmatrix} \text{ in.}$$

$$\mathbf{u}_3 = \Gamma_3\phi_3 D_3 = 0.156\begin{Bmatrix} 1.529 \\ -1.315 \\ 1 \end{Bmatrix}(0.096) = \begin{Bmatrix} 0.022 \\ -0.019 \\ 0.015 \end{Bmatrix} \text{ in.}$$

Combining modal displacements by the SRSS rule gives

$$u_1 \approx \sqrt{(1.608)^2 + (0.146)^2 + (0.022)^2} = 1.615$$

$$u_2 \approx \sqrt{(4.112)^2 + (0.091)^2 + (-0.019)^2} = 4.117$$

$$u_3 \approx \sqrt{(5.89)^2 + (-0.206)^2 + (0.015)^2} = 5.893$$

Observe that the first mode contributes essentially the entire floor displacements.

Part b: *Element forces*

To compute the element forces use the results from Problem 13.12 and replace $A_n(t)$ by the spectral values A_n.

The bending moments in a first-story column due to the first mode are

$$M_{a1} = 0.697mhA_1 = 0.697(100/g)(12)0.828g$$
$$= 692.5 \text{ kip-ft}$$
$$M_{b1} = 0.151mhA_1 = 0.151(100/g)(12)0.828g$$
$$= 150.0 \text{ kip-ft}$$

The bending moments due to the second mode are

$$M_{a2} = 0.102mhA_2 = 0.102(100/g)(12)0.903g$$
$$= 110.5 \text{ kip-ft}$$
$$M_{b2} = 0.073mhA_2 = 0.073(100/g)(12)0.903g$$
$$= 79.10 \text{ kip-ft}$$

Due to the third mode the bending moments are

$$M_{a3} = 0.026mhA_3 = 0.026(100/g)(12)0.819g$$
$$= 25.55 \text{ kip-ft}$$
$$M_{b3} = 0.029mhA_3 = 0.029(100/g)(12)0.819g$$
$$= 28.50 \text{ kip-ft}$$

Combining modal responses by the SRSS rule gives

$$M_a \approx \sqrt{(692.5)^2 + (110.5)^2 + (25.55)^2} = 701.8 \text{ kip-ft}$$

$$M_b \approx \sqrt{(150.0)^2 + (79.10)^2 + (28.5)^2} = 169.6 \text{ kip-ft}$$

The bending moments in the second-story beam due to each of the three modes are computed similarly to obtain

$$M_{a1} = M_{b1} = -973.7 \text{ kip-ft}$$
$$M_{a2} = M_{b2} = 44.43 \text{ kip-ft}$$
$$M_{a3} = M_{b3} = -9.63 \text{ kip-ft}$$

Combining modal displacements by the SRSS rule gives

$$M_a = M_b \approx \sqrt{(-973.7)^2 + (44.43)^2 + (-9.63)^2} = 974.8 \text{ kip-ft}$$

Problem 13.43

Initial calculations

Substituting values of E = 29,000 ksi, I = 1400 in^4, m = 100 kips/g and h = 12 ft in the solution for Problem 13.13 gives

$$\omega_1 = 9.63 \text{ rad/sec} \quad \omega_2 = 25.46 \text{ rad/sec} \quad \omega_3 = 47.54 \text{ rad/sec}$$

$$T_1 = 0.653 \text{ sec} \quad T_2 = 0.247 \text{ sec} \quad T_3 = 0.132 \text{ sec}$$

From the design spectrum of Fig. 6.9.5, scaled to u_{g0}=(1/3)g, the spectral ordinates are

$$D_1 = 3.761 \text{ in.} \quad D_2 = 0.538 \text{ in.} \quad D_3 = 0.154 \text{ in.}$$

$$\frac{A_1}{g} = 0.903 \quad \frac{A_2}{g} = 0.903 \quad \frac{A_3}{g} = 0.903$$

***Part a**: Floor displacements*

Substituting numerical values for Γ_n and ϕ_n from Problem 13.13 and D_n values above in Eq. (13.8.1a) gives the peak displacements due to each of the three modes:

$$\mathbf{u}_1 = \Gamma_1\phi_1 D_1 = 1.426\begin{Bmatrix} 0.234 \\ 0.639 \\ 1 \end{Bmatrix}(3.761) = \begin{Bmatrix} 1.255 \\ 3.427 \\ 5.363 \end{Bmatrix} \text{in.}$$

$$\mathbf{u}_2 = \Gamma_2\phi_2 D_2 = -0.543\begin{Bmatrix} -0.512 \\ -0.591 \\ 1 \end{Bmatrix}(0.538) = \begin{Bmatrix} 0.15 \\ 0.173 \\ -0.292 \end{Bmatrix} \text{in.}$$

$$\mathbf{u}_3 = \Gamma_3\phi_3 D_3 = 0.114\begin{Bmatrix} 3.324 \\ -2.032 \\ 1 \end{Bmatrix}(0.154) = \begin{Bmatrix} 0.058 \\ -0.036 \\ 0.018 \end{Bmatrix} \text{in.}$$

Combining modal displacements by the SRSS rule gives

$$u_1 \approx \sqrt{(1.255)^2 + (0.15)^2 + (0.058)^2} = 1.265$$

$$u_2 \approx \sqrt{(3.427)^2 + (0.173)^2 + (-0.036)^2} = 3.432$$

$$u_3 \approx \sqrt{(5.363)^2 + (-0.292)^2 + (0.018)^2} = 5.371$$

Observe that the first mode contributes essentially the entire floor displacements.

***Part b**: Element forces*

To compute the element forces use the results from Problem 13.13 and replace $A_n(t)$ by the spectral values A_n

The bending moments in a first-story column due to the first mode are

$$M_{a1} = 1555 mhA_1 = 1555(100/g)(12)0.903g$$
$$= 1685 \text{ kip-ft}$$

$$M_{b1} = 0.243 mhA_1 = 0.243(100/g)(12)0.903g$$
$$= 263.3 \text{ kip-ft}$$

The bending moments due to the second mode are

$$M_{a2} = 0.1 mhA_2 = 0.1(100/g)(12)0.903g$$
$$= 108.4 \text{ kip-ft}$$

$$M_{b2} = 0.064 mhA_2 = 0.064(100/g)(12)0.903g$$
$$= 69.35 \text{ kip-ft}$$

Due to the third mode the bending moments are

$$M_{a3} = 0.041 mhA_3 = 0.041(100/g)(12)0.903g$$
$$= 44.43 \text{ kip-ft}$$

$$M_{b3} = 0.050 mhA_3 = 0.050(100/g)(12)0.903g$$
$$= 54.18 \text{ kip-ft}$$

Combining modal responses by the SRSS rule gives

$$M_a \approx \sqrt{(1685)^2 + (108.4)^2 + (44.43)^2} = 1689.1 \text{ kip-ft}$$

$$M_b \approx \sqrt{(263.3)^2 + (69.35)^2 + (54.18)^2} = 277.6 \text{ kip-ft}$$

The bending moments in the second-story beam due to each of the three modes are computed similarly to obtain

$$M_{a1} = M_{b1} = -715.2 \text{ kip-ft}$$

$$M_{a2} = M_{b2} = 54.18 \text{ kip-ft}$$

$$M_{a3} = M_{b3} = -22.76 \text{ kip-ft}$$

Combining modal displacements by the SRSS rule gives

$$M_a = M_b \approx \sqrt{(-715.2)^2 + (54.18)^2 + (-22.76)^2} = 717.6 \text{ kip-ft}$$

Problem 13.44

Initial calculations

Substituting values of $E = 29{,}000$ ksi, $I = 1400\ in^4$, $m = 100$ kips/g and $h = 12$ ft in the solution for Problem 13.14 gives ω_n in rad/sec and T_n in sec:

$$\omega_1 = 7.558 \quad \omega_2 = 22.320 \quad \omega_3 = 45.740$$
$$T_1 = 0.831 \quad T_2 = 0.282 \quad T_3 = 0.137$$

From the design spectrum of Fig. 6.9.5, scaled to $\ddot{u}_{g0} = (1/3)g$, the spectral ordinates are

$$D_1 = 4.877 \text{ in.} \quad D_2 = 0.700 \text{ in.} \quad D_3 = 0.167 \text{ in.}$$
$$\frac{A_1}{g} = 0.722 \quad \frac{A_2}{g} = 0.903 \quad \frac{A_3}{g} = 0.903$$

***Part a**: Floor displacements.*

Substituting numerical values for Γ_n and ϕ_n from Problem 13.14 and D_n values above in Eq. (13.8.1a) gives the peak displacements due to each of the three modes:

$$\mathbf{u}_1 = \Gamma_1\phi_1 D_1 = 1.447\begin{Bmatrix} 0.200 \\ 0.597 \\ 1 \end{Bmatrix}(4.877) = \begin{Bmatrix} 1.410 \\ 4.211 \\ 7.058 \end{Bmatrix} \text{ in.}$$

$$\mathbf{u}_2 = \Gamma_2\phi_2 D_2 = -0.571\begin{Bmatrix} -0.545 \\ -0.656 \\ 1 \end{Bmatrix}(0.700) = \begin{Bmatrix} 0.218 \\ 0.262 \\ -0.400 \end{Bmatrix} \text{ in.}$$

$$\mathbf{u}_3 = \Gamma_3\phi_3 D_3 = 0.124\begin{Bmatrix} 3.220 \\ -1.916 \\ 1 \end{Bmatrix}(0.167) = \begin{Bmatrix} 0.067 \\ -0.040 \\ 0.021 \end{Bmatrix} \text{ in.}$$

Combining modal displacements by the SRSS rule gives

$$u_1 \approx \sqrt{(1.410)^2 + (0.218)^2 + (0.067)^2} = 1.428$$
$$u_2 \approx \sqrt{(4.211)^2 + (0.262)^2 + (-0.040)^2} = 4.219$$
$$u_3 \approx \sqrt{(7.058)^2 + (-0.400)^2 + (0.021)^2} = 7.069$$

Observe that the first mode contributes essentially the entire floor displacements.

***Part b**: Element forces*

To compute the element forces use the results from Problem 13.14 and replace $A_n(t)$ by the spectral values A_n.

The bending moments in a first-story column due to the first mode are

$$M_{a1} = 0.844mhA_1 = 0.844(100/g)(12)0.722g = 730.81 \text{ kip-ft}$$
$$M_{b1} = 0.094mhA_1 = 0.094(100/g)(12)0.722g = 81.51 \text{ kip-ft}$$

Bending moments due to the second mode are

$$M_{a2} = 0.132mhA_2 = 0.132(100/g)(12)0.903g = 143.47 \text{ kip-ft}$$
$$M_{b2} = 0.068mhA_2 = 0.068(100/g)(12)0.903g = 73.50 \text{ kip-ft}$$

Bending moments due to the third mode are

$$M_{a3} = 0.057mhA_3 = 0.057(100/g)(12)0.903g = 62.15 \text{ kip-ft}$$
$$M_{b3} = 0.055mhA_3 = 0.055(100/g)(12)0.903g = 59.13 \text{ kip-ft}$$

Combining modal responses by the SRSS rule gives

$$M_a \approx \sqrt{(730.81)^2 + (143.47)^2 + (62.15)^2} = 747.35 \text{ kip-ft}$$
$$M_b \approx \sqrt{(81.51)^2 + (73.50)^2 + (59.13)^2} = 124.67 \text{ kip-ft}$$

The bending moments in the second-story beam due to each of the three modes are computed similarly to obtain

$$M_{a1} = M_{b1} = -417.50 \text{ kip-ft}$$
$$M_{a2} = M_{b2} = 32.65 \text{ kip-ft}$$
$$M_{a3} = M_{b3} = 12.36 \text{ kip-ft}$$

Combining modal displacements by the SRSS rule gives

$$M_a \approx \sqrt{(-417.50)^2 + (32.65)^2 + (12.36)^2} = 418.96 \text{ kip-ft}$$

Problem 13.45

Initial calculations

From Problem 13.7, the modal properties are

$$\phi_1 = \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix}$$

$$\Gamma_1 = 1.244 \quad \Gamma_2 = -0.3333 \quad \Gamma_3 = 0.0893$$

***Part a**: Spectral ordinates*

From Problem 13.7, the peak values of $D_n(t)$ and $A_n(t)$ are

$$D_1 = 0.8859 \text{ in.} \quad D_2 = 0.1096 \text{ in.} \quad D_3 = 0.0498 \text{ in.}$$

$$A_1 = 0.7746\text{g} \quad A_2 = 0.7153\text{g} \quad A_3 = 0.6065\text{g}$$

***Part b**: Peak modal responses*

Substituting numerical values for Γ_n, ϕ_n, and D_n in Eq. (13.8.1a) gives the peak displacements due to each of the three modes:

$$\mathbf{u}_1 = 1.244 \begin{Bmatrix} 0.5 \\ 0.866 \\ 1 \end{Bmatrix} (0.8859) = \begin{Bmatrix} 0.551 \\ 0.954 \\ 1.102 \end{Bmatrix} \text{ in.}$$

$$\mathbf{u}_2 = -0.3333 \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix} (0.1096) = \begin{Bmatrix} 0.037 \\ 0 \\ -0.037 \end{Bmatrix} \text{ in.}$$

$$\mathbf{u}_3 = 0.0893 \begin{Bmatrix} 0.5 \\ -0.866 \\ 1 \end{Bmatrix} (0.0498) = \begin{Bmatrix} 0.0022 \\ -0.0039 \\ 0.0044 \end{Bmatrix} \text{ in.}$$

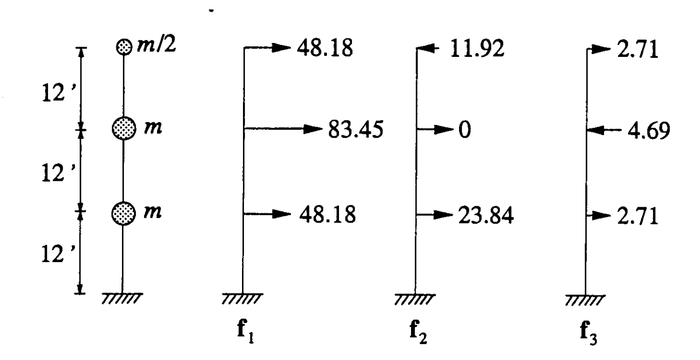
Substituting Γ_n, m_j, ϕ_n and A_n in Eq. (13.8.2) gives the equivalent static forces due to each of the three modes:

$$\mathbf{f}_1 = 1.244 \left(\frac{100}{g}\right) \begin{Bmatrix} 1(0.5) \\ 1(0.866) \\ 0.5(1) \end{Bmatrix} (0.7746\text{g}) = \begin{Bmatrix} 48.18 \\ 83.45 \\ 48.18 \end{Bmatrix} \text{ kips}$$

$$\mathbf{f}_2 = -0.3333 \left(\frac{100}{g}\right) \begin{Bmatrix} 1(-1) \\ 1(0) \\ 0.5(1) \end{Bmatrix} (0.7153\text{g}) = \begin{Bmatrix} 23.84 \\ 0 \\ -11.92 \end{Bmatrix} \text{ kips}$$

$$\mathbf{f}_3 = 0.0893 \left(\frac{100}{g}\right) \begin{Bmatrix} 1(0.5) \\ 1(-0.866) \\ 0.5(1) \end{Bmatrix} (0.6065\text{g}) = \begin{Bmatrix} 2.71 \\ -4.69 \\ 2.71 \end{Bmatrix} \text{ kips}$$

These forces are shown in the following figure.



Static analysis of the structure subjected to forces $\mathbf{f}_n$ gives the responses due to each mode. These computations give Table P13.45a for story shears (in kips) and Table P13.45b for story overturning moments (in kip-ft).

Table P13.45a

Story shear	Mode 1	Mode 2	Mode 3
V_3	48.18	– 11.92	2.71
V_2	131.63	– 11.92	– 1.98
$V_b = V_1$	179.81	11.92	0.73

Table P13.45b

Floor moments	Mode 1	Mode 2	Mode 3
M_2	578.2	– 143.0	32.52
M_1	2157.7	– 286.1	8.76
M_b	4315.5	– 143.0	17.52

The above-determined responses due to each mode should be identical to those determined in Problem 13.7 by RHA; the slight differences are due to numerical truncation errors in the computations for this problem whereas more significant digits were used in the computer work for Problem 13.7.

Part c

The peak modal responses are combined by the SRSS rule:

$$r = \sqrt{\sum_{n=1}^{3} r_n^2}$$

For each response quantity r_n available in Part b are substituted to obtain the total response (Tables P13.45c-d).

Table P13.45c

Floor or story, j	u_j, in.	V_j, kips
3	1.103	49.71
2	0.954	132.15
1	0.552	180.21

Table P13.45d

Floor or story, j	Overturning moment, M_j or M_b (kip-ft)
2	596.5
1	2176.6
Base, b	4317.9

***Part d:** Comments*

Comparison of the RSA results in Tables P13.23c-d with RHA results from Problem 13.9 is summarized in Tables P13.45e-f.

Table P13.45e

Floor or story, j	u_j (in.)		Shear V_j (kips)	
	RSA	RHA	RSA	RHA
3	1.103	1.103	49.71	52.22
2	0.954	0.957	132.15	138.08
1	0.552	0.580	180.21	189.29

Table P13.45f

Floor or story, j	Overturning moment M_j or M_b (kip-ft)	
	RSA	RHA
2	596.5	626.6
1	2176.6	2267.5
Base, b	4317.9	4320.8

For this particular problem, the RSA method gives results that are very close to the RHA results, in part, because most of the response is due to one mode, the first mode.

Problem 13.46

Initial calculations

From Problem 13.6, the modal properties are

$$\phi_1 = \begin{bmatrix} 0.314 \\ 0.686 \\ 1 \end{bmatrix}; \phi_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}; \phi_3 = \begin{bmatrix} 3.189 \\ -2.186 \\ 1 \end{bmatrix}$$

$$\Gamma_1 = 1.403 \qquad \Gamma_2 = -0.5 \qquad \Gamma_3 = 0.0972$$

Part a: Spectral ordinates

From Problem 13.8, the peak values of $D_n(t)$ and $A_n(t)$ are

$$D_1 = 1.086\,\text{in.} \quad D_2 = 0.272\,\text{in.} \quad D_3 = 0.109\,\text{in.}$$
$$A_1 = 0.741g \quad A_2 = 0.887g \quad A_3 = 0.756g$$

Part b: Peak modal responses

Substituting numerical values for Γ_n, ϕ_n, and D_n in Eq. (13.8.1a) gives the peak displacements due to each of the three modes:

$$\mathbf{u}_1 = 1.403 \begin{Bmatrix} 0.314 \\ 0.686 \\ 1 \end{Bmatrix} (1.086) = \begin{Bmatrix} 0.478 \\ 1.045 \\ 1.523 \end{Bmatrix} \text{ in}$$

$$\mathbf{u}_2 = -0.5 \begin{Bmatrix} -1/2 \\ -1/2 \\ 1 \end{Bmatrix} (0.272) = \begin{Bmatrix} 0.068 \\ 0.068 \\ -0.136 \end{Bmatrix} \text{ in}$$

$$\mathbf{u}_3 = 0.0972 \begin{Bmatrix} 3.189 \\ -2.186 \\ 1 \end{Bmatrix} (0.109) = \begin{Bmatrix} 0.034 \\ 0.023 \\ 0.011 \end{Bmatrix} \text{ in}$$

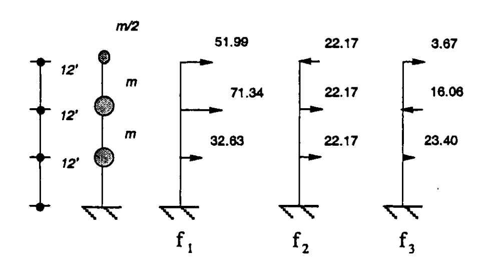
Substituting Γ_n, m_j, ϕ_n and A_n in Eq. (13.8.2) gives the equivalent static forces due to each of the three modes:

$$\mathbf{f}_1 = 1.403 \left(\frac{100}{g} \right) \begin{Bmatrix} 1(0.314) \\ 1(0.686) \\ 0.5(1) \end{Bmatrix} (0.741g) = \begin{Bmatrix} 32.63 \\ 71.34 \\ 51.99 \end{Bmatrix} \text{ kips}$$

$$\mathbf{f}_2 = -0.5 \left(\frac{100}{g} \right) \begin{Bmatrix} 1(-0.5) \\ 1(-0.5) \\ 0.5(1) \end{Bmatrix} (0.887g) = \begin{Bmatrix} 22.17 \\ 22.17 \\ -22.17 \end{Bmatrix} \text{ kips}$$

$$\mathbf{f}_3 = 0.0972 \left(\frac{100}{g} \right) \begin{Bmatrix} 1(3.189) \\ 1(-2.186) \\ 0.5(1) \end{Bmatrix} (0.756g) = \begin{Bmatrix} 23.40 \\ -16.06 \\ 3.67 \end{Bmatrix} \text{ kips}$$

These forces are shown in the following figure.



Static analysis of the structure subjected to forces $\mathbf{f}_n$ gives the responses due to each mode. These computations give Table 13.46a for story shears (in kips) and Table 13.46b for story overturning moments (in kip-ft).

Table 13.46a

Story Shear	Mode 1	Mode 2	Mode 3
V_3	51.99	-22.17	3.67
V_2	123.33	0	-12.38
$V_b = V_1$	155.96	22.17	11.02

Table 13.46b

Floor Moments	Mode 1	Mode 2	Mode 3
M_3	623.8	-266.0	44.1
M_2	2103.8	-266.0	-104.5
$M_b = M_1$	3975.3	0	27.7

The above-determined responses due to each mode should be identical to those determined in Problem 13.8 by RHA; the slight differences are due to numerical truncation errors in the computations for this problem whereas more significant digits were used in the computer work for Problem 13.8.

Part c

The peak modal reponses are combined by the SRSS rule:

$$r = \sqrt{\sum_{n=1}^{3} r_n^2}$$

For each response quantity r_n available in Part b are substituted to obtain the total response (Tables 13.46c-d).

Table 13.46c

Floor or story, j	u_j, in.	V_j, kips
3	1.529	56.64
2	1.047	123.95
1	0.484	157.92

Table 13.46d

Floor j	Overturning moment, M_j or M_b (kip-ft)
3	679.6
2	2123.1
Base, b	3975.4

Part d: Comments

Comparison of the RSA results with RHA results from Problem 13.8 is summarized in Tables 13.46e-f.

Table 13.46e

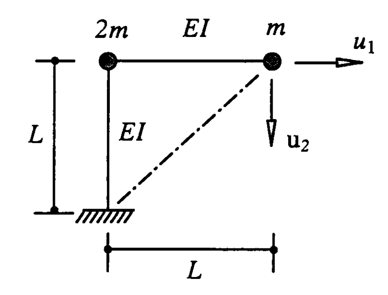
Floor or story, j	u_j (in.)		Shear V_j (kips)	
	RSA	RHA	RSA	RHA
3	1.529	1.4332	56.64	54.85
2	1.047	1.1085	123.95	126.17
1	0.484	0.5281	157.92	172.23

Table 13.46f

Floor or story, j	Overturning moment, M_j or M_b (kip-ft)	
	RSA	RHA
2	679.6	658.2
1	2123.1	2136.0
Base, b	3975.4	3965.9

For this problem, the RSA method gives results that are close to the RHA results, in part, because most of the response is due to one mode, the first mode.

Problem 13.47



1. *Data.*

$L = 120$ in. $\quad E = 29000$ ksi

$I = 28.1 \text{ in}^4 \quad m = 1.5$ kips/g

2. *Natural frequencies and modes.*

$$m = 1.5 \frac{\text{kips}}{\text{g}} = \frac{1.5}{386.4} = 0.00388 \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(29000)(28.1)}{(0.00388)(120)^3}} = 11.02 \text{ s}^{-1}$$

From Example 13.1:

$$\omega_1 = 0.6987\sqrt{\frac{EI}{mL^3}} = 7.70 \frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.815 \text{s}$$

$$\omega_2 = 1.874\sqrt{\frac{EI}{mL^3}} = 20.65 \frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.304 \text{s}$$

$\Gamma_1 = 0.406 \qquad \Gamma_2 = 0.594$

$$\phi_1 = \begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix}$$

3. *Determine correlation coefficient.*

Use both the SRSS and CQC modal combination rules. For CQC, we require ρ_{12}:

$\zeta = 0.05$

$$\beta_{12} = \frac{\omega_1}{\omega_2} = \frac{7.70}{20.65} = 0.373$$

$\therefore$ From Eq. (13.7.10):

$$\rho_{12} = \frac{8(0.05)^2(1+0.373)(0.373)^{1.5}}{[1-(0.373)^2]^2 + 4(0.05)^2(0.373)(1+0.373)^2}$$

$= 0.00836$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.815$ s

$$A_1 = 0.20\left(\frac{1.80}{0.815}\right) = 0.44 \text{ g} = 171 \frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{171}{(7.70)^2} = 2.88 \text{ in}$$

Mode 2: $T_2 = 0.304$ s

$$A_2 = 0.20(2.71) = 0.54 \text{ g} = 209 \frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(20.65)^2} = 0.49 \text{ in}$$

5. *Determine peak displacements.*

$$\mathbf{u}_n = \mathbf{u}_n^{st} A_n \qquad \mathbf{u}_n^{st} = \frac{\Gamma_n}{\omega_n^2}\phi_n$$

$$\therefore \mathbf{u}_n = \Gamma_n \phi_n D_n$$

$$\mathbf{u}_1 = 0.406\begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} 2.88 = \begin{Bmatrix} 1.169 \\ 2.452 \end{Bmatrix}$$

$$\mathbf{u}_2 = 0.594\begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix} 0.49 = \begin{Bmatrix} 0.291 \\ -0.417 \end{Bmatrix}$$

SRSS estimates of the peak displacements:

$$u_1 = \sqrt{(1.169)^2 + (0.291)^2} = 1.205 \text{ in}$$

$$u_2 = \sqrt{(2.452)^2 + (-0.417)^2} = 2.487 \text{ in}$$

CQC estimates of the peak displacements:

$$u_1 = \sqrt{(1.169)^2 + 2(0.00836)(1.169)(0.291) + (0.291)^2}$$

$= 1.207$ in

$$u_2 = \sqrt{(2.452)^2 + 2(0.00836)(2.452)(-0.417) + (-0.417)^2}$$

$= 2.484$ in

6. *Determine peak bending moments.*

Peak modal responses:

$$M_{bn} = M_{bn}^{st} A_n$$

From Figure E13.1:

$$M_{b1}^{st} = 2.069\, mL = 2.069\,(0.00388)\,(120)$$

$$= 0.964 \text{ k} \cdot \text{s}^2$$

$$M_{b2}^{st} = 0.931\, mL = 0.931\,(0.00388)\,(120)$$

$$= 0.434 \text{ k} \cdot \text{s}^2$$

Hence,

$$M_{b1} = 0.964\,(171) = 164.8 \text{ k} \cdot \text{in}$$

$$M_{b2} = 0.434\,(209) = 90.8 \text{ k} \cdot \text{in}$$

SRSS estimate of the peak bending moment:

$$M_b = \sqrt{(164.8)^2 + (90.8)^2} = 188.2 \text{ k} \cdot \text{in}$$

CQC estimate of the peak bending moment:

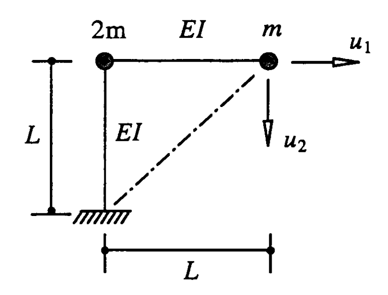
$$M_b = \sqrt{(164.8)^2 + 2\,(0.00836)(164.8)(90.8) + (90.8)^2}$$

$$= 188.8 \text{ k} \cdot \text{in}$$

7. *Comments*

The CQC estimate of the peak response is close to the SRSS result because ω_1 and ω_2 are well separated resulting in ρ_{12} close to zero.

Problem 13.48



1. *Data.*

$L = 120$ in. $\quad E = 29000$ ksi

$I = 28.1\ \text{in}^4 \quad m = 1.5$ kips/g

2. *Natural frequencies and modes.*

$$m = 1.5\ \frac{\text{kips}}{\text{g}} = \frac{1.5}{386.4} = 0.00388\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(29000)(28.1)}{(0.00388)(120)^3}} = 11.02\ \text{s}^{-1}$$

From Problem 13.15:

$$\omega_1 = 0.6987\sqrt{\frac{EI}{mL^3}} = 7.70\ \frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.815\,\text{s}$$

$$\omega_2 = 1.874\sqrt{\frac{EI}{mL^3}} = 20.65\ \frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.304\,\text{s}$$

$\Gamma_1 = 0.2834 \qquad \Gamma_2 = -0.2834$

$$\phi_1 = \begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix}$$

3. *Determine correlation coefficient.*

Use both the SRSS and CQC modal combination rules. For CQC, we require ρ_{12}:

$\varsigma = 0.05$

$$\beta_{12} = \frac{\omega_1}{\omega_2} = \frac{7.70}{20.65} = 0.373$$

$\therefore$ From Eq. (13.7.10):

$$\rho_{12} = \frac{8(0.05)^2(1+0.373)(0.373)^{1.5}}{[1-(0.373)^2]^2 + 4(0.05)^2(0.373)(1+0.373)^2}$$

$$= 0.00836$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $\quad T_1 = 0.815$ s

$$A_1 = 0.20\left(\frac{1.80}{0.815}\right) = 0.44\ \text{g} = 171\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{171}{(7.70)^2} = 2.88\ \text{in}$$

Mode 2: $\quad T_2 = 0.304$ s

$$A_2 = 0.20\,(2.71) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(20.65)^2} = 0.49\ \text{in}$$

5. *Determine peak displacements:*

$$\mathbf{u}_n = \mathbf{u}_n^{st} A_n \qquad \mathbf{u}_n^{st} = \frac{\Gamma_n}{\omega_n^2}\phi_n$$

$$\therefore\ \mathbf{u}_n = \Gamma_n \phi_n D_n$$

$$\mathbf{u}_1 = 0.2834\begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} 2.88 = \begin{Bmatrix} 0.816 \\ 1.712 \end{Bmatrix}$$

$$\mathbf{u}_2 = -0.2834\begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix} 0.49 = \begin{Bmatrix} -0.139 \\ 0.199 \end{Bmatrix}$$

SRSS estimates of the peak displacements:

$$u_1 = \sqrt{(0.816)^2 + (-0.139)^2} = 0.828\ \text{in}$$

$$u_2 = \sqrt{(1.712)^2 + (0.199)^2} = 1.724\ \text{in}$$

CQC estimates of the peak displacements:

$$u_1 = \sqrt{(0.816)^2 + 2(0.00836)(0.816)(-0.139) + (-0.139)^2}$$

$$= 0.827\ \text{in}$$

$$u_2 = \sqrt{(1.712)^2 + 2(0.00836)(1.712)(0.199) + (0.199)^2}$$

$$= 1.725\ \text{in}$$

6. *Determine peak bending moments.*

Peak modal responses:

$$M_{bn} = M_{bn}^{st} A_n$$

From the figure in the solution to Problem 13.15:

$$M_{b1}^{st} = 1.443\,mL = 1.443\,(0.00388)\,(120)$$

$$= 0.672\ \text{k}\cdot\text{s}^2$$

$$M_{b2}^{st} = -0.443\,mL = -0.443\,(0.00388)\,(120)$$

$$= -0.206\ \text{k}\cdot\text{s}^2$$

Hence,

$$M_{b1} = 0.672\,(171) = 114.9\ \text{k}\cdot\text{in}$$

$$M_{b2} = -0.206\,(209) = -43.1\ \text{k}\cdot\text{in}$$

SRSS estimate of the peak bending moment:

$$M_b = \sqrt{(114.9)^2 + (-43.1)^2} = 122.7\ \text{k}\cdot\text{in}$$

CQC estimate of the peak bending moment:

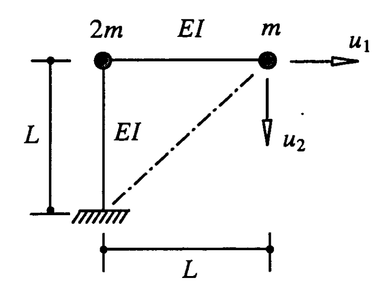
$$M_b = \sqrt{(114.9)^2 + 2\,(0.00836)(114.9)(-43.1) + (-43.1)^2}$$

$$= 122.4\ \text{k}\cdot\text{in}$$

7. *Comments:*

The CQC estimate of the peak response is close to the SRSS result because ω_1 and ω_2 are well separated resulting in ρ_{12} close to zero.

Problem 13.49



1. *Data.*

$L = 120$ in. $\quad E = 29000$ ksi

$I = 28.1 \text{ in}^4 \quad m = 1.5$ kips/g

2. *Natural frequencies and modes.*

$$m = 1.5\ \frac{\text{kips}}{\text{g}} = \frac{1.5}{386.4} = 0.00388\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(29000)(28.1)}{(0.00388)(120)^3}} = 11.02 \text{ s}^{-1}$$

From Problem 13.16:

$$\omega_1 = 0.6987\sqrt{\frac{EI}{mL^3}} = 7.70\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.815\text{s}$$

$$\omega_2 = 1.874\sqrt{\frac{EI}{mL^3}} = 20.65\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.304\text{s}$$

$\Gamma_1 = 0.0863 \qquad \Gamma_2 = 0.6206$

$$\phi_1 = \begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix}$$

3. *Determine correlation coefficient.*

Use both the SRSS and CQC modal combination rules. For CQC, we require ρ_{12}:

$\varsigma = 0.05$

$$\beta_{12} = \frac{\omega_1}{\omega_2} = \frac{7.70}{20.65} = 0.373$$

$\therefore$ From Eq. (13.7.10):

$$\rho_{12} = \frac{8(0.05)^2(1+0.373)(0.373)^{1.5}}{[1-(0.373)^2]^2 + 4(0.05)^2(0.373)(1+0.373)^2}$$

$= 0.00836$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.815$ s

$$A_1 = 0.20\left(\frac{1.80}{0.815}\right) = 0.44\text{ g} = 171\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{171}{(7.70)^2} = 2.88 \text{ in}$$

Mode 2: $T_2 = 0.304$ s

$$A_2 = 0.20(2.71) = 0.54\text{ g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(20.65)^2} = 0.49 \text{ in}$$

5. *Determine peak displacements.*

$$\mathbf{u}_n = \mathbf{u}_n^{st} A_n \qquad \mathbf{u}_n^{st} = \frac{\Gamma_n}{\omega_n^2}\phi_n$$

$$\therefore \mathbf{u}_n = \Gamma_n \phi_n D_n$$

$$\mathbf{u}_1 = 0.0863\begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} 2.88 = \begin{Bmatrix} 0.249 \\ 0.521 \end{Bmatrix}$$

$$\mathbf{u}_2 = 0.6206\begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix} 0.49 = \begin{Bmatrix} 0.304 \\ -0.435 \end{Bmatrix}$$

SRSS estimates of the peak displacements:

$$u_1 = \sqrt{(0.249)^2 + (0.304)^2} = 0.393 \text{ in}$$

$$u_2 = \sqrt{(0.521)^2 + (-0.435)^2} = 0.679 \text{ in}$$

CQC estimates of the peak displacements:

$$u_1 = \sqrt{(0.249)^2 + 2(0.00836)(0.249)(0.304) + (0.304)^2}$$

$= 0.395$ in

$$u_2 = \sqrt{(0.521)^2 + 2(0.00836)(0.521)(-0.435) + (-0.435)^2}$$

$= 0.676$ in

6. *Determine peak bending moments.*

Peak modal responses:

$$M_{bn} = M_{bn}^{st} A_n$$

From the figure in the solution to Problem 13.16:

$$M_{b1}^{st} = 0.440\,mL = 0.440\,(0.00388)\,(120)$$

$$= 0.205\ \text{k}\cdot\text{s}^2$$

$$M_{b2}^{st} = 0.974\,mL = 0.974\,(0.00388)\,(120)$$

$$= 0.453\ \text{k}\cdot\text{s}^2$$

Hence,

$$M_{b1} = 0.205\,(171) = 35.1\ \text{k}\cdot\text{in}$$

$$M_{b2} = 0.453\,(209) = 94.7\ \text{k}\cdot\text{in}$$

SRSS estimate of the peak bending moment:

$$M_b = \sqrt{(35.1)^2 + (94.7)^2} = 101.0\ \text{k}\cdot\text{in}$$

CQC estimate of the peak bending moment:

$$M_b = \sqrt{(35.1)^2 + 2\,(0.00836)(35.1)(94.7) + (94.7)^2}$$

$$= 101.3\ \text{k}\cdot\text{in}$$

7. *Comments.*

The CQC estimate of the peak response is close to the SRSS result because ω_1 and ω_2 are well separated resulting in ρ_{12} close to zero.

Problem 13.50

1. *Determine effect of weight of pipe.*

Pipe weight equals $3 \times 10 \times 18.97 = 569$ lbs. This is negligible compared to the lumped weights equals $5\,mg = 7.5$ kips.

2. *Determine design spectrum ordinates.*

From Problem 10.23,

$$\omega_1 = 0.5259\sqrt{\frac{EI}{mL^3}}$$

$$\omega_2 = 1.6135\sqrt{\frac{EI}{mL^3}}$$

$$\omega_3 = 1.7321\sqrt{\frac{EI}{mL^3}}$$

Substituting the given values of $E, I, m,$ and L:

$$\omega_1 = 0.5259\sqrt{\frac{29 \times 10^3 \times 28.1}{(1.5/386.4) \times (10 \times 12)^3}}$$

$$= 0.5259 \times 11.022 = 5.798 \text{ rad/s}$$

$$T_1 = 1.084 \text{ secs}$$

$$\omega_2 = 17.79 \text{ rad/s}$$

$$T_2 = 0.353 \text{ secs}$$

$$\omega_3 = 19.09 \text{ rad/s}$$

$$T_3 = 0.329 \text{ secs}$$

For these T_n, the design spectrum of Fig. 6.9.5 gives

$$A_1 = 0.2 \times \frac{1.80\text{g}}{1.084} = 0.332\text{g} \qquad D_1 = 3.82 \text{ in.}$$

$$A_2 = 0.2 \times 2.71\text{g} = 0.542\text{g} \qquad D_2 = 0.661 \text{ in.}$$

$$A_3 = 0.2 \times 2.71\text{g} = 0.542\text{g} \qquad D_3 = 0.574 \text{ in.}$$

3. *Determine peak modal responses.*

The modal displacements, available from Problem 13.17, are

$$\mathbf{u}_1(t) = \begin{Bmatrix} 0.397 \\ -0.774 \\ 0.774 \end{Bmatrix} D_1(t)$$

$$\mathbf{u}_2(t) = \begin{Bmatrix} 0.603 \\ 0.774 \\ -0.774 \end{Bmatrix} D_2(t)$$

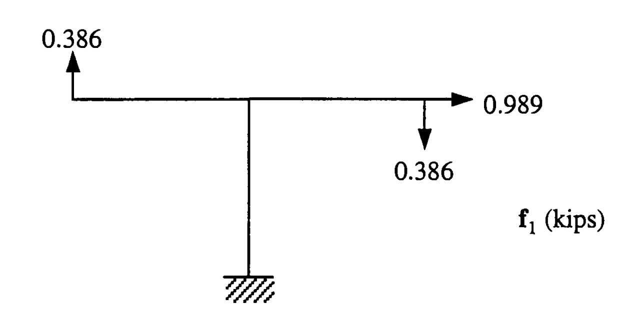
$$\mathbf{u}_3(t) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} D_3(t)$$

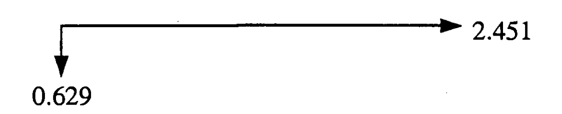
Substituting numerical values for D_1 and D_2, the peak values of $D_1(t)$ and $D_2(t)$, gives the peak values of the modal responses:

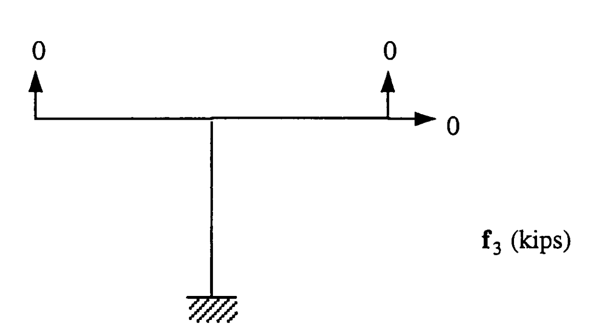
$$\mathbf{u}_1 = \begin{Bmatrix} 1.517 \\ -2.970 \\ 2.970 \end{Bmatrix} \text{in.}$$

$$\mathbf{u}_2 = \begin{Bmatrix} 0.399 \\ 0.512 \\ -0.512 \end{Bmatrix} \text{in.}$$

Substituting numerical values for Γ_n, $\mathbf{m}$, and ϕ_n (available from Problem 13.17) in Eq. (13.8.2) gives the equivalent static forces (in kips) shown in the accompanying figure.







Static analysis of the structure subjected to forces $\mathbf{f}_n$ gives the peak values of the bending moments (kip-ft.) due to each mode (Table P13.50a).

Table P13.50a: Bending Moments (kip-ft.)

	Mode 1	Mode 2	Mode 3
M_a	−3.85	6.29	0
M_b	17.59	11.93	0

4. *Combine modal responses.*

The correlation coefficients, ρ_{in}, in the CQC rule depend on the frequency ratios $\beta_{in} = \omega_i / \omega_n$, computed from the known natural frequencies.

Table P13.50b: Natural Frequency Ratios, β_{in}

Mode, i	$n=1$	$n=2$	$n=3$	ω_i (rad/sec)
1	1.0	0.3259	0.3037	5.798
2	3.0683	1.0	0.9319	17.79
3	3.2925	1.0731	1.0	19.09

Correlation coefficients, ρ_{in}, are calculated from Eq. (13.7.10):

Table P13.50c: Correlation Coefficients, ρ_{in}

Mode, i	$n=1$	$n=2$	$n=3$
1	1.0	0.00613	0.00526
2	0.00613	1.0	0.66731
3	0.00526	0.66731	1.0

Substituting the peak modal responses in Eq. (13.7.3) with $N = 3$ gives the SRSS estimate of the total response, and, in addition, using the ρ_{in} values in Eq. (13.7.5) with $N = 3$ gives the CQC estimate of the total response. Tables P13.50d and P13.50e summarize the results for u_1, u_2 and u_3, and M_a and M_b.

Table P13.50d: Displacements (in.)

	u_1	u_2	u_3
SRSS	1.569	3.014	3.014
CQC	1.571	3.014	3.011

Table P13.50e: Bending Moments (kip-ft.)

	M_a	M_b
SRSS	7.375	21.25
CQC	7.395	21.31

5. *Comments.*

The cross-correlation coefficients ρ_{12} and ρ_{13} are negligible, implying that the 1-2 and 1-3 cross-terms in Eq. (13.7.5) are insignificant. In contrast, $\rho_{23} = 0.66731$, which is large. Thus the 2-3 cross-term in Eq. (13.7.5) may be significant. However, this term turns out to be zero because the response due to the third mode is zero. Thus all the cross-terms are either zero or negligible, which explains why the CQC estimate is very close to the SRSS estimate.

Problem 13.51

1. *Determine design spectrum ordinates.*

From the values obtained in Problem 13.50, we have

$A_1 = 0.332\text{g} \qquad D_1 = 3.82\text{ in.}$

$A_2 = 0.542\text{g} \qquad D_2 = 0.661\text{ in.}$

$A_3 = 0.542\text{g} \qquad D_3 = 0.574\text{ in.}$

2. *Determine peak modal responses.*

(a) The modal displacements, available from Problem 13.18, are

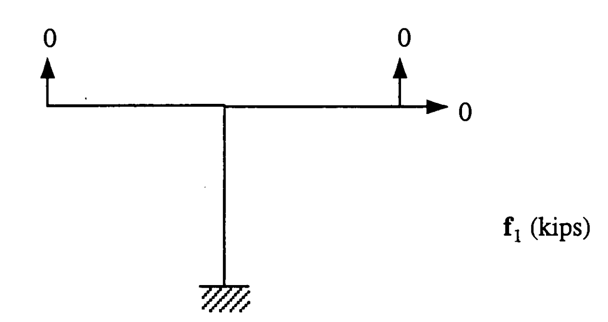
$$\mathbf{u}_1(t) = \mathbf{u}_2(t) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

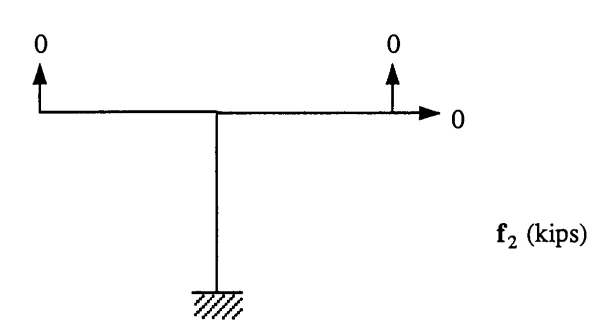
$$\mathbf{u}_3(t) = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} D_3(t)$$

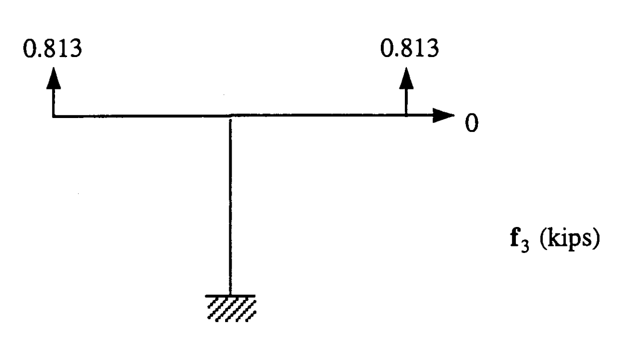
Substituting numerical values for D_3, the peak value of $D_3(t)$, gives the peak values of the modal responses:

$$\mathbf{u}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \mathbf{u}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \mathbf{u}_3 = \begin{Bmatrix} 0 \\ 0.574 \\ 0.574 \end{Bmatrix} \text{in.}$$

(b) Substituting numerical values for Γ_n, $\mathbf{m}$, and ϕ_n (available from Problem 13.18) in Eq. (13.8.2) gives the equivalent static forces (in kips) shown in the accompanying figure.







Static analysis of the structure subjected to forces $\mathbf{f}_n$ gives the peak values of the bending moments (kip-ft.) due to each mode (Table P13.51a)

Table P13.51a: Bending Moments (kip-ft.)

	Mode 1	Mode 2	Mode 3
M_a	0	0	8.13
M_b	0	0	0

3. *Combine modal responses.*

Because the entire response is due to the third mode, we do not need to combine modal responses. The total response is he same as the third mode response:

$u_1 = 0$ in.
$u_2 = 0.574$ in.
$u_3 = 0.574$ in.

$M_a = 8.13$ kip - ft.
$M_b = 0$ kip - ft.

4. *Comments.*

In this case, since only one mode contributes to the response, the peak value of the total response is identical to the peak response in the third mode. Consequently, SRSS and CQC rules give identical results.

Problem 13.52

1. *Determine design spectrum ordinates.*

From the values obtained in Problem 13.50, we have

$A_1 = 0.332\text{g} \qquad D_1 = 3.82 \text{ in.}$

$A_2 = 0.542\text{g} \qquad D_2 = 0.661 \text{ in.}$

$A_3 = 0.542\text{g} \qquad D_3 = 0.574 \text{ in.}$

2. *Determine peak modal responses.*

(a) The modal displacements, available from Problem 13.19, are

$$\mathbf{u}_1(t) = \begin{Bmatrix} 0.281 \\ -0.547 \\ 0.547 \end{Bmatrix} D_1(t)$$

$$\mathbf{u}_2(t) = \begin{Bmatrix} 0.426 \\ 0.547 \\ -0.547 \end{Bmatrix} D_2(t)$$

$$\mathbf{u}_3(t) = \begin{Bmatrix} 0 \\ 0.707 \\ 0.707 \end{Bmatrix} D_3(t)$$

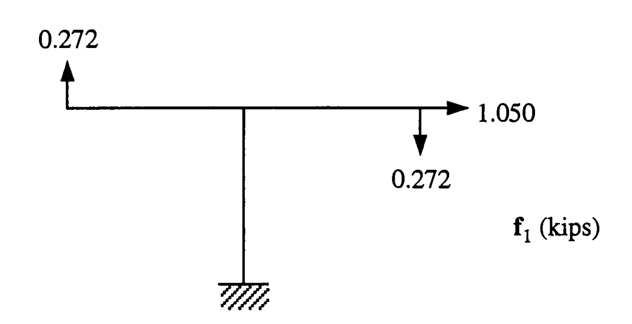
Substituting numerical values for D_1, D_2 and D_3, the peak values of $D_1(t)$, $D_2(t)$ and $D_3(t)$, gives the peak values of the modal responses:

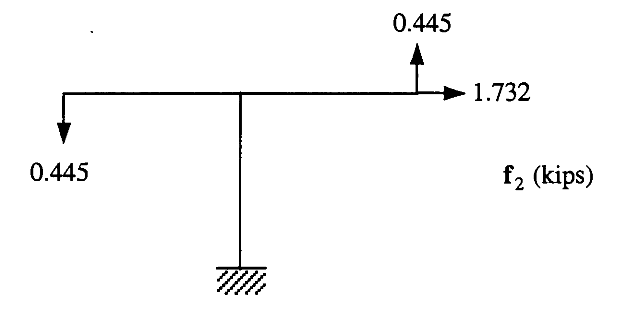
$$\mathbf{u}_1 = \begin{Bmatrix} 1.073 \\ -2.090 \\ 2.090 \end{Bmatrix} \text{in.}$$

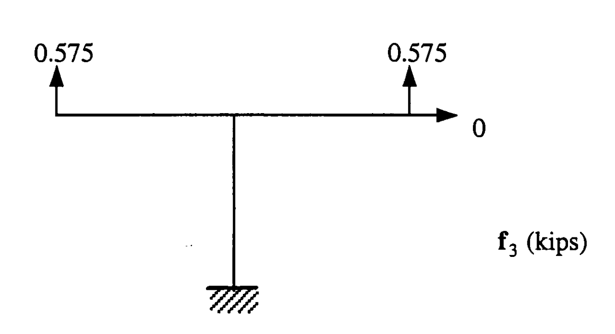
$$\mathbf{u}_2 = \begin{Bmatrix} 0.282 \\ 0.362 \\ -0.362 \end{Bmatrix} \text{in.}$$

$$\mathbf{u}_3 = \begin{Bmatrix} 0 \\ 0.403 \\ 0.403 \end{Bmatrix} \text{in.}$$

(b) Substituting numerical values for Γ_n, $\mathbf{m}$, and ϕ_n (available from Problem 13.19) in Eq. (13.8.2) gives the equivalent static forces (in kips) shown in the accompanying figure.







Static analysis of the structure subjected to forces $\mathbf{f}_n$ gives the peak values of the bending moments (kip-ft.) due to each mode (Table P13.52a)

Table P13.52a: Bending Moments (kip-ft.)

	Mode 1	Mode 2	Mode 3
M_a	–2.724	4.447	5.748
M_b	12.445	8.423	0

3. *Combine modal responses.*

The correlation coefficients, ρ_{in}, in the CQC rule depend on the frequency ratios $\beta_{in} = \omega_i / \omega_n$, computed from the known natural frequencies.

Table P13.52b: Natural Frequency Ratios, β_{in}

Mode, i	$n=1$	$n=2$	$n=3$	ω_i (rad/sec)
1	1.0	0.3259	0.3037	5.798
2	3.0683	1.0	0.9319	17.79
3	3.2925	1.0731	1.0	19.09

Correlation coefficients, ρ_{in}, are calculated from Eq. (13.7.10):

Table P13.52: Correlation Coefficients, ρ_{in}

Mode, i	$n=1$	$n=2$	$n=3$
1	1.0	0.00613	0.00526
2	0.00613	1.0	0.66731
3	0.00526	0.66731	1.0

Substituting the peak modal responses in Eq. (13.7.3) with $N=3$ gives the SRSS estimate of the total response, and, in addition, using the ρ_{in} values in Eq. (13.7.5) with $N=3$ gives the CQC estimate of the total response. Tables P13.52d and P13.52e summarize the results for u_1, u_2 and u_3, and M_a and M_b.

Table P13.52d: Displacements (in.)

	u_1	u_2	u_3
SRSS	1.110	2.159	2.159
CQC	1.112	2.1996	2.113

Table P13.52e: Bending Moments (kip-ft.)

	M_a	M_b
SRSS	7.763	15.028
CQC	9.699	15.070

4. *Comments.*

The cross-correlation coefficients ρ_{12} and ρ_{13} are negligible, implying that the 1-2 and 1-3 cross-terms in Eq. (13.7.5) are insignificant. In contrast, $\rho_{23} = 0.66731$, which is large. Thus the 2-3 cross-term in Eq. (13.7.5) may be significant. This is indeed the case for M_a because the second and third mode contributions are significant. Because the cross-term is positive, the CQC estimate for M_a is significantly larger than its SRSS estimate. For u_1 and M_b the 2-3 cross-term is zero because the third mode response is zero; hence the SRSS and CQC estimates of peak response are close. For u_2 and u_3 the 2-3 cross-term is negligible because the response due to the second and third modes are much smaller than due to the first mode; hence the SRSS and CQC estimates of peak response are close.

Problem 13.53

1. *Determine design spectrum ordinates.*

From the values obtained in Problem 13.50, we have

$A_1 = 0.332\text{g} \qquad D_1 = 3.82 \text{ in.}$

$A_2 = 0.542\text{g} \qquad D_2 = 0.661 \text{ in.}$

$A_3 = 0.542\text{g} \qquad D_3 = 0.574 \text{ in.}$

2. *Determine peak modal responses.*

(a) The modal displacements, available from Problem 13.20, are

$$\mathbf{u}_1(t) = \begin{Bmatrix} -0.281 \\ 0.547 \\ -0.547 \end{Bmatrix} D_1(t)$$

$$\mathbf{u}_2(t) = \begin{Bmatrix} -0.426 \\ -0.547 \\ 0.547 \end{Bmatrix} D_2(t)$$

$$\mathbf{u}_3(t) = \begin{Bmatrix} 0 \\ 0.707 \\ 0.707 \end{Bmatrix} D_3(t)$$

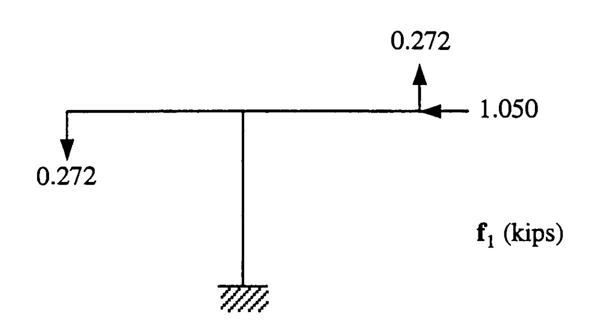
Substituting numerical values for D_1, D_2 and D_3, the peak values of $D_1(t)$, $D_2(t)$ and $D_3(t)$, gives the peak values of the modal responses:

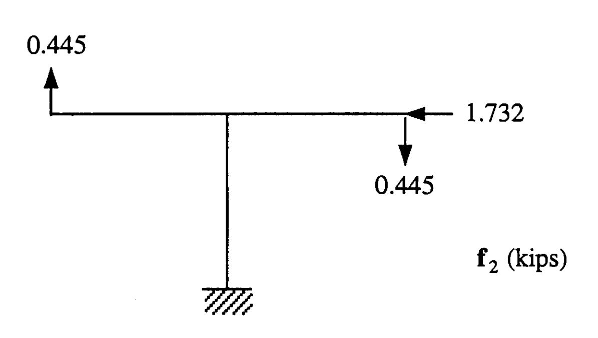
$$\mathbf{u}_1 = \begin{Bmatrix} -1.073 \\ 2.090 \\ -2.090 \end{Bmatrix} \text{in.}$$

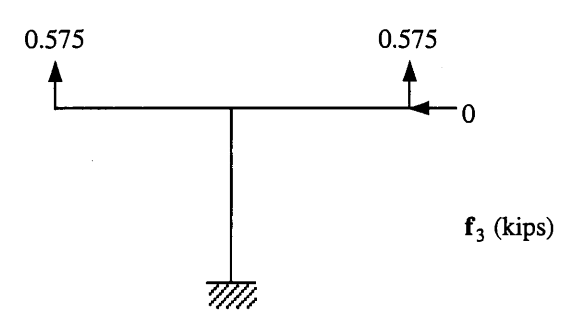
$$\mathbf{u}_2 = \begin{Bmatrix} -0.282 \\ -0.362 \\ 0.362 \end{Bmatrix} \text{in.}$$

$$\mathbf{u}_3 = \begin{Bmatrix} 0 \\ 0.403 \\ 0.403 \end{Bmatrix} \text{in.}$$

(b) Substituting numerical values for Γ_n, $\mathbf{m}$, and ϕ_n (available from Problem 13.20) in Eq. (13.8.2) gives the equivalent static forces (in kips) shown in the accompanying figure.







Static analysis of the structure subjected to forces $\mathbf{f}_n$ gives the peak values of the bending moments (kip-ft.) due to each mode (Table P13.53a)

Table P13.53a: Bending Moments (kip-ft.)

	Mode 1	Mode 2	Mode 3
M_a	2.724	–4.447	5.748
M_b	–12.445	–8.423	0

3. *Combine modal responses.*

The correlation coefficients, ρ_{in}, in the CQC rule depend on the frequency ratios $\beta_{in} = \omega_i / \omega_n$, computed from the known natural frequencies.

Table P13.53b: Natural Frequency Ratios, β_{in}

Mode, i	$n=1$	$n=2$	$n=3$	ω_i (rad/sec)
1	1.0	0.3259	0.3037	5.798
2	3.0683	1.0	0.9319	17.79
3	3.2925	1.0731	1.0	19.09

Correlation coefficients, ρ_{in}, are calculated from Eq. (13.7.10):

Table P13.53c: Correlation Coefficients, ρ_{in}

Mode, i	$n=1$	$n=2$	$n=3$
1	1.0	0.00613	0.00526
2	0.00613	1.0	0.66731
3	0.00526	0.66731	1.0

Substituting the peak modal responses in Eq. (13.7.3) with $N=3$ gives the SRSS estimate of the total response, and, in addition, using the ρ_{in} values in Eq. (13.7.5) with $N=3$ gives the CQC estimate of the total response. Tables P13.53d and P13.53e summarize the results for u_1, u_2 and u_3, and M_a and M_b.

Table P13.53d: Displacements (in.)

	u_1	u_2	u_3
SRSS	1.110	2.159	2.159
CQC	1.112	2.113	2.199

Table P13.53e: Bending Moments (kip-ft.)

	M_a	M_b
SRSS	7.763	15.028
CQC	5.114	15.070

4. *Comments.*

The cross-correlation coefficients ρ_{12} and ρ_{13} are negligible, implying that the 1-2 and 1-3 cross-terms in Eq. (13.7.5) are insignificant. In contrast, $\rho_{23}=0.66731$, which is large. Thus the 2-3 cross-term in Eq. (13.7.5) may be significant. This is indeed the case for M_a because the second and third mode contributions are significant. Because the cross-term is negative, the CQC estimate for M_a is smaller than its SRSS estimate. For u_1 and M_b the 2-3 cross-term is zero because the third mode response is zero; hence the SRSS and CQC estimates of peak response are close. For u_2 and u_3 the 2-3 cross-term is negligible because the response due to the second and third modes are much smaller than due to the first mode; hence the SRSS and CQC estimates of peak response are close.

Problem 13.54

1. *Determine design spectrum ordinates.*

From Problem 13.22,

$$T_1 = 1.027 \text{ sec} \quad T_2 = 0.974 \text{ sec} \quad T_3 = 0.1503 \text{ sec}$$

For these T_n, the design spectrum of Fig. 6.9.5 gives

$$A_1 = \frac{1}{3}\frac{1.80\text{g}}{1.027} = 0.584\text{g} \qquad D_1 = 6.02 \text{ in.}$$

$$A_2 = \frac{1}{3}\frac{1.80\text{g}}{0.974} = 0.616\text{g} \qquad D_2 = 5.70 \text{ in.}$$

$$A_3 = \frac{1}{3}2.71\text{g} = 0.903\text{g} \qquad D_3 = 0.2 \text{ in.}$$

2. *Determine peak modal responses.*

The appendage displacement is

$$u_{3n} = \Gamma_n \phi_{3n} D_n$$

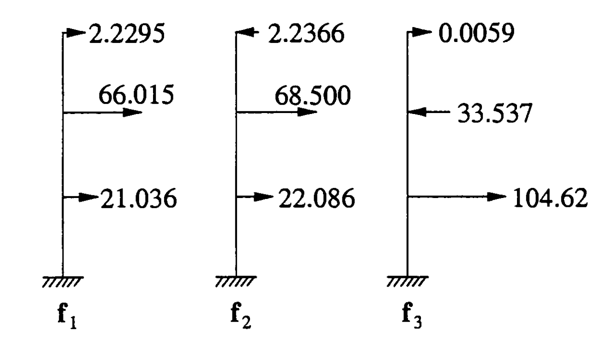
Substituting numerical values for Γ_n and ϕ_{3n} available from Problem 13.22 gives

$$u_{31} = (-0.6296)(-32.34)\,6.02 = 122.57$$

$$u_{32} = (0.6104)(-31.80)\,5.70 = -110.6$$

$$u_{33} = (0.4511)(0.0773)\,0.2 = 0.007$$

Substituting numerical values for Γ_n, $\mathbf{m}$, and ϕ_n (available from Problem 13.22) in Eq. (13.8.2) gives the equivalent static forces shown in the accompanying figure.



Static analysis of the structure subjected to forces $\mathbf{f}_n$ gives the peak value of modal response r_n. These results for the appendage shear and base shear are (in kips):

$$\begin{aligned} V_{a1} &= 2.2295 \quad V_{a2} = -2.2366 \quad V_{a3} = 0.0059 \\ V_{b1} &= 89.281 \quad V_{b2} = 88.349 \quad V_{b3} = 71.089 \end{aligned} \qquad \text{(a)}$$

3. *Combine modal responses.*

Correlation coefficients, ρ_{in}, are calculated from Eq. (13.7.10) using known values of $\beta_{in} = \omega_i/\omega_n$:

Table P13.54a: Correlation Coefficients ρ_{in}

Mode, i	$n=1$	$n=2$	$n=3$
1	1.0	0.7847	0.0013
2	0.7847	1.0	0.0015
3	0.0013	0.0015	1.0

Substituting the peak modal responses in Eq. (13.7.3) with $N=3$ gives the SRSS estimate of the total response and, in addition, using the ρ_{in} values in Eq. (13.7.5) with $N=3$ gives the CQC estimate of the total response. Table P13.54b summarizes the results for u_3, V_a and V_b.

Table P13.54b

	u_3, in.	V_a, kips	V_b, kips
SRSS	165.0	3.158	144.3
CQC	77.27	1.465	182.3

4. *Comments.*

The cross correlation coefficient for the first two modes is 0.7847 which is significant. Therefore, neglecting the 1-2 cross term in the SRSS procedure introduces significant error. The CQC method is more accurate.

Problem 13.55

***Part a**: Response spectrum ordinates*

From Problem 13.23, the peak values of $D_n(t)$ and $A_n(t)$ are

$$D_1 = 4.3641 \text{ in.} \qquad A_1 = 0.4237g$$
$$D_2 = 4.5312 \text{ in.} \qquad A_2 = 0.4884g$$
$$D_3 = 0.1649 \text{ in.} \qquad A_3 = 0.7460g$$

***Part b**: Peak modal responses*

The peak values of the appendage displacement u_3, appendage shear V_a and tower base shear V_b are

$$u_{3n} = u_{3n}^{st} A_n \qquad V_{an} = V_{an}^{st} A_n \qquad V_{bn} = V_{bn}^{st} A_n \qquad \text{(a)}$$

where the modal static responses r_n^{st} are available from the solution to Problem 13.22. Substituting for r_n^{st} and A_n in Eq. (a) gives the results in Table P13.55a.

Table P13.55a

Mode, n	u_3, in.	V_a, kips	V_b, kips
1	88.90	1.619	64.83
2	– 87.93	– 1.778	70.23
3	0.0058	0.0049	58.61

These peak modal responses are essentially identical to those determined in Problem 13.23 by RHA.

Part c

For each response quantity, the peak modal responses (from Table P13.55a) and the correlation coefficients (from Table P13.54a) are substituted in Eq. (13.7.4) with $N = 3$ to obtain peak values of the total response:

$$u_{3o} = 58.02 \text{ in.}$$
$$V_{ao} = 1.125 \text{ kips}$$
$$V_{bo} = 140.5 \text{ kips}$$

The cross-correlation coefficient for the first two modes is 0.7847 (see Problem 13.54), which is significant and contributes importantly to the response. The correlation coefficient between modes 1 and 3 and between 2 and 3 is insignificant because the frequencies are well separated.

Part d

Substituting the peak modal responses (from Table P13.55a) in Eq. (13.7.3) with $N = 3$ gives the SRSS estimate of peak response:

$$u_{3o} = 125.04 \text{ in.}$$
$$V_{ao} = 2.405 \text{ kips}$$
$$V_{bo} = 112.1 \text{ kips}$$

***Part e**: Comments*

Summarized in Table P13.55b are the CQC and SRSS results from Parts c and d and the RHA results from Problem 13.23.

Table P13.55b

	u_{3o}, in.	V_{ao}, kips	V_{bo}, kips
SRSS	125.04	2.405	112.1
CQC	58.02	1.125	140.5
RHA	44.58	0.879	159.8

Clearly the peak response from the CQC method is much closer to that from RHA because the CQC method considers the correlation between modes. The SRSS method should not be used when frequencies are closely spaced.

Problem 13.56

From Problem 13.24, **m**, ϕ_n, and Γ_n are available.

***Part a**: Spectral ordinates*

From Problem 13.26,

$$D_1 = 4.291 \text{ in.} \qquad A_1 = 0.3047g$$

$$D_2 = 4.354 \text{ in.} \qquad A_2 = 0.4354g$$

$$D_3 = 2.604 \text{ in.} \qquad A_3 = 0.8020g$$

***Part b**: Peak modal responses*

Displacements:

$$\mathbf{u} = \Gamma_n \phi_n D_n \qquad \text{(a)}$$

Equivalent static forces:

$$\mathbf{f}_n = \Gamma_n \mathbf{m} \phi_n A_n \qquad \text{(b)}$$

Base shear and torque:

$$V_{bn} = f_{yn} \qquad T_{bn} = f_{\theta n} \qquad \text{(c)}$$

Substituting **m**, ϕ_n, and Γ_n from Problem 13.24 and D_n and A_n from Part a in Eqs. (a)-(c) gives the peak modal responses in Table P13.56a.

Table P13.56a

Mode, n	u_y	$(b/2)\,u_\theta$	V_b	T_b
1	4.132	0.9931	34.21	822.2
2	0	0	0	0
3	0.0966	− 0.6027	2.677	− 1671

Part c

The correlation coefficients ρ_{in} are calculated from Eq. (13.7.10) using known values of $\beta_{in} = \omega_i / \omega_n$:

Table P13.56b: Correlation Coefficients ρ_{in}

Mode, i	$n = 1$	$n = 2$	$n = 3$
1	1.0	0.8513	0.0247
2	0.8513	1.0	0.0287
3	0.0247	0.0287	1.0

The modal peaks r_n (or r_{no}) and ρ_{in} are substituted in SRSS and CQC formulas [Eqs. (13.7.3) and (13.7.4) with $N = 3$], specialized for each response quantity, to obtain estimates of the peak value of the total response. The exact values are available from RHA in Problem 13.26.

Table P13.56c

	u_y, in.	$(b/2)\,u_\theta$, in.	V_b, kips	T_b, kip-ft
SRSS	4.133	1.162	34.3	1862
CQC	4.135	1.149	34.4	1844
RHA	4.182	1.222	35.6	1899

***Part d**: Comments*

The accuracy of both combination rules (SRSS and CQC) is good in this case because the two modes that contribute to the response have well separated frequencies. Both methods are similarly accurate in this case.

Problem 13.58

Solution to this problem closely follows that to Problem 13.57; therefore many details are omitted.

1. *Spectral ordinates (from Problem 13.57).*

$$D_1 = 9.26 \text{ in.} \quad D_2 = 8.89 \text{ in.} \quad D_3 = 4.40 \text{ in.} \qquad \text{(a)}$$

2. *Determine peak modal responses.*

From Problem 13.25,

$$\Gamma_1 = 0.3350 \quad \Gamma_2 = 0.3414 \quad \Gamma_3 = -0.0658 \qquad \text{(b)}$$

Displacements:

$$\begin{Bmatrix} u_{xn} \\ u_{yn} \\ u_{\theta n} \end{Bmatrix} = \Gamma_n \begin{Bmatrix} \phi_{xn} \\ \phi_{yn} \\ \phi_{\theta n} \end{Bmatrix} D_n \qquad \text{(c)}$$

Base forces:

$$V_{bxn} = f_{xn} = \Gamma_n \, m \, \phi_{xn} \, A_n$$

$$V_{byn} = f_{yn} = \Gamma_n \, m \, \phi_{yn} \, A_n \qquad \text{(d)}$$

$$T_{bn} = f_{\theta n} = \Gamma_n \, (m r^2) \, \phi_{\theta n} \, A_n$$

In Eqs. (c) and (d), substituting for m, r, ϕ_{xn}, ϕ_{yn} and $\phi_{\theta n}$ from Problem 13.24, Γ_n from Eq. (b), D_n from Eq. (a), and $A_n = \omega_n^2 D_n$ gives the results in Table P13.58a.

Table P13.58a

Response	Peak modal responses			Peak total response	
	$n = 1$	$n = 2$	$n = 3$	CQC	SRSS
u_x, in.	0	6.286	0	6.286	6.286
u_y, in.	6.305	0	0.115	6.309	6.306
$(b/2)u_\theta$, in.	1.516	0	- 0.720	1.662	1.678
V_x, kips	0	56.6	0	56.6	56.6
V_y, kips	52.2	0	3.2	52.4	52.3
T, kip-in.	1254.7	0	- 1996.2	2331.4	2357.8

3. *Combine peak modal responses.*

The modal peaks (from Table P13.58a) and correlation coefficients ρ_{in} (from Table P13.56b) are substituted in Eqs. (13.7.3) and (13.7.4) for each response quantity to obtain estimates of the peak value of the total response. The results obtained by SRSS and CQC methods are presented in Table P13.58a.

4. *Determine bending moments in columns.*

Following the solution of Problem 13.57 for each (x and y) component of bending moment in each column, the peak values of modal responses and the peak value of total response are calculated. The results are summarized in Table P13.58b.

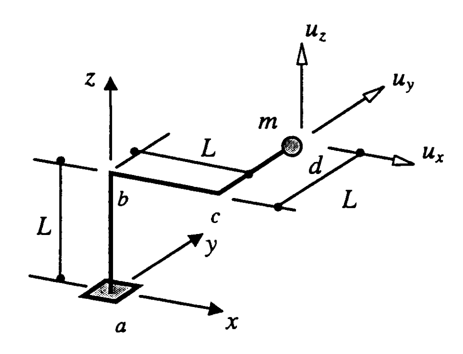
Table P13.58b: Column Moments in kip-in.

Response	Peak modal responses			Peak total response	
	$n = 1$	$n = 2$	$n = 3$	CQC	SRSS
M_{ay}	- 327.4	1357.8	155.5	1108.1	1405.4
M_{ax}	1034.5	0	180.5	1054.5	1050.1
M_{by}	- 163.7	678.9	77.8	554.0	702.7
M_{bx}	844.6	0	- 65.3	845.5	847.1
M_{cy}	163.7	678.9	- 77.8	824.2	702.7
M_{cx}	844.6	0	- 65.3	845.5	847.1
M_{dy}	327.4	1357.8	- 155.5	1648.4	1405.4
M_{dx}	1034.5	0	180.5	1054.5	1050.1

It is interesting to note that the differences between the SRSS and CQC methods only show up in those bending moments that have contributions of the first and second modes. These modes have closely spaced frequencies and, hence, the cross correlation term for these two modes is significant. Thus, all moments about the y-axis will be in significant error if estimated using the SRSS method.

Therefore, we conclude that, depending on the response quantity selected, the cross-correlation terms may or may not be important.

Problem 13.59



1. *Data.*

$L = 36$ in. $\quad m = 1.0$ kips/g

$E = 30000$ ksi $\quad I = 3.017\ \text{in}^4$

$G = 12000$ ksi $\quad J = 6.034\ \text{in}^4$

Note: $GJ = \dfrac{4}{5}EI$

2. *Natural frequencies and modes.*

$$m = 1.0\frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38\ \text{s}^{-1}$$

From Problems 10.28 and 13.27:

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}} = 13.24\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475\ \text{s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = 0.7767$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}} = 13.66\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460\ \text{s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = -0.2084$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}} = 40.59\frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155\ \text{s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.5943$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore\ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore\ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore\ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20\ \text{in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12\ \text{in}$$

Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127\ \text{in}$$

5. *Determine peak modal displacements.*

$$\mathbf{u}_n = \mathbf{u}_n^{st} A_n \qquad \mathbf{u}_n^{st} = \frac{\Gamma_n}{\omega_n^2}\phi_n$$

$$\therefore\ \mathbf{u}_n = \Gamma_n \phi_n D_n \qquad \text{(a)}$$

$$\mathbf{u}_1 = 0.7767 \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} 1.20 = \begin{Bmatrix} 0.7212 \\ -0.4572 \\ -0.3647 \end{Bmatrix}$$

$$\mathbf{u}_2 = -0.2084 \begin{Bmatrix} -0.2084 \\ 0.3975 \\ -0.8980 \end{Bmatrix} 1.12 = \begin{Bmatrix} 0.0488 \\ -0.0906 \\ 0.2100 \end{Bmatrix}$$

$$\mathbf{u}_3 = 0.5943 \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} 0.127 = \begin{Bmatrix} 0.0449 \\ 0.0589 \\ 0.0150 \end{Bmatrix}$$

6. *Combine peak modal displacements.*

Using Eqs. (13.7.3) and (13.7.4), the SRSS and CQC estimates for the peak displacements of the mass can be calculated. These estimates are summarized in the following table.

Displacement	SRSS rule (inches)	CQC rule (inches)
u_x	0.724	0.767
u_y	0.470	0.544
u_z	0.421	0.195

7. *Determine peak modal responses.*

$$M_{xn} = M_{xn}^{st} A_n \qquad M_{yn} = M_{yn}^{st} A_n \qquad T_n = T_n^{st} A_n \quad \text{(b)}$$

where M_{xn}^{st}, M_{yn}^{st} and T_n^{st} are given in the solution to Problem 13.27 as:

$$M_{xn}^{st} = [-0.0773mL, -0.2679mL, \; 0.3453mL]$$

$$M_{yn}^{st} = [-0.9084mL, \; 0.1437mL, -0.2353mL]$$

$$T_n^{st} = [\; 0.9858mL, \; 0.1242mL, \; -0.1100mL]$$

Substituting numerical values for M_{xn}^{st}, M_{yn}^{st}, T_n^{st} and A_n in Eq. (b) gives:

$M_{x1} = -1.509 \text{ k}\cdot\text{in}$ $\qquad M_{y1} = -17.725 \text{ k}\cdot\text{in}$

$M_{x2} = -5.228 \text{ k}\cdot\text{in}$ $\qquad M_{y2} = 2.804 \text{ k}\cdot\text{in}$

$M_{x3} = 6.737 \text{ k}\cdot\text{in}$ $\qquad M_{y3} = -4.591 \text{ k}\cdot\text{in}$

$T_1 = 19.234 \text{ k}\cdot\text{in}$

$T_2 = 2.424 \text{ k}\cdot\text{in}$

$T_3 = -2.146 \text{ k}\cdot\text{in}$

8. *Combine peak modal responses.*

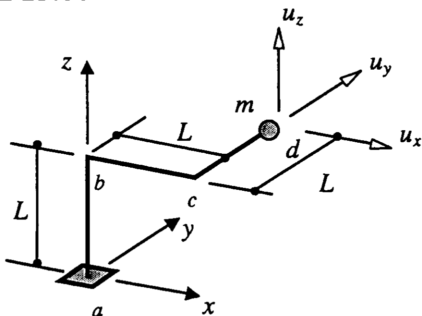
Using Eqs. (13.7.3) and (13.7.4), the SRSS and CQC estimates for the peak values of these responses can be calculated. These estimates are summarized in the following table.

Response	SRSS rule (k · in)	CQC rule (k · in)
M_x	8.66	9.42
M_y	18.52	15.92
T	19.50	21.55

9. *Comments.*

Modes 1 and 2 are strongly correlated with each other (ρ_{12}=0.9089), but weakly correlated with mode 3. Consequently, significant differences between the SRSS and CQC estimates should result only when both modes 1 and 2 contribute significantly to the total response. Examining the above results, we see that this is the case. Only the SRSS and CQC estimates for the displacement u_z differ significantly. The effect of modal correlation is less pronounced for the other displacements and responses either because (1) the contribution from mode 3 is larger than that of modes 1 and 2 (e.g. M_x) or (2) the contributions from modes 1 and 2 are not comparable (e.g. u_x, u_y, M_y and T).

Problem 13.60



1. *Data.*

$L = 36$ in. $\quad m = 1.0$ kips/g

$E = 30000$ ksi $\quad I = 3.017\ \text{in}^4$

$G = 12000$ ksi $\quad J = 6.034\ \text{in}^4$

Note: $GJ = \dfrac{4}{5}EI$

2. *Natural frequencies and modes.*

$$m = 1.0\frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38\ \text{s}^{-1}$$

From Problems 10.28 and 13.28:

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}} = 13.24\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475\ \text{s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = -0.4923$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}} = 13.66\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460\ \text{s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = 0.3875$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}} = 40.59\frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155\ \text{s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.7794$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore\ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore\ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore\ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20\ \text{in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12\ \text{in}$$

Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127\ \text{in}$$

5. *Determine peak modal displacements.*

$$\mathbf{u}_n = \mathbf{u}_n^{st} A_n \qquad \mathbf{u}_n^{st} = \frac{\Gamma_n}{\omega_n^2}\phi_n \qquad \text{(a)}$$

$$\therefore\ \mathbf{u}_n = \Gamma_n \phi_n D_n$$

$$\mathbf{u}_1 = -0.4923 \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} 1.20 = \begin{Bmatrix} -0.4572 \\ 0.2898 \\ 0.2312 \end{Bmatrix}$$

$$\mathbf{u}_2 = 0.3875 \begin{Bmatrix} -0.2084 \\ 0.3975 \\ -0.8980 \end{Bmatrix} 1.12 = \begin{Bmatrix} -0.0906 \\ 0.1685 \\ -0.3905 \end{Bmatrix}$$

$$\mathbf{u}_3 = 0.7794 \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} 0.127 = \begin{Bmatrix} 0.0589 \\ 0.0772 \\ 0.0197 \end{Bmatrix}$$

6. *Combine peak modal displacements.*

Using Eqs. (13.7.3) and (13.7.4), the SRSS and CQC estimates for the peak displacements of the mass can be calculated. These estimates are summarized in the following table.

Displacement	SRSS rule (inches)	CQC rule (inches)
u_x	0.470	0.544
u_y	0.344	0.456
u_z	0.454	0.205

7. *Determine peak modal responses.*

$$M_{xn} = M_{xn}^{st} A_n \qquad M_{yn} = M_{yn}^{st} A_n \qquad T_n = T_n^{st} A_n \quad \text{(b)}$$

where M_{xn}^{st}, M_{yn}^{st} and T_n^{st} are given in the solution to Problem 13.28 as:

$$M_{xn}^{st} = [\ 0.0490mL, \quad 0.4982mL, \quad 0.4528mL]$$

$$M_{yn}^{st} = [\ 0.5758mL, \ -0.2672mL, \ -0.3086mL]$$

$$T_n^{st} = [-0.6248mL, -0.2310mL, \ -0.1442mL]$$

Substituting numerical values for M_{xn}^{st}, M_{yn}^{st}, T_n^{st} and A_n in Eq. (b) gives:

$M_{x1} = 0.956$ k·in $\qquad M_{y1} = 11.235$ k·in

$M_{x2} = 9.721$ k·in $\qquad M_{y2} = -5.214$ k·in

$M_{x3} = 8.835$ k·in $\qquad M_{y3} = -6.021$ k·in

$T_1 = -12.191$ k·in

$T_2 = -4.507$ k·in

$T_3 = -2.814$ k·in

8. *Combine peak modal responses.*

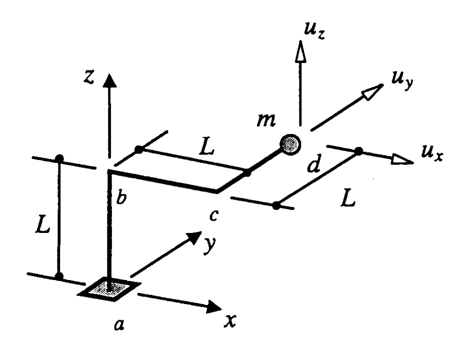
Using Eqs. (13.7.3) and (13.7.4), the SRSS and CQC estimates for the peak values of these responses can be calculated. These estimates are summarized in the following table.

Response	SRSS rule (k · in)	CQC rule (k · in)
M_x	13.17	13.84
M_y	13.77	9.10
T	13.30	16.65

9. *Comments.*

Modes 1 and 2 are strongly correlated with each other (ρ_{12}=0.9089), but weakly correlated with mode 3. Consequently, significant differences between the SRSS and CQC estimates should result only when both modes 1 and 2 contribute significantly to the total response. Examining the above results, we see that this is the case. The SRSS and CQC estimates for the displacements u_y and u_z and responses M_y and T differ significantly. The effect of modal correlation is less pronounced for u_x and M_x because the contributions from modes 1 and 2 are not comparable.

Problem 13.61



1. *Data.*

$L = 36$ in. $\quad m = 1.0$ kips/g

$E = 30000$ ksi $\quad I = 3.017\ \text{in}^4$

$G = 12000$ ksi $\quad J = 6.034\ \text{in}^4$

Note: $GJ = \dfrac{4}{5}EI$

2. *Natural frequencies and modes.*

$$m = 1.0\frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38\ \text{s}^{-1}$$

From Problems 10.28 and 13.29:

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}} = 13.24\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475\ \text{s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = -0.3928$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}} = 13.66\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460\ \text{s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = -0.8980$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}} = 40.59\frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155\ \text{s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.1984$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore\ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore\ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore\ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20\ \text{in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12\ \text{in}$$

Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127\ \text{in}$$

5. *Determine peak modal displacements.*

$$\mathbf{u}_n = \mathbf{u}_n^{st} A_n \qquad \mathbf{u}_n^{st} = \frac{\Gamma_n}{\omega_n^2}\phi_n \tag{a}$$

$$\therefore\ \mathbf{u}_n = \Gamma_n \phi_n D_n$$

$$\mathbf{u}_1 = -0.3928 \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} 1.20 = \begin{Bmatrix} -0.3647 \\ 0.2312 \\ 0.1844 \end{Bmatrix}$$

$$\mathbf{u}_2 = -0.8980 \begin{Bmatrix} -0.2084 \\ 0.3975 \\ -0.8980 \end{Bmatrix} 1.12 = \begin{Bmatrix} 0.2100 \\ -0.3905 \\ 0.9049 \end{Bmatrix}$$

$$\mathbf{u}_3 = 0.1984 \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} 0.127 = \begin{Bmatrix} 0.0150 \\ 0.0197 \\ 0.0050 \end{Bmatrix}$$

6. *Combine peak modal displacements.*

Using Eqs. (13.7.3) and (13.7.4), the SRSS and CQC estimates for the peak displacements of the mass can be calculated. These estimates are summarized in the following table.

Displacement	SRSS rule (inches)	CQC rule (inches)
u_x	0.421	0.195
u_y	0.454	0.205
u_z	0.923	1.075

7. *Determine peak modal responses.*

$$M_{xn} = M_{xn}^{st} A_n \qquad M_{yn} = M_{yn}^{st} A_n \qquad T_n = T_n^{st} A_n \quad \text{(b)}$$

where M_{xn}^{st}, M_{yn}^{st} and T_n^{st} are given in the solution to Problem 13.29 as:

$$M_{xn}^{st} = \begin{bmatrix} 0.0391mL, & -1.1544mL, & 0.1153mL \end{bmatrix}$$

$$M_{yn}^{st} = \begin{bmatrix} 0.4594mL, & 0.6192mL, & -0.0786mL \end{bmatrix}$$

$$T_n^{st} = \begin{bmatrix} -0.4985mL, & 0.5352mL, & -0.0367mL \end{bmatrix}$$

Substituting numerical values for M_{xn}^{st}, M_{yn}^{st}, T_n^{st} and A_n in Eq. (b) gives:

$M_{x1} = 0.763$ k·in $\qquad M_{y1} = 8.963$ k·in

$M_{x2} = -22.524$ k·in $\qquad M_{y2} = 12.082$ k·in

$M_{x3} = 2.249$ k·in $\qquad M_{y3} = -1.533$ k·in

$T_1 = -9.726$ k·in

$T_2 = 10.442$ k·in

$T_3 = -0.716$ k·in

8. *Combine peak modal responses.*

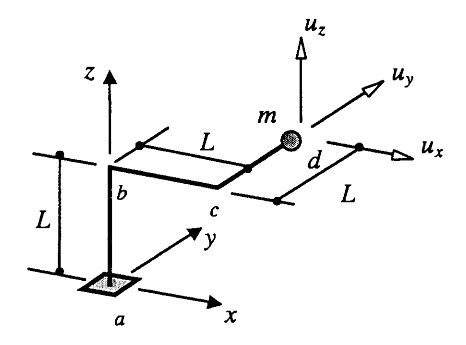
Using Eqs. (13.7.3) and (13.7.4), the SRSS and CQC estimates for the peak values of these responses can be calculated. These estimates are summarized in the following table.

Response	SRSS rule (k · in)	CQC rule (k · in)
M_x	22.65	21.93
M_y	15.12	20.62
T	14.29	4.42

9. *Comments.*

Modes 1 and 2 are strongly correlated with each other (ρ_{12}=0.9089), but weakly correlated with mode 3. Consequently, significant differences between the SRSS and CQC estimates should result only when both modes 1 and 2 contribute significantly to the total response. Examining the above results, we see that this is the case. The SRSS and CQC estimates for the displacements u_x and u_y and responses M_y and T differ significantly. The effect of modal correlation is less pronounced for u_z and M_x because the contributions from modes 1 and 2 are not comparable.

Problem 13.62



1. *Data.*

$L = 36$ in. $\qquad m = 1.0$ kips/g

$E = 30000$ ksi $\qquad I = 3.017$ in^4

$G = 12000$ ksi $\qquad J = 6.034$ in^4

Note: $GJ = \dfrac{4}{5}EI$

2. *Natural frequencies and modes.*

$$m = 1.0\frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38\ \text{s}^{-1}$$

From Problems 10.28 and 13.30:

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}} = 13.24\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475\text{ s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = -0.0626$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}} = 13.66\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460\text{ s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = -0.4150$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}} = 40.59\frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155\text{ s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.9077$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore\ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore\ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore\ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54\text{ g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20\text{ in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54\text{ g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12\text{ in}$$

Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54\text{ g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127\text{ in}$$

5. *Determine peak modal displacements.*

$$\mathbf{u}_n = \mathbf{u}_n^{st} A_n \qquad \mathbf{u}_n^{st} = \frac{\Gamma_n}{\omega_n^2}\phi_n \qquad \text{(a)}$$

$$\therefore\ \mathbf{u}_n = \Gamma_n \phi_n D_n$$

$$\mathbf{u}_1 = -0.0626\begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix}1.20 = \begin{Bmatrix} -0.0581 \\ 0.0368 \\ 0.0294 \end{Bmatrix}$$

$$\mathbf{u}_2 = -0.4150 \begin{Bmatrix} -0.2084 \\ 0.3975 \\ -0.8980 \end{Bmatrix} 1.12 = \begin{Bmatrix} 0.0971 \\ -0.1805 \\ 0.4182 \end{Bmatrix}$$

$$\mathbf{u}_3 = 0.9077 \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} 0.127 = \begin{Bmatrix} 0.0686 \\ 0.0899 \\ 0.0229 \end{Bmatrix}$$

6. *Combine peak modal displacements.*

Using Eqs. (13.7.3) and (13.7.4), the SRSS and CQC estimates for the peak displacements of the mass can be calculated. These estimates are summarized in the following table.

Displacement	SRSS rule (inches)	CQC rule (inches)
u_x	0.132	0.085
u_y	0.205	0.173
u_z	0.420	0.446

7. *Determine peak modal responses.*

$$M_{xn} = M_{xn}^{st} A_n \qquad M_{yn} = M_{yn}^{st} A_n \qquad T_n = T_n^{st} A_n \quad \text{(b)}$$

where M_{xn}^{st}, M_{yn}^{st} and T_n^{st} are given in the solution to Problem 13.30 as:

$$M_{xn}^{st} = [\ 0.0062mL, \ -0.5335mL, \ 0.5273mL]$$
$$M_{yn}^{st} = [\ 0.0732mL, \ 0.0262mL, \ -0.03594mL]$$
$$T_n^{st} = [-0.0794mL, \ 0.2474mL, \ -0.1680mL]$$

Substituting numerical values for M_{xn}^{st}, M_{yn}^{st}, T_n^{st} and A_n in Eq. (b) gives:

$M_{x1} = 0.122$ k · in
$M_{x2} = -10.410$ k · in
$M_{x3} = 10.289$ k · in

$M_{y1} = 1.428$ k · in
$M_{y2} = 5.584$ k · in
$M_{y3} = -7.012$ k · in

$T_1 = -1.549$ k · in
$T_2 = 4.826$ k · in
$T_3 = -3.277$ k · in

8. *Combine peak modal responses.*

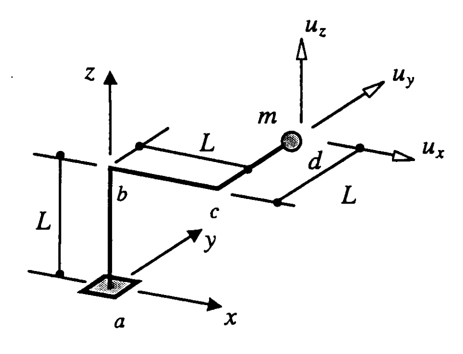
Using Eqs. (13.7.3) and (13.7.4), the SRSS and CQC estimates for the peak values of these responses can be calculated. These estimates are summarized in the following table.

Response	SRSS rule (k · in)	CQC rule (k · in)
M_x	14.64	14.51
M_y	9.08	9.81
T	6.04	4.76

9. *Comments.*

Modes 1 and 2 are strongly correlated with each other (ρ_{12}=0.9089), but weakly correlated with mode 3. Consequently, significant differences between the SRSS and CQC estimates should result only when both modes 1 and 2 contribute significantly to the total response. Examining the above results, we see that this is the case. The SRSS and CQC estimates for the displacement u_x and response T differ significantly. The effect of modal correlation is less pronounced for the other displacements and responses either because (1) the contribution from mode 3 is larger than that of modes 1 and 2 (e.g. M_y) or (2) the contributions from modes 1 and 2 are not comparable (e.g. u_y, u_z and M_x).

Problem 13.63



1. *Data.*

$L = 36$ in. $\quad m = 1.0$ kips/g

$E = 30000$ ksi $\quad I = 3.017\ \text{in}^4$

$G = 12000$ ksi $\quad J = 6.034\ \text{in}^4$

Note: $GJ = \dfrac{4}{5}EI$

2. *Natural frequencies and modes.*

$$m = 1.0\frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38\ \text{s}^{-1}$$

From Problems 10.28 and 13.27:

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}} = 13.24\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475\ \text{s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = 0.7767$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}} = 13.66\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460\ \text{s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = -0.2084$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}} = 40.59\frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155\ \text{s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.5943$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore\ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore\ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore\ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20\ \text{in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12\ \text{in}$$

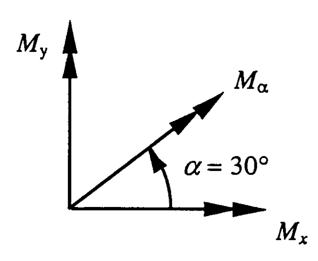
Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127\ \text{in}$$

5. *Determine modal static responses for M_α.*

The bending moment, M_α, about an axis oriented at an angle $\alpha = 30°$ counterclockwise from the *x*-axis can be expressed in terms of the bending moments, M_x and M_y, about the *x* and *y* axes as:



$$M_\alpha = M_x \cos\alpha + M_y \sin\alpha$$
$$= 0.866 M_x + 0.500 M_y$$

Therefore,

$$M_{\alpha n} = 0.866 M_{xn} + 0.500 M_{yn}$$
$$= (0.866 M_{xn}^{st} + 0.500 M_{yn}^{st}) A_n \qquad \text{(a)}$$
$$= M_{\alpha n}^{st} A_n$$

Thus,

$M_{\alpha n}^{st} = 0.866 M_{xn}^{st} + 0.500 M_{yn}^{st}$ where M_{xn}^{st} and M_{yn}^{st} are given in the solution to Problem 13.27:

$$M_{x1}^{st} = -0.0773 mL \qquad M_{y1}^{st} = -0.9085 mL$$
$$M_{x2}^{st} = -0.2680 mL \qquad M_{y2}^{st} = 0.1438 mL \qquad \text{(b)}$$
$$M_{x3}^{st} = 0.3453 mL \qquad M_{y3}^{st} = -0.2353 mL$$

Equation (a), after substituting Eq. (b) becomes

$$M_{\alpha 1}^{st} = -0.5212 mL$$
$$M_{\alpha 2}^{st} = -0.1602 mL$$
$$M_{\alpha 3}^{st} = 0.1814 mL$$

6. *Determine peak modal responses* $M_{\alpha n}$.

The peak values, $M_{\alpha n}$, of the modal contributions $M_{\alpha n}(t)$ to response M_α are determined next

$$M_{\alpha 1} = -0.5212\, mLA_1 = -10.17 \text{ k}\cdot\text{in}$$
$$M_{\alpha 2} = -0.1602\, mLA_2 = -3.13 \text{ k}\cdot\text{in}$$
$$M_{\alpha 3} = 0.1814\, mLA_3 = 3.54 \text{ k}\cdot\text{in}$$

7. *Combine peak modal responses.*

(a) SRSS rule

Using Eq. (13.7.3), the SRSS estimate for the peak value of M_α can be calculated:

$$M_\alpha = \left(M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2\right)^{1/2}$$
$$= 11.21 \text{ k}\cdot\text{in}$$

(b) CQC rule

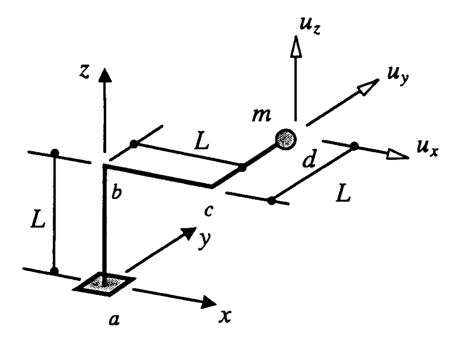
Using Eq. (13.7.4), the CQC estimate for the peak value of M_α can be calculated :

$$M_\alpha = (M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2 + 2\rho_{12} M_{\alpha 1} M_{\alpha 2}$$
$$+ 2\rho_{13} M_{\alpha 1} M_{\alpha 3} + 2\rho_{23} M_{\alpha 2} M_{\alpha 3})^{1/2}$$
$$= 13.52 \text{ k}\cdot\text{in}$$

8. *Comments.*

Because modes 1 and 2 are strongly correlated ($\rho_{12} = 0.9089$) and both contribute significantly to the total response, the SRSS and CQC estimates differ significantly.

Problem 13.64



1. *Data.*

$L = 36$ in. $\quad m = 1.0$ kips/g

$E = 30000$ ksi $\quad I = 3.017\text{ in}^4$

$G = 12000$ ksi $\quad J = 6.034\text{ in}^4$

Note: $GJ = \frac{4}{5}EI$

2. *Natural frequencies and modes.*

$$m = 1.0\frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38\text{ s}^{-1}$$

From Problems 10.28 and 13.28:

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}} = 13.24\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475\text{ s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = -0.4923$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}} = 13.66\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460\text{ s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = 0.3875$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}} = 40.59\frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155\text{ s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.7794$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore\ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore\ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore\ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54\text{ g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20\text{ in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54\text{ g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12\text{ in}$$

Mode 3: $T_3 = 0.155$ s

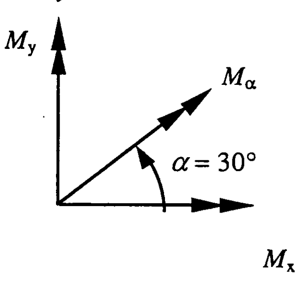
$$A_3 = 0.20\,(2.71\text{g}) = 0.54\text{ g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127\text{ in}$$

5. *Determine modal static responses for M_α.*

The bending moment, M_α, about an axis oriented at an angle α=30° counterclockwise from the x-axis can be

expressed in terms of the bending moments, M_x and M_y, about the x and y axes as:



$$M_\alpha = M_x \cos\alpha + M_y \sin\alpha$$
$$= 0.866 M_x + 0.500 M_y$$

Therefore,

$$M_{\alpha n} = 0.866 M_{xn} + 0.500 M_{yn}$$
$$= (0.866 M_{xn}^{st} + 0.500 M_{yn}^{st}) A_n$$
$$= M_{\alpha n}^{st} A_n$$

Thus,

$$M_{\alpha n}^{st} = 0.866 M_{xn}^{st} + 0.500 M_{yn}^{st} \qquad \text{(a)}$$

where M_{xn}^{st} and M_{yn}^{st} are given in the solution to Problem 13.28:

$$M_{x1}^{st} = 0.0490mL \qquad M_{y1}^{st} = 0.5758mL$$
$$M_{x2}^{st} = 0.4982mL \qquad M_{y2}^{st} = -0.2672mL \qquad \text{(b)}$$
$$M_{x3}^{st} = 0.4528mL \qquad M_{y3}^{st} = -0.3086mL$$

Equation (a), after substituting Eq. (b) becomes

$$M_{\alpha 1}^{st} = 0.3303mL$$
$$M_{\alpha 2}^{st} = 0.2978mL$$
$$M_{\alpha 3}^{st} = 0.2378mL$$

6. *Determine peak modal responses $M_{\alpha n}$.*

The peak values, $M_{\alpha n}$, of the modal contributions $M_{\alpha n}(t)$ to response M_α are determined next

$$M_{\alpha 1} = 0.3303\, mLA_1 = 6.45 \text{ k}\cdot\text{in}$$
$$M_{\alpha 2} = 0.2978\, mLA_2 = 5.81 \text{ k}\cdot\text{in}$$
$$M_{\alpha 3} = 0.2378\, mLA_3 = 4.64 \text{ k}\cdot\text{in}$$

7. *Combine peak modal responses.*

(a) SRSS rule

Using Eq. (13.7.3), the SRSS estimate for the peak value of M_α can be calculated:

$$M_\alpha = \left(M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2\right)^{1/2}$$
$$= 9.84 \text{ k}\cdot\text{in}$$

(b) CQC rule

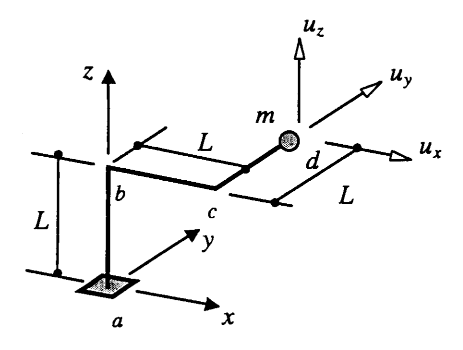
Using Eq. (13.7.4), the CQC estimate for the peak value of M_α can be calculated :

$$M_\alpha = (M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2 + 2\rho_{12} M_{\alpha 1} M_{\alpha 2}$$
$$+ 2\rho_{13} M_{\alpha 1} M_{\alpha 3} + 2\rho_{23} M_{\alpha 2} M_{\alpha 3})^{1/2}$$
$$= 12.87 \text{ k}\cdot\text{in}$$

8. *Comments.*

Because modes 1 and 2 are strongly correlated ($\rho_{12} = 0.9089$) and both contribute significantly to the total response, the SRSS and CQC estimates differ significantly.

Problem 13.65



1. *Data.*

$L = 36$ in. $\quad m = 1.0$ kips/g

$E = 30000$ ksi $\quad I = 3.017\text{ in}^4$

$G = 12000$ ksi $\quad J = 6.034\text{ in}^4$

Note: $GJ = \frac{4}{5}EI$

2. *Natural frequencies and modes.*

$$m = 1.0\frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38\text{ s}^{-1}$$

From Problems 10.28 and 13.29:

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}} = 13.24\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475\text{ s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = -0.3928$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}} = 13.66\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460\text{ s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = -0.8980$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}} = 40.59\frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155\text{ s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.1984$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore\ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore\ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore\ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54\text{ g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20\text{ in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54\text{ g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12\text{ in}$$

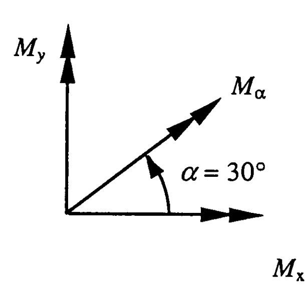
Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54\text{ g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127\text{ in}$$

5. *Determine modal static responses for M_α.*

The bending moment, M_α, about an axis oriented at an angle α=30° counterclockwise from the *x*-axis can be expressed in terms of the bending moments, M_x and M_y, about the *x* and *y* axes as:



$$M_\alpha = M_x \cos\alpha + M_y \sin\alpha$$

$$= 0.866 M_x + 0.500 M_y$$

Therefore,

$$M_{\alpha n} = 0.866 M_{xn} + 0.500 M_{yn}$$

$$= (0.866 M_{xn}^{st} + 0.500 M_{yn}^{st}) A_n$$

$$= M_{\alpha n}^{st} A_n$$

Thus,

$$M_{\alpha n}^{st} = 0.866 M_{xn}^{st} + 0.500 M_{yn}^{st} \qquad \text{(a)}$$

where M_{xn}^{st} and M_{yn}^{st} are given in the solution to Problem 13.29:

$$\begin{aligned} M_{x1}^{st} &= 0.0392 mL & M_{y1}^{st} &= 0.4593 mL \\ M_{x2}^{st} &= -1.1544 mL & M_{y2}^{st} &= 0.6192 mL \\ M_{x3}^{st} &= 0.1152 mL & M_{y3}^{st} &= -0.0785 mL \end{aligned} \qquad \text{(b)}$$

Equation (a), after substituting Eq. (b) becomes

$$M_{\alpha 1}^{st} = 0.2635 mL$$

$$M_{\alpha 2}^{st} = -0.6901 mL$$

$$M_{\alpha 3}^{st} = 0.0605 mL$$

6. *Determine peak modal responses $M_{\alpha n}$.*

The peak values, $M_{\alpha n}$, of the modal contributions $M_{\alpha n}(t)$ to response M_α are determined next

$$M_{\alpha 1} = 0.2635\, mLA_1 = 5.14 \text{ k}\cdot\text{in}$$

$$M_{\alpha 2} = -0.6901\, mLA_2 = -13.47 \text{ k}\cdot\text{in}$$

$$M_{\alpha 3} = 0.0605\, mLA_3 = 1.18 \text{ k}\cdot\text{in}$$

7. *Combine peak modal responses.*

(a) SRSS rule

Using Eq. (13.7.3), the SRSS estimate for the peak value of M_α can be calculated:

$$M_\alpha = \left(M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2\right)^{1/2}$$

$$= 14.46 \text{ k}\cdot\text{in}$$

(b) CQC rule

Using Eq. (13.7.4), the CQC estimate for the peak value of M_α can be calculated :

$$M_\alpha = (M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2 + 2\rho_{12} M_{\alpha 1} M_{\alpha 2}$$

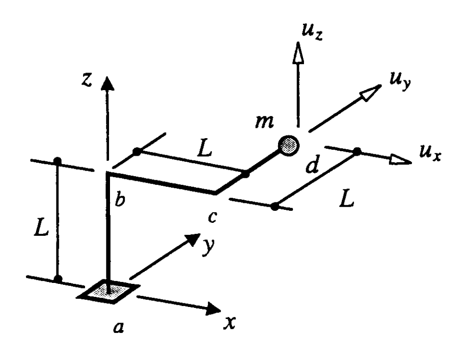
$$+ 2\rho_{13} M_{\alpha 1} M_{\alpha 3} + 2\rho_{23} M_{\alpha 2} M_{\alpha 3})^{1/2}$$

$$= 9.12 \text{ k}\cdot\text{in}$$

8. *Comments.*

Because modes 1 and 2 are strongly correlated ($\rho_{12} = 0.9089$) and both contribute significantly to the total response, the SRSS and CQC estimates differ significantly.

Problem 13.66



1. *Data.*

$L = 36$ in. $\quad m = 1.0$ kips/g

$E = 30000$ ksi $\quad I = 3.017\ \text{in}^4$

$G = 12000$ ksi $\quad J = 6.034\ \text{in}^4$

Note: $GJ = \frac{4}{5}EI$

2. *Natural frequencies and modes.*

$$m = 1.0\frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38\ \text{s}^{-1}$$

From Problems 10.28 and 13:30:

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}} = 13.24\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475\ \text{s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = -0.0626$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}} = 13.66\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460\ \text{s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = -0.4150$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}} = 40.59\frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155\ \text{s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.9077$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore\ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore\ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore\ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20\ \text{in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12\ \text{in}$$

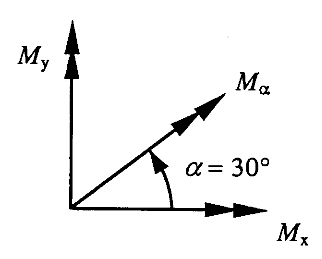
Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127\ \text{in}$$

5. *Determine modal static responses for M_α*

The bending moment, M_α, about an axis oriented at an angle α=30° counterclockwise from the x-axis can be expressed in terms of the bending moments, M_x and M_y, about the x and y axes as:



$$M_\alpha = M_x \cos\alpha + M_y \sin\alpha$$

$$= 0.866 M_x + 0.500 M_y$$

Therefore,

$$M_{\alpha n} = 0.866 M_{xn} + 0.500 M_{yn}$$

$$= (0.866 M_{xn}^{st} + 0.500 M_{yn}^{st}) A_n$$

$$= M_{\alpha n}^{st} A_n$$

Thus,

$$M_{\alpha n}^{st} = 0.866 M_{xn}^{st} + 0.500 M_{yn}^{st} \qquad \text{(a)}$$

where M_{xn}^{st} and M_{yn}^{st} are given in the solution to Problem 13.30:

$$\begin{aligned} M_{x1}^{st} &= 0.0062mL & M_{y1}^{st} &= 0.0731mL \\ M_{x2}^{st} &= -0.5335mL & M_{y2}^{st} &= 0.2862mL \qquad \text{(b)} \\ M_{x3}^{st} &= 0.5273mL & M_{y3}^{st} &= -0.3593mL \end{aligned}$$

Equation (a), after substituting Eq. (b) becomes

$$M_{\alpha 1}^{st} = 0.0420mL$$

$$M_{\alpha 2}^{st} = -0.3190mL$$

$$M_{\alpha 3}^{st} = 0.2770mL$$

6. *Determine peak modal responses $M_{\alpha n}$.*

The peak values, $M_{\alpha n}$, of the modal contributions $M_{\alpha n}(t)$ to response M_α are determined next

$$M_{\alpha 1} = 0.0420\, mLA_1 = 0.82 \text{ k}\cdot\text{in}$$

$$M_{\alpha 2} = -0.3190\, mLA_2 = -6.22 \text{ k}\cdot\text{in}$$

$$M_{\alpha 3} = 0.2770\, mLA_3 = 5.40 \text{ k}\cdot\text{in}$$

7. *Combine peak modal responses.*

(a) SRSS rule

Using Eq. (13.7.3), the SRSS estimate for the peak value of M_α can be calculated:

$$M_\alpha = \left(M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2\right)^{1/2}$$

$$= 8.28 \text{ k}\cdot\text{in}$$

(b) CQC rule

Using Eq. (13.7.4), the CQC estimate for the peak value of M_α can be calculated :

$$M_\alpha = (M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2 + 2\rho_{12} M_{\alpha 1} M_{\alpha 2}$$

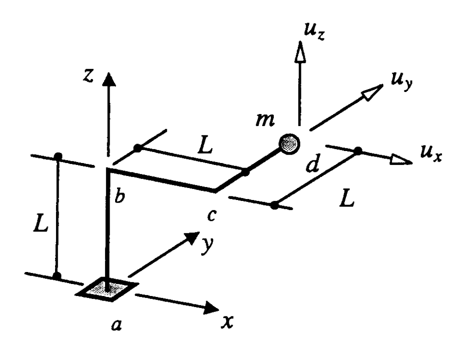
$$+ 2\rho_{13} M_{\alpha 1} M_{\alpha 3} + 2\rho_{23} M_{\alpha 2} M_{\alpha 3})^{1/2}$$

$$= 7.68 \text{ k}\cdot\text{in}$$

8. *Comments.*

The contributions to the total response from modes 2 and 3 are much larger than the contribution from mode 1. Because modes 2 and 3 are only weakly correlated (ρ_{23} = 0.0066), the SRSS and CQC estimates do not differ significantly.

Problem 13.67



1. *Data.*

$L = 36$ in. $\qquad m = 1.0$ kips/g

$E = 30000$ ksi $\qquad I = 3.017\ \text{in}^4$

$G = 12000$ ksi $\qquad J = 6.034\ \text{in}^4$

Note: $GJ = \frac{4}{5}EI$

2. *Natural frequencies and modes.*

$$m = 1.0\frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38\ \text{s}^{-1}$$

From Problems 10.28 and 13.27:

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}} = 13.24\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475\ \text{s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = 0.7767$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}} = 13.66\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460\ \text{s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = -0.2084$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}} = 40.59\frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155\ \text{s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.5943$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore\ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore\ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore\ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20\ \text{in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12\ \text{in}$$

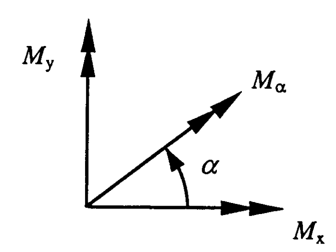
Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127\ \text{in}$$

5. *Determine modal static responses for M_α.*

The bending moment, M_α, about an axis oriented at an angle α counterclockwise from the x-axis can be expressed in terms of the bending moments, M_x and M_y, about the x and y axes as:



$$M_\alpha = M_x \cos\alpha + M_y \sin\alpha$$

Therefore,

$$M_{\alpha n} = M_{xn} \cos\alpha + M_{yn} \sin\alpha$$
$$= (M_{xn}^{st} \cos\alpha + M_{yn}^{st} \sin\alpha) A_n$$
$$= M_{\alpha n}^{st} A_n$$

Thus,

$$M_{\alpha n}^{st} = M_{xn}^{st} \cos\alpha + M_{yn}^{st} \sin\alpha \qquad \text{(a)}$$

where M_{xn}^{st} and M_{yn}^{st} are given in the solution to Problem 13.27:

$$\begin{aligned} M_{x1}^{st} &= -0.0773mL & M_{y1}^{st} &= -0.9085mL \\ M_{x2}^{st} &= -0.2680mL & M_{y2}^{st} &= 0.1438mL \\ M_{x3}^{st} &= 0.3453mL & M_{y3}^{st} &= -0.2353mL \end{aligned} \qquad \text{(b)}$$

Equation (a), after substituting Eq. (b) becomes

$$M_{\alpha 1}^{st} = (-0.0773\cos\alpha - 0.9085\sin\alpha)mL$$

$$M_{\alpha 2}^{st} = (-0.2680\cos\alpha + 0.1438\sin\alpha)mL$$

$$M_{\alpha 3}^{st} = (0.3453\cos\alpha - 0.2353\sin\alpha)mL$$

6. *Determine peak modal responses $M_{\alpha n}$.*

The peak values, $M_{\alpha n}$, of the modal contributions $M_{\alpha n}(t)$ to response M_α are determined next

$$M_{\alpha 1} = [(-0.0773\cos\alpha - 0.9085\sin\alpha)\ mL]A_1$$
$$= -1.51\cos\alpha - 17.73\sin\alpha$$

$$M_{\alpha 2} = [(-0.2680\cos\alpha + 0.1438\sin\alpha)\ mL]A_2$$
$$= -5.23\cos\alpha + 2.80\sin\alpha$$

$$M_{\alpha 3} = [(0.3453\cos\alpha - 0.2353\sin\alpha)\ mL]A_3$$
$$= 6.74\cos\alpha - 4.59\sin\alpha$$

7. *Combine peak modal responses.*

(a) SRSS rule

Using Eq. (13.7.3), the SRSS estimate for the peak value of M_α can be calculated:

$$\begin{aligned} M_\alpha &= \left(M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2\right)^{1/2} \\ &= \big[(-1.51\cos\alpha - 17.73\sin\alpha)^2 \\ &\quad + (-5.23\cos\alpha + 2.80\sin\alpha)^2 \\ &\quad + (6.74\cos\alpha - 4.59\sin\alpha)^2\big]^{1/2} \\ &= \left(A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha\right)^{1/2} \end{aligned} \qquad \text{(c)}$$

where

$A = 343.12\ \text{k}\cdot\text{in} \quad B = -37.70\ \text{k}\cdot\text{in} \quad C = 75.00\ \text{k}\cdot\text{in}$

(b) CQC rule

Using Eq. (13.7.4), the CQC estimate for the peak value of M_α can be calculated :

$$\begin{aligned} M_\alpha &= (M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2 + 2\rho_{12}M_{\alpha 1}M_{\alpha 2} \\ &\quad + 2\rho_{13}M_{\alpha 1}M_{\alpha 3} + 2\rho_{23}M_{\alpha 2}M_{\alpha 3})^{1/2} \\ &= (A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha)^{1/2} \end{aligned} \qquad \text{(d)}$$

where

$A = 253.59\ \text{k}\cdot\text{in} \quad B = 122.25\ \text{k}\cdot\text{in} \quad C = 88.75\ \text{k}\cdot\text{in}$

8. *Determine maximum of M_α over α.*

To find the value of α corresponding to the maximum value of M_α, solve

$$\frac{dM_\alpha}{d\alpha} = 0 \quad \text{for} \quad \alpha$$

$$\frac{dM_\alpha}{d\alpha} = \frac{1}{2}\frac{2(A-C)\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha)}{\sqrt{A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha}}$$
$$= \frac{1}{2}\frac{(A-C)\sin 2\alpha + B\cos 2\alpha}{\sqrt{A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha}}$$

Therefore, the maximum value of M_α occurs for α that satisfies

$$(A-C)\sin 2\alpha + B\cos 2\alpha = 0$$

Hence,

$$\tan 2\alpha = \frac{-B}{A-C} \qquad \text{(e)}$$

(a) SRSS rule

Substituting A, B and C from Eq. (c) into Eq. (e) gives

$$\tan 2\alpha = 0.1406$$

$$\alpha = \frac{1}{2}\tan^{-1}(0.1406) = 0.0698 \text{ rad}, \quad -1.5010 \text{ rad}$$

The two values of α obtained correspond to the maximum and minimum values of M_α. The value of α corresponding to the maximum value of M_α can be ascertained by substituting each value of α into Eq. (c)

for $\alpha = 0.0698$ rad, $\quad M_\alpha = 8.58 \text{ k} \cdot \text{in}$

for $\alpha = -1.5010$ rad, $\quad M_\alpha = 18.56 \text{ k} \cdot \text{in}$

Thus, the peak value of M_α estimated by the SRSS combination rule is $M_\alpha = 18.56$ k·in about an axis oriented 1.5010 rad (86.0°) clockwise from the x-axis.

(b) CQC rule

Substituting A, B and C from Eq. (d) into Eq. (e) gives

$$\tan 2\alpha = -0.7416$$

$$\alpha = \frac{1}{2}\tan^{-1}(-0.7416) = -0.3191 \text{ rad}, \quad 1.2517 \text{ rad}$$

for $\alpha = -0.3191$ rad, $\quad M_\alpha = 8.28 \text{ k} \cdot \text{in}$

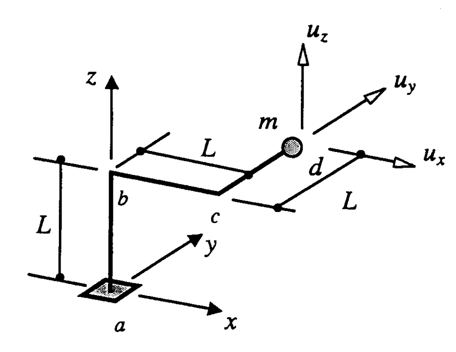
for $\alpha = 1.2517$ rad, $\quad M_\alpha = 16.55 \text{ k} \cdot \text{in}$

Thus, the peak value of M_α estimated by the CQC combination rule is $M_\alpha = 16.55$ k·in about an axis oriented 1.2517 rad (71.7°) counterclockwise from the x-axis.

9. *Comments.*

Modes 1 and 2 are strongly correlated ($\rho_{12} = 0.9089$); hence, the CQC estimate, which accounts for this correlation, is expected to be more accurate than the SRSS estimate. Note that the SRSS and CQC estimates for the peak value of M_α differ by 11.4%. The values of α predicted by the two combination rules differ by 14.3°.

Problem 13.68



1. *Data.*

$L = 36$ in. $\quad m = 1.0$ kips/g

$E = 30000$ ksi $\quad I = 3.017$ in^4

$G = 12000$ ksi $\quad J = 6.034$ in^4

Note: $GJ = \dfrac{4}{5}EI$

2. *Natural frequencies and modes.*

$$m = 1.0\frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259\ \frac{\text{k}\cdot\text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38\ \text{s}^{-1}$$

From Problems 10.28 and 13.28:

$$\omega_1 = 0.4834\sqrt{\frac{EI}{mL^3}} = 13.24\frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475\ \text{s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = -0.4923$$

$$\omega_2 = 0.4990\sqrt{\frac{EI}{mL^3}} = 13.66\frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460\ \text{s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = 0.3875$$

$$\omega_3 = 1.4827\sqrt{\frac{EI}{mL^3}} = 40.59\frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155\ \text{s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.7794$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore\ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore\ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore\ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20\ \text{in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12\ \text{in}$$

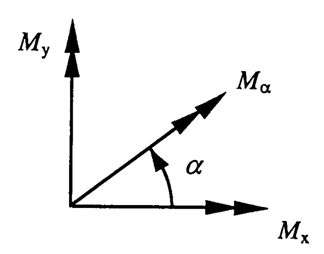
Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54\ \text{g} = 209\frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127\ \text{in}$$

5. *Determine modal static responses for M_α.*

The bending moment, M_α, about an axis oriented at an angle α counterclockwise from the x-axis can be expressed in terms of the bending moments, M_x and M_y, about the x and y axes as:



$$M_\alpha = M_x \cos\alpha + M_y \sin\alpha$$

Therefore,

$$M_{\alpha n} = M_{xn}\cos\alpha + M_{yn}\sin\alpha$$
$$= (M_{xn}^{st}\cos\alpha + M_{yn}^{st}\sin\alpha)A_n$$
$$= M_{\alpha n}^{st} A_n$$

Thus,

$$M_{\alpha n}^{st} = M_{xn}^{st}\cos\alpha + M_{yn}^{st}\sin\alpha \qquad \text{(a)}$$

where M_{xn}^{st} and M_{yn}^{st} are given in the solution to Problem 13.28:

$$\begin{aligned} M_{x1}^{st} &= 0.0490mL & M_{y1}^{st} &= 0.5758mL \\ M_{x2}^{st} &= 0.4982mL & M_{y2}^{st} &= -0.2672mL \\ M_{x3}^{st} &= 0.4528mL & M_{y3}^{st} &= -0.3086mL \end{aligned} \qquad \text{(b)}$$

Equation (a) after substituting Eq. (b) becomes

$$M_{\alpha 1}^{st} = (0.0490\cos\alpha + 0.5758\sin\alpha)mL$$
$$M_{\alpha 2}^{st} = (0.4982\cos\alpha - 0.2672\sin\alpha)mL$$
$$M_{\alpha 3}^{st} = (0.4528\cos\alpha - 0.3086\sin\alpha)mL$$

6. *Determine peak modal responses $M_{\alpha n}$.*

The peak values, $M_{\alpha n}$, of the modal contributions $M_{\alpha n}(t)$ to response M_α are determined next

$$M_{\alpha 1} = [(0.0490\cos\alpha + 0.5758\sin\alpha)\, mL]A_1$$
$$= 0.96\cos\alpha + 11.23\sin\alpha$$
$$M_{\alpha 2} = [(0.4982\cos\alpha - 0.2672\sin\alpha)\, mL]A_2$$
$$= 9.72\cos\alpha - 5.21\sin\alpha$$
$$M_{\alpha 3} = [(0.4528\cos\alpha - 0.3086\sin\alpha)\, mL]A_3$$
$$= 8.83\cos\alpha - 6.02\sin\alpha$$

7. *Combine peak modal responses.*

(a) SRSS rule

Using Eq. (13.7.3), the SRSS estimate for the peak value of M_α can be calculated:

$$\begin{aligned} M_\alpha &= \left(M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2\right)^{1/2} \\ &= \big[(0.96\cos\alpha + 11.23\sin\alpha)^2 \\ &\quad + (9.72\cos\alpha - 5.21\sin\alpha)^2 \\ &\quad + (8.83\cos\alpha - 6.02\sin\alpha)^2\big]^{1/2} \\ &= \left(A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha\right)^{1/2} \end{aligned} \qquad \text{(c)}$$

where

$$A = 189.66 \text{ k}\cdot\text{in} \quad B = -186.27 \text{ k}\cdot\text{in} \quad C = 173.46 \text{ k}\cdot\text{in}$$

(b) CQC rule

Using Eq. (13.7.4), the CQC estimate for the peak value of M_α can be calculated :

$$\begin{aligned} M_\alpha &= (M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2 + 2\rho_{12}M_{\alpha 1}M_{\alpha 2} \\ &\quad + 2\rho_{13}M_{\alpha 1}M_{\alpha 3} + 2\rho_{23}M_{\alpha 2}M_{\alpha 3})^{1/2} \\ &= (A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha)^{1/2} \end{aligned} \qquad \text{(d)}$$

where

$$A = 82.75 \text{ k}\cdot\text{in} \quad B = 2.97 \text{ k}\cdot\text{in} \quad C = 191.60 \text{ k}\cdot\text{in}$$

8. *Determine maximum of M_α over α.*

To find the value of α corresponding to the maximum value of M_α, solve

$$\frac{dM_\alpha}{d\alpha} = 0 \quad \text{for} \quad \alpha$$

$$\frac{dM_\alpha}{d\alpha} = \frac{1}{2}\frac{2(A-C)\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha)}{\sqrt{A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha}}$$
$$= \frac{1}{2}\frac{(A-C)\sin 2\alpha + B\cos 2\alpha}{\sqrt{A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha}}$$

Therefore, the maximum value of M_α occurs for α that satisfies

$$(A-C)\sin 2\alpha + B\cos 2\alpha = 0$$

Hence,

$$\tan 2\alpha = \frac{-B}{A-C} \qquad \text{(e)}$$

(a) SRSS rule

Substituting *A, B* and *C* from Eq. (c) into Eq. (e) gives

$$\tan 2\alpha = 11.5000$$

$$\alpha = \frac{1}{2}\tan^{-1}(11.5000) = 0.7420 \text{ rad}, \quad -0.8288 \text{ rad}$$

The two values of α obtained correspond to the maximum and minimum values of M_α. The value of α corresponding to the maximum value of M_α can be ascertained by substituting each value of α into Eq. (c)

$$\text{for } \alpha = 0.7420 \text{ rad}, \qquad M_\alpha = 9.38 \text{ k}\cdot\text{in}$$

$$\text{for } \alpha = -0.8288 \text{ rad}, \qquad M_\alpha = 16.58 \text{ k}\cdot\text{in}$$

Thus, the peak value of M_α estimated by the SRSS combination rule is $M_\alpha = 16.58$ k·in about an axis oriented 0.8288 rad (47.5°) clockwise from the x-axis.

(b) CQC rule

Substituting *A, B* and *C* from Eq. (d) into Eq. (e) gives

$$\tan 2\alpha = 0.0273$$

$$\alpha = \frac{1}{2}\tan^{-1}(0.0273) = 0.0136 \text{ rad}, \quad -1.5572 \text{ rad}$$

$$\text{for } \alpha = 0.0136 \text{ rad}, \qquad M_\alpha = 13.84 \text{ k}\cdot\text{in}$$

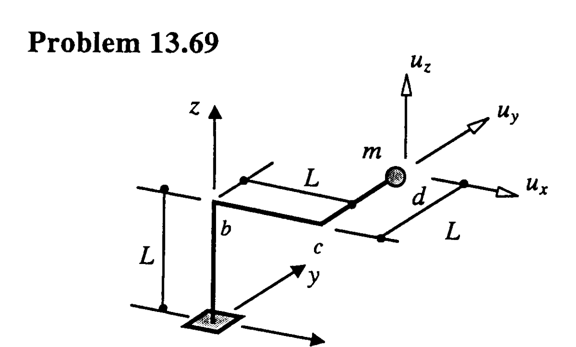
$$\text{for } \alpha = -1.5572 \text{ rad}, \qquad M_\alpha = 9.10 \text{ k}\cdot\text{in}$$

Thus, the peak value of M_α estimated by the CQC combination rule is $M_\alpha = 13.84$ k·in about an axis oriented 0.0136 rad (0.78°) counterclockwise from the x-axis.

9. *Comments.*

Modes 1 and 2 are strongly correlated ($\rho_{12} = 0.9089$); hence, the CQC estimate, which accounts for this correlation, is expected to be more accurate than the SRSS estimate. Note that the SRSS and CQC estimates for the peak value of M_α differ by 18.0%. The values of α predicted by the two combination rules differ by 46.7°.

Problem 13.69



1. *Data.*

$L = 36$ in. $\quad m = 1.0$ kips/g

$E = 30000$ ksi $\quad I = 3.017 \text{ in}^4$

$G = 12000$ ksi $\quad J = 6.034 \text{ in}^4$

Note: $GJ = \dfrac{4}{5} EI$

2. *Natural frequencies and modes.*

$$m = 1.0 \frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259 \ \frac{\text{k} \cdot \text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38 \text{ s}^{-1}$$

From Problems 10.28 and 13.29:

$$\omega_1 = 0.4834 \sqrt{\frac{EI}{mL^3}} = 13.24 \frac{\text{rad}}{\text{s}} \Rightarrow T_1 = 0.475 \text{ s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = -0.3928$$

$$\omega_2 = 0.4990 \sqrt{\frac{EI}{mL^3}} = 13.66 \frac{\text{rad}}{\text{s}} \Rightarrow T_2 = 0.460 \text{ s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = -0.8980$$

$$\omega_3 = 1.4827 \sqrt{\frac{EI}{mL^3}} = 40.59 \frac{\text{rad}}{\text{s}} \Rightarrow T_3 = 0.155 \text{ s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.1984$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore \ \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore \ \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore \ \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54 \text{ g} = 209 \frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20 \text{ in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54 \text{ g} = 209 \frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12 \text{ in}$$

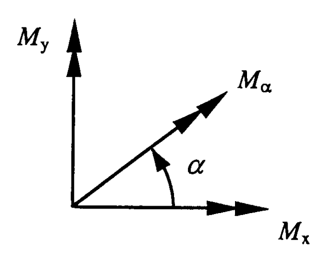
Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54 \text{ g} = 209 \frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127 \text{ in}$$

5. *Determine modal static responses for M_α*

The bending moment, M_α, about an axis oriented at an angle α counterclockwise from the x-axis can be expressed in terms of the bending moments, M_x and M_y, about the x and y axes as:



$$M_\alpha = M_x \cos\alpha + M_y \sin\alpha$$

Therefore,

$$M_{\alpha n} = M_{xn} \cos\alpha + M_{yn} \sin\alpha$$
$$= (M_{xn}^{st} \cos\alpha + M_{yn}^{st} \sin\alpha) A_n$$
$$= M_{\alpha n}^{st} A_n$$

Thus,

$$M_{\alpha n}^{st} = M_{xn}^{st} \cos\alpha + M_{yn}^{st} \sin\alpha \qquad \text{(a)}$$

where M_{xn}^{st} and M_{yn}^{st} are given in the solution to Problem 13.29:

$$M_{x1}^{st} = 0.0392 mL \qquad M_{y1}^{st} = 0.4593 mL$$
$$M_{x2}^{st} = -1.1544 mL \qquad M_{y2}^{st} = 0.6192 mL \qquad \text{(b)}$$
$$M_{x3}^{st} = 0.1152 mL \qquad M_{y3}^{st} = -0.0785 mL$$

Equation (a) after substituting Eq. (b) becomes

$$M_{\alpha 1}^{st} = (0.0392\cos\alpha + 0.4593\sin\alpha) mL$$
$$M_{\alpha 2}^{st} = (-1.1544\cos\alpha + 0.6192\sin\alpha) mL$$
$$M_{\alpha 3}^{st} = (0.1152\cos\alpha - 0.0785\sin\alpha) mL$$

6. *Determine peak modal responses $M_{\alpha n}$.*

The peak values, $M_{\alpha n}$, of the modal contributions $M_{\alpha n}(t)$ to response M_α are determined next

$$M_{\alpha 1} = [(0.0392\cos\alpha + 0.4593\sin\alpha)\, mL] A_1$$
$$= 0.76\cos\alpha + 8.96\sin\alpha$$
$$M_{\alpha 2} = [(-1.1544\cos\alpha + 0.6192\sin\alpha)\, mL] A_2$$
$$= -22.52\cos\alpha + 12.08\sin\alpha$$
$$M_{\alpha 3} = [(0.1152\cos\alpha - 0.0785\sin\alpha)\, mL] A_3$$
$$= 2.25\cos\alpha - 1.53\sin\alpha$$

7. *Combine peak modal responses.*

(a) SRSS rule

Using Eq. (13.7.3), the SRSS estimate for the peak value of M_α can be calculated:

$$\begin{aligned} M_\alpha &= \left(M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2\right)^{1/2} \\ &= \big[(0.76\cos\alpha + 8.96\sin\alpha)^2 \\ &\quad + (-22.52\cos\alpha + 12.08\sin\alpha)^2 \\ &\quad + (2.25\cos\alpha - 1.53\sin\alpha)^2\big]^{1/2} \\ &= \left(A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha\right)^{1/2} \end{aligned} \qquad \text{(c)}$$

where

$A = 228.65$ k·in $\quad B = -537.47$ k·in $\quad C = 512.97$ k·in

(b) CQC rule

Using Eq. (13.7.4), the CQC estimate for the peak value of M_α can be calculated:

$$\begin{aligned} M_\alpha &= (M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2 + 2\rho_{12} M_{\alpha 1} M_{\alpha 2} \\ &\quad + 2\rho_{13} M_{\alpha 1} M_{\alpha 3} + 2\rho_{23} M_{\alpha 2} M_{\alpha 3})^{1/2} \\ &= (A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha)^{1/2} \end{aligned} \qquad \text{(d)}$$

where

$A = 425.09$ k·in $\quad B = -886.66$ k·in $\quad C = 481.09$ k·in

8. *Determine maximum of M_α over α.*

To find the value of α corresponding to the maximum value of M_α, solve

$$\frac{dM_\alpha}{d\alpha} = 0 \quad \text{for} \quad \alpha$$

$$\frac{dM_\alpha}{d\alpha} = \frac{1}{2} \frac{2(A-C)\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha)}{\sqrt{A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha}}$$
$$= \frac{1}{2} \frac{(A-C)\sin 2\alpha + B\cos 2\alpha}{\sqrt{A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha}}$$

Therefore, the maximum value of M_α occurs for α that satisfies

$$(A-C)\sin 2\alpha + B\cos 2\alpha = 0$$

Hence,

$$\tan 2\alpha = \frac{-B}{A-C} \qquad \text{(e)}$$

(a) SRSS rule

Substituting *A, B* and *C* from Eq. (c) into Eq. (e) gives

$$\tan 2\alpha = -1.8904$$

$$\alpha = \frac{1}{2}\tan^{-1}(-1.8904) = -0.5421 \text{ rad}, \quad 1.0287 \text{ rad}$$

The two values of α obtained correspond to the maximum and minimum values of M_α. The value of α corresponding to the maximum value of M_α can be ascertained by substituting each value of α into Eq. (c)

for $\alpha = -0.5421$ rad, $M_\alpha = 25.98$ k·in

for $\alpha = 1.0287$ rad, $M_\alpha = 8.17$ k·in

Thus, the peak value of M_α estimated by the SRSS combination rule is $M_\alpha = 25.98$ k·in about an axis oriented 0.5421 rad (31.1°) clockwise from the x-axis.

(b) CQC rule

Substituting *A, B* and *C* from Eq. (d) into Eq. (e) gives

$$\tan 2\alpha = -15.8335$$

$$\alpha = \frac{1}{2}\tan^{-1}(-1.8335) = -0.7539 \text{ rad}, \quad 0.8169 \text{ rad}$$

for $\alpha = -0.7539$ rad, $M_\alpha = 29.96$ k·in

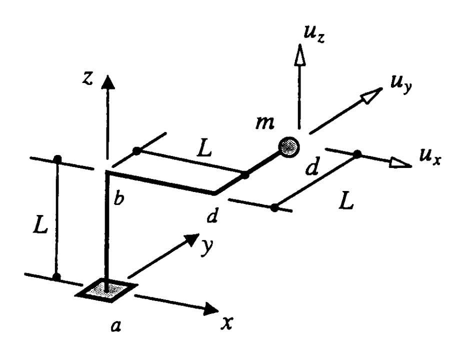
for $\alpha = 0.8169$ rad, $M_\alpha = 2.98$ k·in

Thus, the peak value of M_α estimated by the CQC combination rule is $M_\alpha = 29.96$ k·in about an axis oriented 0.7539 rad (43.2°) clockwise from the x-axis.

9. *Comments.*

Modes 1 and 2 are strongly correlated ($\rho_{12} = 0.9089$); hence, the CQC estimate, which accounts for this correlation, is expected to be more accurate than the SRSS estimate. Note that the SRSS and CQC estimates for the peak value of M_α differ by 14.2%. The values of α predicted by the two combination rules differ by 12.1°.

Problem 13.70



1. *Data.*

$L = 36$ in. $\quad m = 1.0$ kips/g

$E = 30000$ ksi $\quad I = 3.017 \text{ in}^4$

$G = 12000$ ksi $\quad J = 6.034 \text{ in}^4$

Note: $GJ = \dfrac{4}{5} EI$

2. *Natural frequencies and modes.*

$$m = 1.0 \frac{\text{kips}}{\text{g}} = \frac{1.0}{386.4} = 0.00259 \ \frac{\text{k} \cdot \text{s}^2}{\text{in}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{(30000)(3.017)}{(0.00259)(36)^3}} = 27.38 \text{ s}^{-1}$$

From Problems 10.28 and 13.30:

$$\omega_1 = 0.4834 \sqrt{\frac{EI}{mL^3}} = 13.24 \frac{\text{rad}}{\text{s}} \quad \Rightarrow \quad T_1 = 0.475 \text{ s}$$

$$\phi_1 = \begin{Bmatrix} 0.7767 \\ -0.4923 \\ -0.3928 \end{Bmatrix} \qquad \Gamma_1 = -0.0626$$

$$\omega_2 = 0.4990 \sqrt{\frac{EI}{mL^3}} = 13.66 \frac{\text{rad}}{\text{s}} \quad \Rightarrow \quad T_2 = 0.460 \text{ s}$$

$$\phi_2 = \begin{Bmatrix} -0.2084 \\ 0.3875 \\ -0.8980 \end{Bmatrix} \qquad \Gamma_2 = -0.4150$$

$$\omega_3 = 1.4827 \sqrt{\frac{EI}{mL^3}} = 40.59 \frac{\text{rad}}{\text{s}} \quad \Rightarrow \quad T_3 = 0.155 \text{ s}$$

$$\phi_3 = \begin{Bmatrix} 0.5943 \\ 0.7794 \\ 0.1984 \end{Bmatrix} \qquad \Gamma_3 = 0.9077$$

3. *Determine correlation coefficients.*

Use Eq. (13.7.10) to compute ρ_{ij} for $\zeta = 0.05$:

$$\beta_{12} = \frac{\omega_1}{\omega_2} = 0.969 \qquad \therefore \quad \rho_{12} = 0.9089$$

$$\beta_{13} = \frac{\omega_1}{\omega_3} = 0.326 \qquad \therefore \quad \rho_{13} = 0.0061$$

$$\beta_{23} = \frac{\omega_2}{\omega_3} = 0.336 \qquad \therefore \quad \rho_{23} = 0.0066$$

4. *Determine spectral ordinates.*

From Fig. 6.9.5:

Mode 1: $T_1 = 0.475$ s

$$A_1 = 0.20\,(2.71\text{g}) = 0.54 \text{ g} = 209 \frac{\text{in}}{\text{s}^2}$$

$$D_1 = \frac{A_1}{\omega_1^2} = \frac{209}{(13.24)^2} = 1.20 \text{ in}$$

Mode 2: $T_2 = 0.460$ s

$$A_2 = 0.20\,(2.71\text{g}) = 0.54 \text{ g} = 209 \frac{\text{in}}{\text{s}^2}$$

$$D_2 = \frac{A_2}{\omega_2^2} = \frac{209}{(13.66)^2} = 1.12 \text{ in}$$

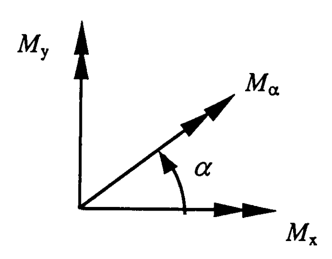
Mode 3: $T_3 = 0.155$ s

$$A_3 = 0.20\,(2.71\text{g}) = 0.54 \text{ g} = 209 \frac{\text{in}}{\text{s}^2}$$

$$D_3 = \frac{A_3}{\omega_3^2} = \frac{209}{(40.59)^2} = 0.127 \text{ in}$$

5. *Determine modal static responses for M_α.*

The bending moment, M_α, about an axis oriented at an angle α counterclockwise from the x-axis can be expressed in terms of the bending moments, M_x and M_y, about the x and y axes as:



$$M_\alpha = M_x \cos\alpha + M_y \sin\alpha$$

Therefore,

$$M_{\alpha n} = M_{xn} \cos\alpha + M_{yn} \sin\alpha$$
$$= (M_{xn}^{st} \cos\alpha + M_{yn}^{st} \sin\alpha) A_n$$
$$= M_{\alpha n}^{st} A_n$$

Thus,

$$M_{\alpha n}^{st} = M_{xn}^{st} \cos\alpha + M_{yn}^{st} \sin\alpha \qquad \text{(a)}$$

where M_{xn}^{st} and M_{yn}^{st} are given in the solution to Problem 13.30:

$$M_{x1}^{st} = 0.0062mL \qquad M_{y1}^{st} = 0.0731mL$$
$$M_{x2}^{st} = -0.5335mL \qquad M_{y2}^{st} = 0.2862mL \qquad \text{(b)}$$
$$M_{x3}^{st} = 0.5273mL \qquad M_{y3}^{st} = -0.3593mL$$

Equation (a) after substituting Eq. (b) becomes

$$M_{\alpha 1}^{st} = (0.0062\cos\alpha + 0.0731\sin\alpha)mL$$

$$M_{\alpha 2}^{st} = (-0.5335\cos\alpha + 0.2862\sin\alpha)mL$$

$$M_{\alpha 3}^{st} = (0.5273\cos\alpha - 0.3593\sin\alpha)mL$$

6. *Determine peak modal responses $M_{\alpha n}$.*

The peak values, $M_{\alpha n}$, of the modal contributions $M_{\alpha n}(t)$ to response M_α are determined next

$$M_{\alpha 1} = [(0.0062\cos\alpha + 0.0731\sin\alpha)\, mL]A_1$$
$$= 0.12\cos\alpha + 1.43\sin\alpha$$
$$M_{\alpha 2} = [(-0.5335\cos\alpha + 0.2862\sin\alpha)\, mL]A_2$$
$$= -10.41\cos\alpha + 5.58\sin\alpha$$
$$M_{\alpha 3} = [(0.5273\cos\alpha - 0.3593\sin\alpha)\, mL]A_3$$
$$= 10.29\cos\alpha - 7.01\sin\alpha$$

7. *Combine peak modal responses.*

(a) SRSS rule

Using Eq. (13.7.3), the SRSS estimate for the peak value of M_α can be calculated:

$$M_\alpha = \left(M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2\right)^{1/2}$$
$$= \left[(0.12\cos\alpha + 1.43\sin\alpha)^2 + (-10.41\cos\alpha + 5.58\sin\alpha)^2 + (10.29\cos\alpha - 7.01\sin\alpha)^2\right]^{1/2} \qquad \text{(c)}$$
$$= \left(A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha\right)^{1/2}$$

where

$$A = 82.38 \text{ k}\cdot\text{in} \quad B = -260.20 \text{ k}\cdot\text{in} \quad C = 214.25 \text{ k}\cdot\text{in}$$

(b) CQC rule

Using Eq. (13.7.4), the CQC estimate for the peak value of M_α can be calculated :

$$M_\alpha = (M_{\alpha 1}^2 + M_{\alpha 2}^2 + M_{\alpha 3}^2 + 2\rho_{12}M_{\alpha 1}M_{\alpha 2} + 2\rho_{13}M_{\alpha 1}M_{\alpha 3} + 2\rho_{23}M_{\alpha 2}M_{\alpha 3})^{1/2} \qquad \text{(d)}$$
$$= (A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha)^{1/2}$$

where

$$A = 96.24 \text{ k}\cdot\text{in} \quad B = -284.10 \text{ k}\cdot\text{in} \quad C = 210.56 \text{ k}\cdot\text{in}$$

8. *Determine maximum of M_α over α.*

To find the value of α corresponding to the maximum value of M_α, solve

$$\frac{dM_\alpha}{d\alpha} = 0 \quad \text{for} \quad \alpha$$

$$\frac{dM_\alpha}{d\alpha} = \frac{1}{2}\frac{2(A-C)\sin\alpha\cos\alpha + B(\cos^2\alpha - \sin^2\alpha)}{\sqrt{A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha}}$$
$$= \frac{1}{2}\frac{(A-C)\sin 2\alpha + B\cos 2\alpha}{\sqrt{A\sin^2\alpha + B\sin\alpha\cos\alpha + C\cos^2\alpha}}$$

Therefore, the maximum value of M_α occurs for α that satisfies

$$(A-C)\sin 2\alpha + B\cos 2\alpha = 0$$

Hence,

$$\tan 2\alpha = \frac{-B}{A-C} \qquad \text{(e)}$$

(a) SRSS rule

Substituting A, B and C from Eq. (c) into Eq. (e) gives

$$\tan 2\alpha = -1.9732$$

$$\alpha = \frac{1}{2}\tan^{-1}(-1.9732) = -0.5509 \text{ rad}, \quad 1.0199 \text{ rad}$$

The two values of α obtained correspond to the maximum and minimum values of M_α. The value of α corresponding to the maximum value of M_α can be ascertained by substituting each value of α into Eq. (c)

for $\alpha = -0.5509$ rad, $\quad M_\alpha = 17.15 \text{ k}\cdot\text{in}$

for $\alpha = 1.0199$ rad, $\quad M_\alpha = 1.57 \text{ k}\cdot\text{in}$

Thus, the peak value of M_α estimated by the SRSS combination rule is $M_\alpha = 17.15$ k·in about an axis oriented 0.5509 rad (31.6°) clockwise from the x-axis.

(b) CQC rule

Substituting A, B and C from Eq. (d) into Eq. (e) gives

$$\tan 2\alpha = -2.4851$$

$$\alpha = \frac{1}{2}\tan^{-1}(-2.4851) = -0.5941 \text{ rad}, \quad 0.9767 \text{ rad}$$

for $\alpha = -0.5941$ rad, $\quad M_\alpha = 17.51 \text{ k}\cdot\text{in}$

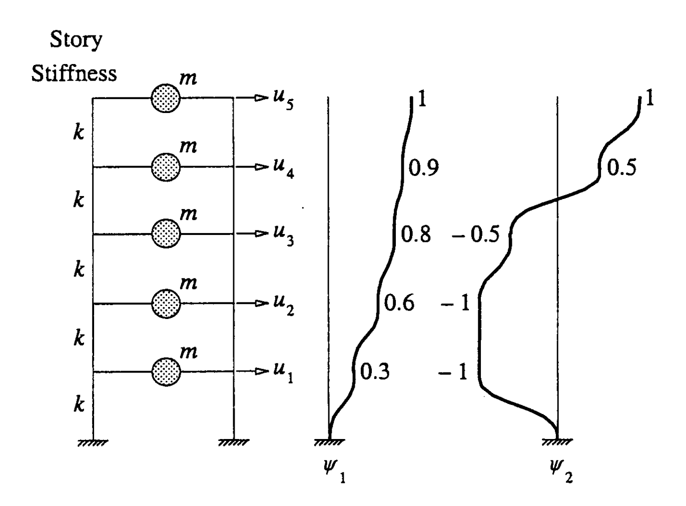
for $\alpha = 0.9767$ rad, $\quad M_\alpha = 0.53 \text{ k}\cdot\text{in}$

Thus, the peak value of M_α estimated by the CQC combination rule is $M_\alpha = 17.51$ k·in about an axis oriented 0.5941 rad (34.0°) clockwise from the x-axis.

9. *Comments.*

Modes 1 and 2 are strongly correlated ($\rho_{12} = 0.9089$); hence, the CQC estimate, which accounts for this correlation, is expected to be more accurate than the SRSS estimate. Note that the SRSS and CQC estimates for the peak value of M_α differ by 2.1%. The values of α predicted by the two combination rules differ by 2.4°.

Problem 14.1



$$\Psi = \begin{bmatrix}\psi_1 & \psi_2\end{bmatrix} = \begin{bmatrix} 0.3 & -1 \\ 0.6 & -1 \\ 0.8 & -0.5 \\ 0.9 & 0.5 \\ 1.0 & 1 \end{bmatrix}$$

1. *Determine mass and stiffness matrices.*

   Available in Example 14.1.

2. *Compute* $\tilde{\mathbf{k}}$ *and* $\tilde{\mathbf{m}}$.

$$\tilde{\mathbf{k}} = \Psi^T \mathbf{k} \Psi = k\begin{bmatrix} 0.24 & -0.05 \\ -0.05 & 2.5 \end{bmatrix}$$

$$\tilde{\mathbf{m}} = \Psi^T \mathbf{m} \Psi = m\begin{bmatrix} 2.9 & 0.15 \\ 0.15 & 3.5 \end{bmatrix}$$

3. *Determine approximate frequencies and modes.*

$$\left(\tilde{\mathbf{k}} - \tilde{\omega}^2 \tilde{\mathbf{m}}\right)\mathbf{z} = \mathbf{0}$$

$$\tilde{\omega}_1 = 0.2866\sqrt{\frac{k}{m}} \qquad \tilde{\omega}_2 = 0.8474\sqrt{\frac{k}{m}}$$

$$\mathbf{z}_1 = \begin{Bmatrix} 0.9996 \\ 0.0282 \end{Bmatrix} \qquad \mathbf{z}_2 = \begin{Bmatrix} -0.0853 \\ 0.9964 \end{Bmatrix}$$

$$\tilde{\Phi} = \Psi \mathbf{z} = \begin{bmatrix} 0.3101 & -1.2091 \\ 0.6524 & -1.2394 \\ 0.8966 & -0.6702 \\ 1.0428 & 0.4986 \\ 1.1730 & 1.0780 \end{bmatrix}$$

Comparing these $\tilde{\omega}_n$ and $\tilde{\phi}_n$ with the approximate results of Example 14.1 and the exact results in Section 12.8, we note that the selection of Ritz vectors has little influence on the two vibration frequencies or the first mode but affects considerably the second mode.

Problem 14.2

The **k** and **m** are available in Example 14.1 and

$$\mathbf{s} = \langle 0 \quad 0 \quad 0 \quad 0 \quad 1 \rangle^T$$

1. *Determine the first Ritz vector.*

Solve

$$\mathbf{k}\,\mathbf{y}_1 = \mathbf{s} \;\Rightarrow\; \psi_1 = \frac{\mathbf{y}_1}{\left(\mathbf{y}_1^T\,\mathbf{m}\,\mathbf{y}_1\right)^{1/2}} = \begin{Bmatrix} 0.2649 \\ 0.5298 \\ 0.7948 \\ 1.0597 \\ 1.3246 \end{Bmatrix}$$

2. *Determine the second Ritz vector.*

Solve

$$\mathbf{k}\,\mathbf{y}_2 = \mathbf{m}\,\psi_1 \;\Rightarrow\; \mathbf{y}_2 = \begin{Bmatrix} 0.0061 \\ 0.0100 \\ 0.0096 \\ 0.0026 \\ -0.0130 \end{Bmatrix}$$

$$\hat{\psi}_2 = \mathbf{y}_2 - \left(\psi_1^T\,\mathbf{m}\,\mathbf{y}_2\right)\psi_1 \;\Rightarrow$$

$$\psi_2 = \frac{\hat{\psi}_2}{\left(\hat{\psi}_2^T\,\mathbf{m}\,\hat{\psi}_2\right)^{1/2}} = \begin{Bmatrix} 0.5939 \\ 0.9757 \\ 0.9333 \\ 0.2545 \\ -1.2726 \end{Bmatrix}$$

3. *Compute* $\tilde{\mathbf{k}}$ *and* $\tilde{\mathbf{m}}$.

$$\Psi = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix}$$

$$\tilde{\mathbf{k}} = \Psi^T\,\mathbf{k}\,\Psi = \begin{bmatrix} 11.075 & -10.64 \\ -10.64 & 103.932 \end{bmatrix}$$

$$\tilde{\mathbf{m}} = \Psi^T \mathbf{m} \Psi = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix}$$

4. *Solve* $\left(\tilde{\mathbf{k}} - \tilde{\omega}^2\,\tilde{\mathbf{m}}\right)\mathbf{z} = \mathbf{0}$.

$$\tilde{\omega}_1 = 3.1418 \text{ rads/sec} \qquad \tilde{\omega}_2 = 10.2536 \text{ rads/sec}$$

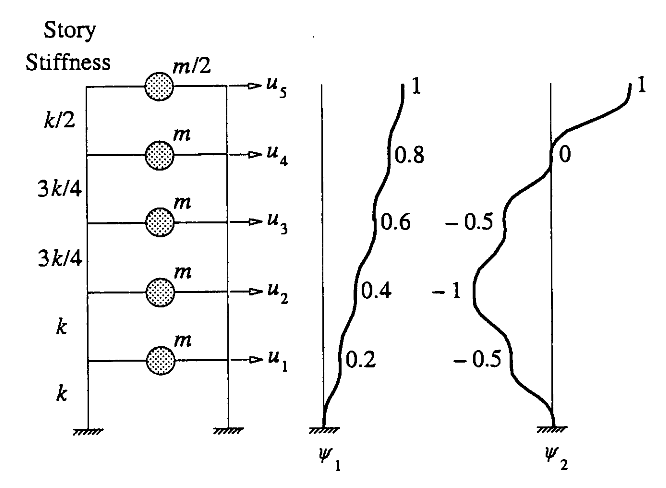
$$\mathbf{z}_1 = \begin{Bmatrix} 0.9937 \\ 0.1124 \end{Bmatrix} \qquad \mathbf{z}_2 = \begin{Bmatrix} -0.1124 \\ 0.9937 \end{Bmatrix}$$

5. *Compute natural vibration modes.*

$$\tilde{\Phi} = \Psi\,\mathbf{z} = \begin{bmatrix} 0.330 & 0.560 \\ 0.636 & 0.910 \\ 0.895 & 0.838 \\ 1.082 & 0.134 \\ 1.173 & -1.414 \end{bmatrix}$$

We note that this selection of **s** gives essentially the same values for ω_1 and ϕ_1 as from $\mathbf{s} = m\mathbf{1}$ (Example 14.2) and both are close to the exact results. However, this selection of **s** gives less accurate values for ω_2 and ϕ_2 than from $\mathbf{s} = m\mathbf{1}$ (Example 14.2).

Problem 14.3



$k = 200$ kips/in.

$m = 100/386 = 0.259 \text{ kip-sec}^2/\text{in.}$

$$\Psi = \begin{bmatrix} 0.2 & -0.5 \\ 0.4 & -1 \\ 0.6 & -0.5 \\ 0.8 & 0 \\ 1.0 & 1 \end{bmatrix}$$

1. *Determine mass and stiffness matrices.*

$$\mathbf{k} = \begin{bmatrix} 400 & -200 & 0 & 0 & 0 \\ & 350 & -150 & 0 & 0 \\ & & 300 & -150 & 0 \\ & (sym) & & 250 & -100 \\ & & & & 100 \end{bmatrix}$$

$$\mathbf{m} = \begin{bmatrix} 0.259 & & & & \\ & 0.259 & & & \\ & & 0.259 & & \\ & & & 0.259 & \\ & & & & 0.1295 \end{bmatrix}$$

2. *Compute* $\tilde{\mathbf{k}}$ *and* $\tilde{\mathbf{m}}$.

$$\tilde{\mathbf{k}} = \Psi^T \mathbf{k} \Psi = \begin{bmatrix} 32 & 10 \\ 10 & 275 \end{bmatrix}$$

$$\tilde{\mathbf{m}} = \Psi^T \mathbf{m} \Psi = \begin{bmatrix} 0.4404 & -0.0777 \\ -0.0777 & 0.5181 \end{bmatrix}$$

3. *Solve eigenvalue problem.*

$$\tilde{\mathbf{k}}\,\mathbf{z} = \tilde{\omega}^2\, \tilde{\mathbf{m}}\,\mathbf{z}$$

$\tilde{\omega}_1 = 8.389$ rads/sec $\qquad \tilde{\omega}_2 = 23.59$ rads/sec

$$\mathbf{z}_1 = \begin{Bmatrix} 0.9979 \\ -0.0647 \end{Bmatrix} \qquad \mathbf{z}_2 = \begin{Bmatrix} 0.2425 \\ 0.9702 \end{Bmatrix}$$

4. *Determine natural vibration modes.*

$$\tilde{\Phi} = \Psi\,\mathbf{z} = \begin{bmatrix} 0.2319 & -0.4366 \\ 0.4639 & -0.8732 \\ 0.6311 & -0.3396 \\ 0.7983 & 0.1940 \\ 0.9332 & 1.2126 \end{bmatrix}$$

5. *Compare with exact values.*

$\omega_1 = 8.262$ rads/sec $\qquad \omega_2 = 22.347$ rads/sec

$$\Phi = \begin{bmatrix} 0.2319 & -0.4366 \\ 0.4433 & -0.5908 \\ 0.6728 & -0.2868 \\ 0.8231 & 0.2646 \\ 0.9029 & 0.7493 \end{bmatrix}$$

Problem 14.4

The **k** and **m** are available in the solution to Problem 14.3 and

$$\mathbf{s} = \langle 1 \quad 1 \quad 1 \quad 1 \quad 0.5 \rangle^T$$

1. *Determine the first Ritz vector.*

Solve

$$\mathbf{k}\,\mathbf{y}_1 = \mathbf{s} \;\Rightarrow\; \psi_1 = \frac{\mathbf{y}_1}{\left(\mathbf{y}_1^T \mathbf{m}\, \mathbf{y}_1\right)^{1/2}} = \begin{Bmatrix} 0.3982 \\ 0.7078 \\ 1.0027 \\ 1.1797 \\ 1.2682 \end{Bmatrix}$$

2. *Determine the second Ritz vector.*

Solve $\mathbf{k}\mathbf{y}_2 = \mathbf{m}\psi_1 \;\Rightarrow\; \mathbf{y}_2$

$$\hat{\psi}_2 = \mathbf{y}_2 - \left(\psi_1^T \mathbf{m}\, \mathbf{y}_2\right)\psi_1 = \begin{Bmatrix} -0.7396 \\ -0.7016 \\ -0.1490 \\ 0.3965 \\ 0.7456 \end{Bmatrix}$$

$$\psi_2 = \frac{\hat{\psi}_2}{\left(\hat{\psi}_2^T \mathbf{m}\, \hat{\psi}_2\right)^{1/2}} = \begin{Bmatrix} -1.1879 \\ -1.1269 \\ -0.2393 \\ 0.6368 \\ 1.1976 \end{Bmatrix}$$

3. *Compute* $\tilde{\mathbf{k}}$ *and* $\tilde{\mathbf{m}}$.

$$\Psi = \left[\psi_1 \quad \psi_2\right]$$

$$\tilde{\mathbf{k}} = \Psi^T \mathbf{k}\, \Psi = \begin{bmatrix} 69.413 & -23.333 \\ -23.333 & 547.737 \end{bmatrix}$$

$$\tilde{\mathbf{m}} = \Psi^T \mathbf{m}\, \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. *Solve* $\left(\tilde{\mathbf{k}} - \tilde{\omega}^2\, \tilde{\mathbf{m}}\right)\mathbf{z} = \mathbf{0}$.

$$\tilde{\omega}_1 = 8.263 \text{ rads/sec} \qquad \tilde{\omega}_2 = 23.428 \text{ rads/sec}$$

$$\mathbf{z}_1 = \begin{Bmatrix} 0.9988 \\ 0.04861 \end{Bmatrix} \qquad \mathbf{z}_2 = \begin{Bmatrix} -0.04861 \\ 0.9988 \end{Bmatrix}$$

5. *Compute natural vibration modes.*

$$\tilde{\Phi} = \Psi\, \mathbf{z} = \begin{bmatrix} 0.2319 & -0.4366 \\ 0.4449 & -0.4200 \\ 0.6753 & -0.1042 \\ 0.8249 & 0.2095 \\ 0.9038 & 0.4108 \end{bmatrix}$$

6. *Comment on accuracy.*

The results using force dependent Ritz vectors are better than those obtained using the vectors guessed in Problem 14.3.

Problem 14.5

The $\mathbf{k}$ and $\mathbf{m}$ are given in the solution to Problem 14.3 and

$$\mathbf{s} = \langle 0 \quad 0 \quad 0 \quad 0 \quad 1 \rangle^T$$

1. *Determine the first Ritz vector,* ψ_1.

Solve $\mathbf{k}\,\mathbf{y}_1 = \mathbf{s}$ to obtain

$$\mathbf{y}_1 = \langle 0.0050 \quad 0.0100 \quad 0.0167 \quad 0.0233 \quad 0.0333 \rangle^T$$

Divide $\mathbf{y}_1$ by $\left(\mathbf{y}_1^T \mathbf{m}\,\mathbf{y}_1\right)^{1/2} = 0.0197$ to obtain the normalized vector:

$$\psi_1 = \langle 0.2534 \quad 0.5068 \quad 0.8447 \quad 1.1826 \quad 1.6894 \rangle^T$$

2. *Determine the second Ritz vector,* ψ_2.

Solve $\mathbf{k}\,\mathbf{y}_2 = \mathbf{m}\,\psi_1$ to obtain:

$$\mathbf{y}_2 = \langle 0.0047 \quad 0.0091 \quad 0.0140 \quad 0.0175 \quad 0.0197 \rangle^T$$

Orthogonalize $\mathbf{y}_2$ with respect to ψ_1:

$$a_{12} = \psi_1^T \mathbf{m}\,\mathbf{y}_2 = 0.0143$$

$$\hat{\psi}_2 = \mathbf{y}_2 - 0.0143\,\psi_1$$
$$= 10^{-2} \langle 0.1089 \quad 0.1851 \quad 0.1991 \quad 0.0672 \quad 0.4370 \rangle^T$$

Divide $\hat{\psi}_2$ by $\left(\hat{\psi}_2^T \mathbf{m}\,\hat{\psi}_2\right)^{1/2} = 0.0022$ to get the normalized vector:

$$\psi_2 = \langle 0.4967 \quad 0.8438 \quad 0.9076 \quad 0.3064 \quad -1.9918 \rangle^T$$

3. *Compute* $\tilde{\mathbf{k}}$ *and* $\tilde{\mathbf{m}}$.

$$\Psi = [\psi_1 \quad \psi_2]$$

$$\tilde{\mathbf{k}} = \Psi^T \mathbf{k} \Psi = \begin{bmatrix} 85.619 & -100.948 \\ -100.948 & 656.450 \end{bmatrix}$$

$$\tilde{\mathbf{m}} = \Psi^T \mathbf{m} \Psi = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix}$$

4. *Solve the reduced eigenvalue problem, Eq. (14.3.11).*

$$\tilde{\omega}_1 = 8.2640 \qquad \tilde{\omega}_2 = 25.9572$$

$$z_1 = \begin{bmatrix} 0.9856 \\ 0.1692 \end{bmatrix} \qquad z_2 = \begin{bmatrix} -0.1692 \\ 0.9856 \end{bmatrix}$$

5. *Determine the natural modes,* $\tilde{\phi}_n = \Psi z_n$, $n = 1, 2$.

$$\tilde{\phi}_1 = \langle 0.3338 \quad 0.6422 \quad 0.9860 \quad 1.2173 \quad 1.3281 \rangle^T$$

$$\tilde{\phi}_2 = \langle 0.4467 \quad 0.7459 \quad 0.7516 \quad 0.1019 \quad -2.2489 \rangle^T$$

6. *Compare with exact results:*

The exact values of the first two frequencies and modes are:

$$\omega_1 = 8.2619 \qquad \omega_2 = 22.3473$$

$$\phi_1 = \langle 0.3407 \quad 0.6513 \quad 0.9886 \quad 1.2093 \quad 1.3266 \rangle^T$$

$$\phi_2 = \langle 0.8698 \quad 1.1769 \quad 0.5713 \quad -0.5271 \quad -1.4927 \rangle^T$$

The approximate values of the first frequency and mode from Problems 14.4 and 14.5 are similarly accurate, whereas the use of $\mathbf{s} = \langle 1 \quad 1 \quad 1 \quad 1 \quad 0.5 \rangle^T$ in Problem 14.4 provides better accuracy for the second frequency and mode than from Problem 14.5.

Problem 14.6

The stiffness and mass matrices , k and m, are given in Problem 14.3 and

$$\mathbf{s} = \langle 0 \quad 0 \quad 0 \; -2 \quad 1 \rangle^T$$

1. *Determine the first Ritz vector,* ψ_1.

Solve $\mathbf{k}\,\mathbf{y}_1 = \mathbf{s}$ to obtain

$$\mathbf{y}_1 = \langle -0.0050 \quad -0.0100 \quad -0.0167 \quad -0.0233 \quad -0.0133 \rangle^T$$

Divide $\mathbf{y}_1$ by $\left(\mathbf{y}_1^T \mathbf{m}\,\mathbf{y}_1\right)^{1/2} =$ 0.0164 to obtain the normalized vector:

$$\psi_1 = \langle -0.3052 \quad -0.6104 \quad -1.0173 \quad -1.4242 \quad -0.8138 \rangle^T$$

2. *Determine the second Ritz vector,* ψ_2.

Solve $\mathbf{k}\,\mathbf{y}_2 = \mathbf{m}\,\psi_1$ to obtain:

$$\mathbf{y}_2 = \langle -0.0049 \quad -0.0094 \quad -0.0143 \quad -0.0174 \quad -0.0185 \rangle^T$$

Orthogonalize $\mathbf{y}_2$ with respect to ψ_1:

$$a_{12} = \psi_1^T \mathbf{m}\mathbf{y}_2 = 0.0140$$

$$\begin{aligned} \psi_2 &= \mathbf{y}_2 - 0.0140\psi_1 \\ &= 10^{-2}\langle -0.0600 \quad -0.0804 \quad -0.0023 \quad 0.2515 \quad -0.7090 \rangle^T \end{aligned}$$

Divide $\hat{\psi}_2$ by $\left(\hat{\psi}_2^T \mathbf{m}\,\hat{\psi}_2\right)^{1/2} = 4.7518 \cdot 10^{-5}$ to get the normalized vector:

$$\psi_2 = \langle -0.2068 \quad -0.2774 \quad -0.0079 \quad 0.8673 \quad -2.4446 \rangle^T$$

3. *Compute* $\tilde{\mathbf{k}}$ *and* $\tilde{\mathbf{m}}$.

$$\Psi = \left[\psi_1 \quad \psi_2 \right]$$

$$\tilde{\mathbf{k}} = \Psi^T \mathbf{k} \Psi = \begin{bmatrix} 85.619 & -100.948 \\ -100.948 & 656.450 \end{bmatrix}$$

$$\tilde{\mathbf{m}} = \Psi^T \mathbf{m} \Psi = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix}$$

4. *Solve the reduced eigenvalue problem, Eq. (14.3.11).*

$$\tilde{\omega}_1 = 8.2630 \qquad \tilde{\omega}_2 = 35.8902 \qquad \text{rad/sec}$$

$$z_1 = \begin{bmatrix} 0.9768 \\ 0.2141 \end{bmatrix} \qquad z_2 = \begin{bmatrix} -0.2141 \\ 0.9768 \end{bmatrix}$$

5. *Determine the natural modes,* $\tilde{\phi}_n = \Psi\, z_n$, *n=1,2.*

$$\tilde{\phi}_1 = \langle 0.3424 \quad 0.6556 \quad 0.9954 \quad 1.2055 \quad 1.3183 \rangle^T$$

$$\tilde{\phi}_2 = \langle -0.1367 \quad -0.1403 \quad 0.2100 \quad 1.1921 \quad -2.2137 \rangle^T$$

6. *Compare with exact results.*

The first two exact modes and frequencies of the structure are:

$$\omega_1 = 8.2619 \qquad \omega_2 = 22.3473$$

$$\phi_1 = \langle 0.3407 \quad 0.6513 \quad 0.9886 \quad 1.2093 \quad 1.3266 \rangle^T$$

$$\phi_2 = \langle 0.8698 \quad 1.1769 \quad 0.5713 \quad -0.5271 \quad -1.4927 \rangle^T$$

Problems 14.4, 14.5 and 14.6 lead to almost the same accuracy for the first frequency and mode. The second frequency and mode differ among the three solutions, being the best from Problem 14.4 and the worst from Problem 14.6.

Problem 14.7

The stiffness and mass matrices , **k** and **m**, are given in Problem 14.3.

1. *Compute the Ritz vectors following the steps in Table 14.4.1*

- For the force distribution $\mathbf{s}_a$:

$$\Psi = [\psi_1 \quad \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5] = \begin{bmatrix} 0.2534 & 0.4967 & 0.8386 & 0.9932 & 1.3636 \\ 0.5068 & 0.8438 & 1.0229 & 0.3766 & -1.3050 \\ 0.8447 & 0.9076 & 0.0358 & -1.4282 & 0.5307 \\ 1.1826 & 0.3064 & -1.2992 & 0.8148 & -0.1258 \\ 1.6894 & -1.9918 & 0.9178 & -0.2364 & 0.0194 \end{bmatrix}$$

- For the force distribution $\mathbf{s}_b$:

$$\Psi = [\psi_1 \quad \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5] = \begin{bmatrix} -0.3052 & -0.2068 & 0.3007 & 1.2005 & 1.4807 \\ -0.6104 & -0.2774 & 0.7990 & 1.1183 & -1.2335 \\ -1.0173 & -0.0079 & 1.2630 & -1.0426 & 0.3780 \\ -1.4242 & 0.8673 & -1.0313 & 0.1138 & -0.0552 \\ -0.8138 & -2.4446 & -0.9721 & -0.3694 & -0.0123 \end{bmatrix}$$

- For the force distribution $\mathbf{s}_c$:

$$\Psi = [\psi_1 \quad \psi_2 \quad \psi_3 \quad \psi_4 \quad \psi_5]$$

$$= \begin{bmatrix} 0.3981 & -1.1878 & 1.2697 & -0.796 & 0.2122 \\ 0.7077 & -1.1268 & -0.3903 & 1.2493 & -0.6136 \\ 1.0026 & -0.2393 & -1.0788 & -0.5555 & 1.1511 \\ 1.1796 & 0.6367 & -0.0110 & -0.7105 & -1.2483 \\ 1.2680 & 1.1974 & 1.3650 & 1.3056 & 1.0538 \end{bmatrix}$$

2. *Compute* e_J.

The error norm, e_J, is defined as:

$$e_J = \frac{\mathbf{s}^T \mathbf{e}_J}{\mathbf{s}^T \mathbf{s}}$$

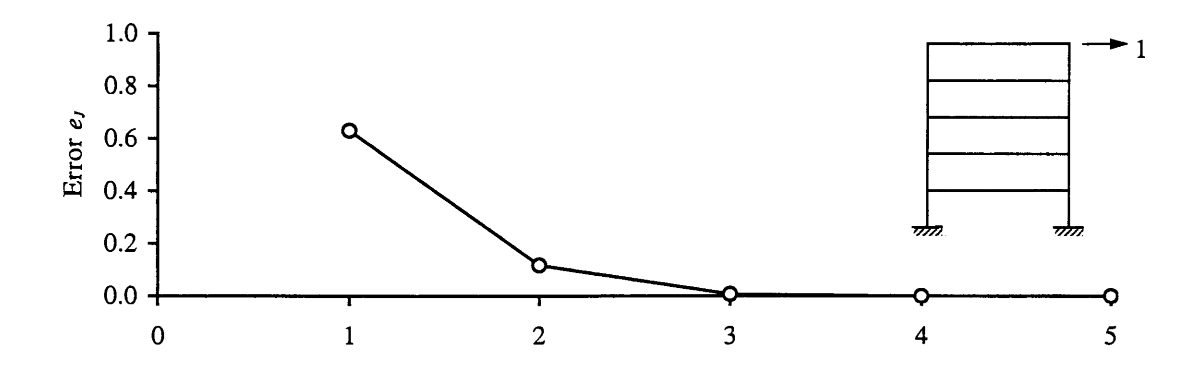
where

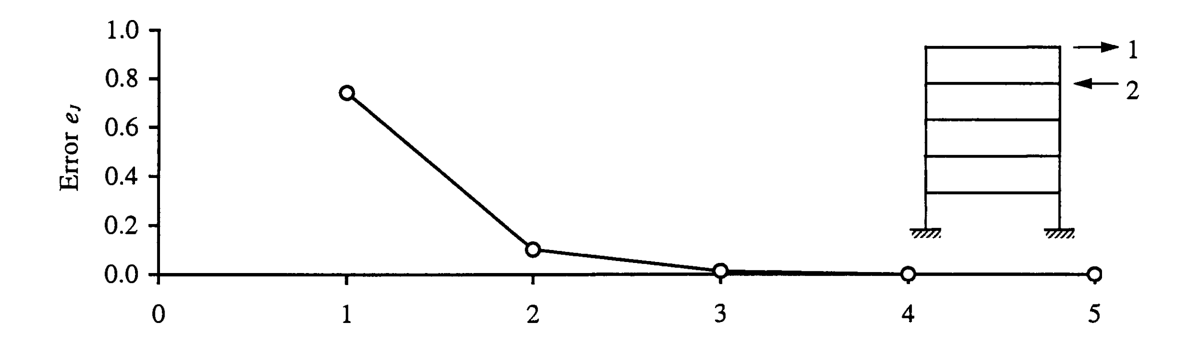
$$\mathbf{e}_J = \mathbf{s} - \sum_{n=1}^{J} \tilde{\Gamma}_n \mathbf{m} \psi_n \qquad \tilde{\Gamma}_n = \psi_n^T \mathbf{s}$$

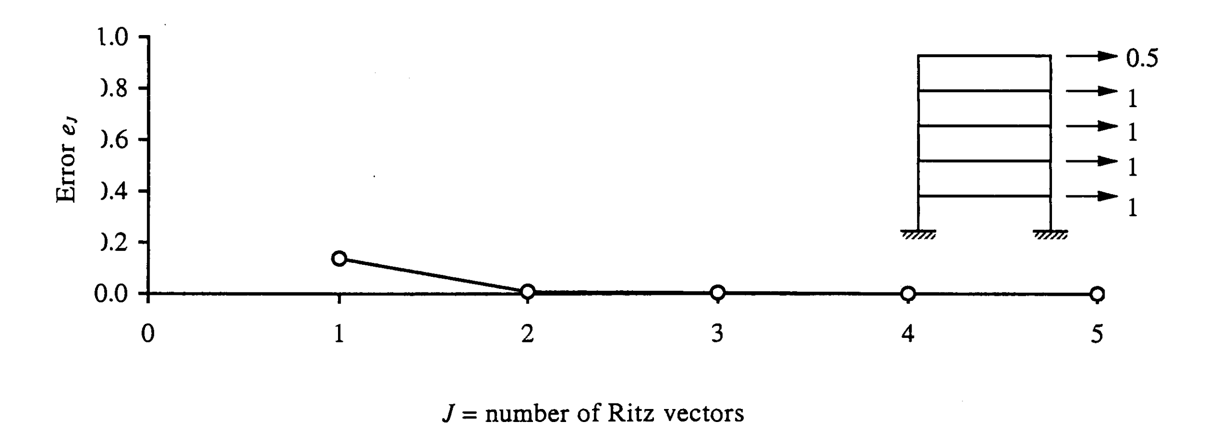
3. *Comments.*

The error norm, e_J, computed using the above formula is plotted versus the number of Ritz vectors in Fig. P14.7.

For a given force distribution, the error decreases as more Ritz vectors are included, and is zero when all five Ritz vectors are included. For a fixed number of Ritz vectors, the error is smallest for the force distribution $\mathbf{s}_c$, and is larger for force distributions $\mathbf{s}_a$ and $\mathbf{s}_b$.







J = number of Ritz vectors

Fig. P14.7

Problem 14.8

The stiffness and mass matrices, **k** and **m**, are given in Problem 14.3.

1. *Determine the natural frequencies and modes.*

$\omega_1 = 8.2619 \qquad \omega_2 = 22.3473 \qquad \omega_3 = 32.9203$
$\omega_4 = 41.7423 \qquad \omega_5 = 48.9507 \quad \text{rad/sec}$

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 \end{bmatrix}$$
$$= \begin{bmatrix} 0.3407 & 0.8698 & 0.9108 & 0.8706 & 1.1832 \\ 0.6513 & 1.1769 & 0.5430 & -0.2238 & -1.3061 \\ 0.9886 & 0.5713 & -0.9638 & -1.0095 & 0.7800 \\ 1.2093 & -0.5271 & -0.6666 & 1.2427 & -0.3620 \\ 1.3266 & -1.4927 & 1.6507 & -0.9886 & 0.1721 \end{bmatrix}$$

2. *Compute* e_J.

The error e_J is defined as:

$$e_J = \frac{\mathbf{s}^T \mathbf{e}_J}{\mathbf{s}^T \mathbf{s}}$$

where

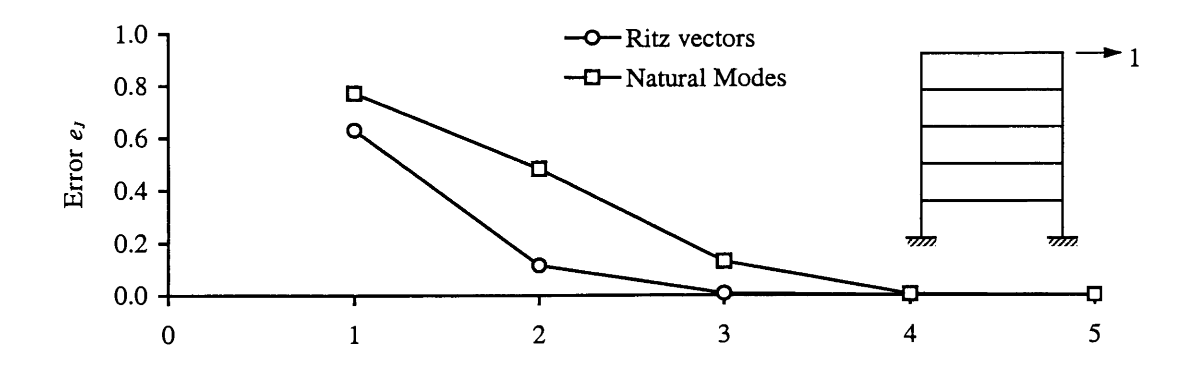
$$\mathbf{e}_J = \mathbf{s} - \sum_{n=1}^{J} \Gamma_n \, \mathbf{m} \, \phi_n \qquad \Gamma_n = \phi_n^T \mathbf{s}$$

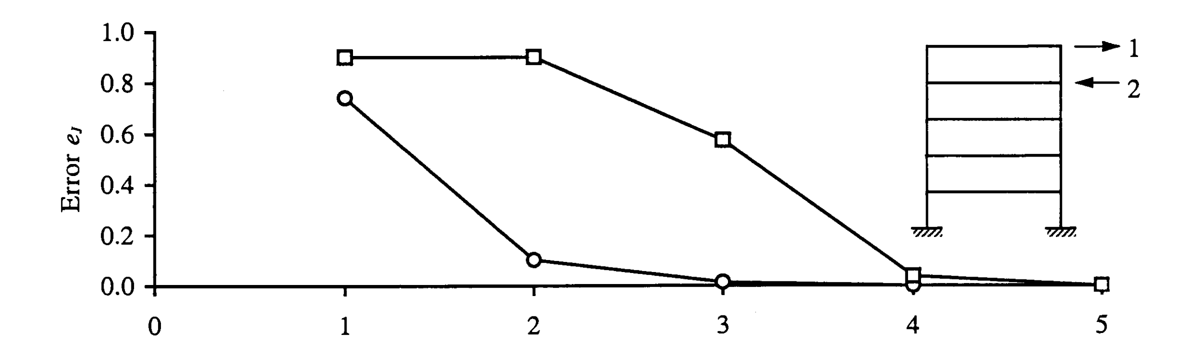
The error norm, e_J, computed using the above formula is plotted versus the number of natural modes in Fig. P14.8.

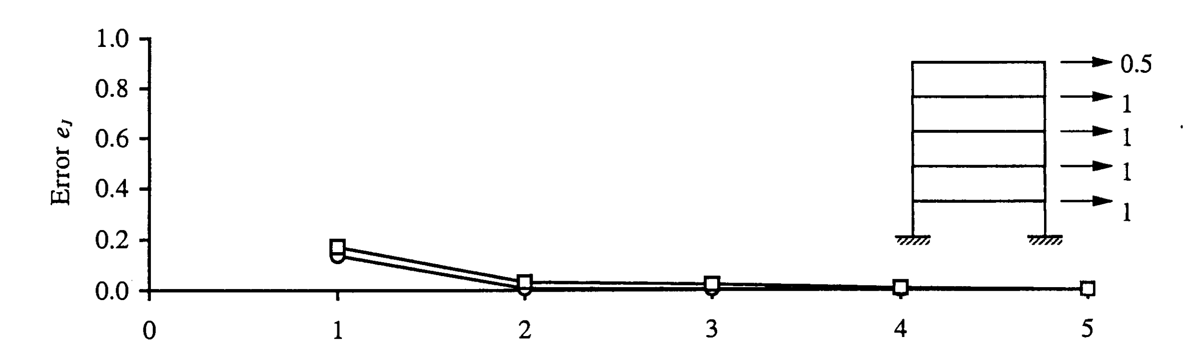
3. *Comments.*

For a given force distribution, the error decreases as more modes are included, and is zero then all five modes are included. For a fixed number of modes, the error is smallest for the force distribution $\mathbf{s}_c$, largest for $\mathbf{s}_b$, and has an intermediate value for $\mathbf{s}_a$.

Figure P14.8 also shows the error norm, e_J, versus the number of Ritz vectors (from Problem 14.7). The error is smaller when Ritz vectors are used, because they are derived from the force distribution. Ritz vectors are useful for dynamic analysis of large systems with classical damping, since the vibration properties of the system can be obtained by solving, a smaller eigenvalue problem of order *J*, instead of original eigenvalue problem of size *N*. It must be noted that the resulting frequencies and mode shapes are approximations to the first *J* natural vibration frequencies and modes of the system. While this property indicates that Ritz vectors are preferable to natural modes, the latter leads to uncoupled modal equations of motion, which have several advantages. In particular: (1) these uncoupled equations are easier to solve in response history analysis; (2) they permit estimation of the peak value of the earthquake response of a structure by response spectrum analysis.







J = number of natural modes or Ritz vectors

Fig. P4.8

Problem 14.9

The stiffness and mass matrices , **k** and **m**, are given in Problem 14.3. From Problem 14.7 the first two Ritz vectors and approximate natural frequencies using the force distribution $s = \langle 1 \quad 1 \quad 1 \quad 1 \quad 0.5 \rangle^T$ are:

$$\Psi = [\psi_1 \quad \psi_2] = \begin{bmatrix} 0.3981 & -1.1878 \\ 0.7077 & -1.1268 \\ 1.0026 & -0.2393 \\ 1.1796 & 0.6367 \\ 1.2680 & 1.1974 \end{bmatrix}$$

***Part a:** Rayleigh-Ritz analysis*

1. *Compute,* $\tilde{\mathbf{k}}$, $\tilde{\mathbf{m}}$ *and* $\tilde{\mathbf{L}}$.

$$\tilde{\mathbf{k}} = \Psi^T \mathbf{k} \Psi = \begin{bmatrix} 69.3929 & -23.3226 \\ -23.3226 & 547.6113 \end{bmatrix} \qquad \text{(a)}$$

$$\tilde{\mathbf{m}} = \Psi^T \mathbf{m} \Psi = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix} \qquad \text{(b)}$$

$$\tilde{\mathbf{L}} = \Psi^T \mathbf{m} \mathbf{1} = \begin{Bmatrix} 1.0161 \\ -0.3416 \end{Bmatrix} \qquad \text{(c)}$$

2. *Set up equations in generalized coordinates, Eq. (14.3.3) is specialized for this problem.*

$$\tilde{\mathbf{m}}\ddot{\mathbf{z}} + \tilde{\mathbf{k}}\mathbf{z} = -\tilde{\mathbf{L}}\ddot{u}_g(t) \qquad \text{(d)}$$

where $\tilde{\mathbf{m}}$, $\tilde{\mathbf{k}}$, and $\tilde{\mathbf{L}}$ are given by Eqs. (a)-(c), and $\ddot{u}_g(t) = 0.2g \sin \omega t$, where $\omega = 15$ rad/sec.

3. *Determine steady-state response.*

Equation (d) is specialized for the steady-state part of the response. By substituting $\mathbf{z}(t) = \mathbf{z}_0 \sin \omega t$ and $\ddot{u}_g(t) = 0.2g \sin \omega t$ and canceling the $\sin \omega t$ terms on both sides:

$$\left[\mathbf{k} - (15)^2 \tilde{\mathbf{m}}\right] \mathbf{z}_0 = -0.2g\,\tilde{\mathbf{L}} \qquad \text{(e)}$$

Solving the two coupled algebraic equations gives

$$\mathbf{z}_0 = \begin{Bmatrix} 0.4866 \\ 0.1169 \end{Bmatrix} \qquad \text{(f)}$$

The peak values of the displacements are

$$\mathbf{u}_0 = \Psi \mathbf{z}_0 \qquad \text{(g)}$$

Substituting for Ψ and $\mathbf{z}_0$ gives

$$\mathbf{u}_0 = \langle 0.0548 \quad 0.2126 \quad 0.4599 \quad 0.6484 \quad 0.7570 \rangle^T \qquad \text{(h)}$$

***Part b:** Modal analysis*

4. *Determine frequencies and modes.*

From Problem 14.8 the natural frequencies and modes of the structure are:

$$\omega_1 = 8.2619 \quad \omega_2 = 22.3473 \quad \omega_3 = 32.9203$$
$$\omega_4 = 41.7423 \quad \omega_5 = 48.9507 \text{ rad/sec} \qquad \text{(i)}$$

$$\Phi = [\phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4 \quad \phi_5]$$
$$= \begin{bmatrix} 0.3407 & 0.8698 & 0.9108 & 0.8706 & 1.1832 \\ 0.6513 & 1.1769 & 0.5430 & -0.2238 & -1.3061 \\ 0.9886 & 0.5713 & -0.9638 & -1.0095 & 0.7800 \\ 1.2093 & -0.5271 & -0.6666 & 1.2427 & -0.3620 \\ 1.3266 & -1.4927 & 1.6507 & -0.9886 & 0.1721 \end{bmatrix} \qquad \text{(j)}$$

5. *Determine the steady-state response.*

The response of the structure to ground motion $\ddot{u}_g(t)$ using first J modes of vibration is given by Eq.(13.1.15).

$$\mathbf{u}(t) = \sum_{n=1}^{J} \Gamma_n \phi_n D_n(t) \qquad \text{(k)}$$

where

$$\Gamma_n = \frac{\phi_n^T \mathbf{m} \mathbf{1}}{\phi_n^T \mathbf{m} \phi_n} \qquad \text{(l)}$$

and $D_n(t)$ is governed by

$$\ddot{D}_n + \omega_n^2 D_n = -\ddot{u}_g(t) = -0.2g \sin \omega t \qquad \text{(m)}$$

The steady state solution is

$$D_n(t) = -\frac{0.2g}{\omega_n^2 - \omega^2} \sin \omega t \qquad \text{(n)}$$

where $\omega = 15$ rad/sec and ω_n are given by Eq. (i).

Substituting ϕ_n and **m** into Eq. (1) gives

$$\Gamma_1 = 0.9982, \quad \Gamma_2 = 0.3483, \quad \Gamma_3 = 0.1681,$$
$$\Gamma_4 = 0.0999, \quad \Gamma_5 = 0.0988 \qquad \text{(o)}$$

Substituting Γ_n from Eq. (o), ϕ_n from Eq. (j) and $D_n(t)$ from Eq. (n) into Eq. (k) gives the displacement response:

$$\mathbf{u}(t) = \mathbf{u}_0 \sin \omega t \qquad \text{(p)}$$

where $\mathbf{u}_0$ is given as follows. Considering two modes only, i.e., $J = 2$:

$$\mathbf{u}_0 = \langle 0.0823 \quad 0.2049 \quad 0.4301 \quad 0.6462 \quad 0.7985 \rangle^T \qquad \text{(q)}$$

Considering all five modes, i.e., $J = 5$:

$$\mathbf{u}_0 = \langle 0.0599 \quad 0.2024 \quad 0.4470 \quad 0.6130 \quad 0.7780 \rangle^T \qquad \text{(r)}$$

6. *Comments.*

Comparison of Eqs. (h) and (q) with the exact solutions, Eq. (r), indicates that Rayleigh-Ritz method using two force-dependent Ritz vetors gives a roughly similar accuracy as modal analysis with two modes for responses u_2, u_3, u_4, and u_5. However, the error in u_1 determined by modal analysis is large. This becomes obvious from the following table:

Percentage Error

Response	Reyleigh-Ritz Method, $J = 2$	Modal Analysis, $J = 2$
u_5	-2.7 %	2.6 %
u_4	5.8 %	5.4 %
u_3	2.9 %	-3.8 %
u_2	5.0 %	1.2 %
u_1	8.5 %	37.4 %

Problem 15.1

First we set up modal equations. These are given by Eq. (15.2.3) with **M**, **K** and **P**(t) available from Example 15.1:

$$\mathbf{M} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 2.895 & 0 \\ 0 & 104.4 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix} \tag{a}$$

The first two natural frequencies and modes are given by Eq. (c) of Example 15.1. We now implement the procedure of Table 15.2.1.

1.0 *Initial calculations*

1.1 $\mathbf{u}_0 = \dot{\mathbf{u}}_0 = \mathbf{0}$; therefore, $\mathbf{q}_0 = \dot{\mathbf{q}}_0 = \mathbf{0}$

1.2 $\mathbf{p}_0 = \mathbf{0}$; therefore, $\mathbf{P}_0 = \mathbf{0}$

1.3 $\ddot{\mathbf{q}}_0 = \mathbf{0}$

1.4 $\Delta t = 0.1 \text{ sec}$

1.5 $\mathbf{q}_{-1} = \mathbf{0}$

1.6 Substituting **M**, Δt and $\mathbf{C} = \mathbf{0}$ in step 1.6 gives

$$\hat{\mathbf{K}} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

1.7 Substituting **M**, **K**, Δt and $\mathbf{C} = \mathbf{0}$ in step 1.7 gives

$$\mathbf{a} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -197.10 & 0 \\ 0 & 95.6 \end{bmatrix}$$

2.0 *Calculations for each time step i*

For the parameters of this problem, computational steps 2.1 through 2.5 are specialized and implemented for each time step i as follows.

2.1 $\mathbf{P}_i = \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix}$

2.2 $\hat{\mathbf{P}}_i = \mathbf{P}_i - \mathbf{a}\,\mathbf{q}_{i-1} - \mathbf{b}\,\mathbf{q}_i$

$$= \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix}_i - \begin{bmatrix} 100q_1 \\ 100q_2 \end{bmatrix}_{i-1} - \begin{bmatrix} -197.10q_1 \\ 95.6q_2 \end{bmatrix}_i$$

2.3 Solve: $\begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_{i+1} = \begin{bmatrix} \hat{P}_1 \\ \hat{P}_2 \end{bmatrix}_i$

$$\Rightarrow \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_{i+1}$$

The resulting modal displacements q_i are given in Table P15.1.

2.5 $$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}_{i+1} = 10^{-1} \begin{bmatrix} 0.0386 & 0.1727 \\ 0.1391 & 0.4020 \\ 0.2796 & 0.3697 \\ 0.4411 & 0.0041 \\ 0.6098 & -0.5502 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_{i+1}$$

These displacements are also presented in Table P15.1.

Table P15.1: Numerical solution of modal equations by the central difference method

Time	q_1	q_2	u_1	u_2	u_3	u_4	u_5
0.10	0.6098	− 0.5502	− 0.0072	− 0.0136	− 0.0033	0.0267	0.0675
0.20	1.8117	− 1.0762	− 0.0116	− 0.0181	0.0109	0.0795	0.1697
0.30	3.5710	− 1.0288	− 0.0040	0.0083	0.0618	0.1571	0.2744
0.40	5.8368	− 0.4576	0.0146	0.0628	0.1463	0.2573	0.3811
0.50	8.5433	0.0412	0.0337	0.1205	0.2404	0.3769	0.5187
0.60	11.6120	− 0.0532	0.0439	0.1594	0.3227	0.5122	0.7110
0.70	14.9550	− 0.6423	0.0466	0.1822	0.3944	0.6594	0.9743
0.80	18.4740	− 1.1110	0.0521	0.2123	0.4755	0.8145	1.1877
0.90	22.0690	− 0.9700	0.0684	0.2680	0.5812	0.9731	1.3991
1.00	25.6340	− 0.3665	0.0926	0.3418	0.7032	1.1306	1.5833
1.10	29.0670	0.0694	0.1134	0.4071	0.8153	1.2822	1.7687
1.20	32.2690	− 0.1173	0.1225	0.4441	0.8979	1.4233	1.9742
1.30	35.1460	− 0.7318	0.1230	0.4595	0.9556	1.5500	2.1834
1.40	37.6150	− 1.1324	0.1256	0.4777	1.0098	1.6587	2.3561
1.50	39.6050	− 0.9011	0.1373	0.5147	1.0740	1.7466	2.4647
1.60	41.0580	− 0.2792	0.1537	0.5599	1.1377	1.8110	2.5191
1.70	41.9330	0.0840	0.1633	0.5867	1.1756	1.8497	2.5525
1.80	42.2030	− 0.1908	0.1596	0.5794	1.1730	1.8615	2.5841
1.90	41.8620	− 0.8165	0.1475	0.5495	1.1403	1.8462	2.5977
2.00	40.9180	− 1.1401	0.1382	0.5233	1.1019	1.8044	2.5579

Problem 15.2

The relevant data are available from the solution to Problem 15.1. The same procedure is implemented.

1.0 *Initial calculations*

1.1 $\mathbf{u}_0 = \dot{\mathbf{u}}_0 = \mathbf{0}$; therefore, $\mathbf{q}_0 = \dot{\mathbf{q}}_0 = \mathbf{0}$

1.2 $\mathbf{p}_0 = \mathbf{0}$; therefore, $\mathbf{P}_0 = \mathbf{0}$

1.3 $\ddot{\mathbf{q}}_0 = \mathbf{0}$

1.4 $\Delta t = 0.05$ sec

1.5 $\mathbf{q}_{-1} = \mathbf{0}$

1.6 Substituting $\mathbf{M}$, Δt and $\mathbf{C} = \mathbf{0}$ in step 1.6 gives

$$\hat{\mathbf{K}} = \begin{bmatrix} 400 & 0 \\ 0 & 400 \end{bmatrix}$$

1.7 Substituting $\mathbf{M}$, $\mathbf{K}$, Δt and $\mathbf{C} = \mathbf{0}$ in step 1.7 gives

$$\mathbf{a} = \begin{bmatrix} 400 & 0 \\ 0 & 400 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -797.1 & 0 \\ 0 & -695.6 \end{bmatrix}$$

2.0 *Calculations for each time step i*

For the parameters of this problem, computational steps 2.1 through 2.5 are specialized and implemented for each time step i as follows.

2.1 $$\mathbf{P}_i = \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix}$$

2.2 $\hat{\mathbf{P}}_i = \mathbf{P}_i - \mathbf{a}\,\mathbf{q}_{i-1} - \mathbf{b}\,\mathbf{q}_i$

$$= \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix}_i - \begin{bmatrix} 400q_1 \\ 400q_2 \end{bmatrix}_{i-1} - \begin{bmatrix} -797.10q_1 \\ -695.6q_2 \end{bmatrix}_i$$

2.3 Solve: $$\begin{bmatrix} 400 & 0 \\ 0 & 400 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_{i+1} = \begin{bmatrix} \hat{P}_1 \\ \hat{P}_2 \end{bmatrix}_i$$

$$\Rightarrow \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_{i+1}$$

The resulting modal displacements q_i are given in Table P15.2a.

2.5 $$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}_{i+1} = 10^{-1} \begin{bmatrix} 0.0386 & 0.1727 \\ 0.1391 & 0.4020 \\ 0.2796 & 0.3697 \\ 0.4411 & 0.0041 \\ 0.6098 & -0.5502 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_{i+1}$$

These displacements are also presented in Table P15.2a.

3. *Comments.*

The solutions to Problems 15.1 and 15.2 are compared in Table P15.2b which shows that the smaller time step leads to more accurate results.

Table P15.2a: Numerical solution of modal equations by the central difference method

Time	q_1	q_2	u_1	u_2	u_3	u_4	u_5
0.10	0.4563	−0.3767	−0.0047	−0.0088	−0.0012	0.0200	0.0486
0.20	1.5080	−0.9001	−0.0097	−0.0152	0.0089	0.0661	0.1415
0.30	3.1249	−1.0594	−0.0062	0.0009	0.0482	0.1374	0.2488
0.40	5.2602	−0.6991	0.0082	0.0451	0.1212	0.2317	0.3592
0.50	7.8521	−0.1709	0.0274	0.1023	0.2132	0.3463	0.4882
0.60	10.8260	0.0098	0.0420	0.1510	0.3031	0.4775	0.6596
0.70	14.0950	−0.3334	0.0486	0.1827	0.3818	0.6216	0.8779
0.80	17.5660	−0.8655	0.0529	0.2096	0.4592	0.7745	1.1188
0.90	21.1390	−1.0673	0.0632	0.2511	0.5516	0.9320	1.3478
1.00	24.7080	−0.7419	0.0826	0.3139	0.6634	1.0896	1.5475
1.10	28.1730	−0.2067	0.1052	0.3836	0.7801	1.2426	1.7294
1.20	31.4320	0.0159	0.1216	0.4379	0.8794	1.3865	1.9159
1.30	34.3920	−0.2914	0.1277	0.4667	0.9508	1.5169	2.1132
1.40	36.9660	−0.8285	0.1284	0.4809	1.0029	1.6302	2.2998
1.50	39.0810	−1.0715	0.1323	0.5005	1.0531	1.7234	2.4421
1.60	40.6750	−0.7831	0.1435	0.5343	1.1083	1.7939	2.5235
1.70	41.7030	−0.2448	0.1567	0.5702	1.1570	1.8394	2.5565
1.80	42.1340	0.0181	0.1629	0.5868	1.1787	1.8585	2.5683
1.90	41.9560	−0.2510	0.1576	0.5735	1.1638	1.8506	2.5723
2.00	41.1740	−0.7894	0.1453	0.5410	1.1220	1.8159	2.5542

Table P15.2b

Time	u_5 ($\Delta t = 0.1$)	u_5 ($\Delta t = 0.05$)	u_5 (Theoretical)
0.10	0.0675	0.0486	0.03747
0.20	0.1697	0.1415	0.11710
0.30	0.2744	0.2488	0.22582
0.40	0.3811	0.3592	0.33496
0.50	0.5187	0.4882	0.45697
0.60	0.7110	0.6596	0.61781
0.70	0.9743	0.8779	0.81922
0.80	1.1877	1.1188	1.06098
0.90	1.3991	1.3478	1.29169
1.00	1.5833	1.5475	1.50588
1.10	1.7687	1.7294	1.68850
1.20	1.9742	1.9159	1.87169
1.30	2.1834	2.1132	2.06365
1.40	2.3561	2.2998	2.25139
1.50	2.4647	2.4421	2.41334
1.60	2.5191	2.5235	2.51023
1.70	2.5525	2.5565	2.56092
1.80	2.5841	2.5683	2.56900
1.90	2.5977	2.5723	2.57275
2.00	2.5579	2.5542	2.55760

Problem 15.3

First we set up modal equations. These are given by Eq. (15.2.3) with **M**, **K** and **P**(t) available from Example 15.1:

$$\mathbf{M} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 2.895 & 0 \\ 0 & 104.4 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix} \tag{a}$$

We now implement the procedure of Table 15.2.2.

1.0 *Initial calculations*

1.1 $\mathbf{u}_0 = \dot{\mathbf{u}}_0 = \mathbf{0}$; therefore, $\mathbf{q}_0 = \dot{\mathbf{q}}_0 = \mathbf{0}$

1.2 $\mathbf{p}_0 = \mathbf{0}$; therefore, $\mathbf{P}_0 = \mathbf{0}$

1.3 $\ddot{\mathbf{q}}_0 = \mathbf{0}$

1.4 $\Delta t = 0.1$ sec

1.5 Substituting **M**, **K**, Δt, $\mathbf{C} = \mathbf{0}$, and $\beta = 1/4$ in step 1.5 gives

$$\hat{\mathbf{K}} = \begin{bmatrix} 402.9 & 0 \\ 0 & 504.4 \end{bmatrix}$$

1.6 Substituting **M**, Δt, $\mathbf{C} = \mathbf{0}$, $\beta = 1/4$, and $\gamma = 1/2$ in step 1.6 gives

$$\mathbf{a} = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

2.0 *Calculations for each time step i*

For the parameters of this problem, computational steps 2.1 through 2.5 are specialized and implemented for each time step i as follows.

2.1 $\mathbf{P}_i = \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix}$

2.2 $\Delta\hat{\mathbf{P}}_i = \Delta\mathbf{P}_i + \mathbf{a}\,\dot{\mathbf{q}}_i + \mathbf{b}\,\ddot{\mathbf{q}}_i$; $\Delta\mathbf{P}_1 = \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix}$

and $\Delta\mathbf{P}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $i > 1$; and

$$\begin{bmatrix} \Delta\hat{P}_1 \\ \Delta\hat{P}_2 \end{bmatrix}_i = \begin{bmatrix} \Delta P_1 + 40\dot{q}_1 + 2q_1 \\ \Delta P_2 + 40\dot{q}_2 + 2q_2 \end{bmatrix}_i$$

2.3 Solve: $\begin{bmatrix} 402.9 & 0 \\ 0 & 504.4 \end{bmatrix} \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}_i = \begin{bmatrix} \Delta\hat{P}_1 \\ \Delta\hat{P}_2 \end{bmatrix}_i$

$\Rightarrow \Delta\mathbf{q}_i$

2.4 $\begin{bmatrix} \Delta\dot{q}_1 \\ \Delta\dot{q}_2 \end{bmatrix}_i = 20 \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}_i - 2 \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}_i$

Table P15.3a: Numerical solution of modal equations by the average acceleration method

Time	q_1	q_2	u_1	u_2	u_3	u_4	u_5
0.10	0.1514	– 0.1091	– 0.0013	– 0.0023	0.0002	0.0066	0.0152
0.20	0.7524	– 0.4551	– 0.0050	– 0.0078	0.0042	0.0330	0.0709
0.30	1.9373	– 0.8607	– 0.0074	– 0.0077	0.0224	0.0851	0.1655
0.40	3.6719	– 0.9901	– 0.0030	0.0113	0.0661	0.1616	0.2784
0.50	5.9064	– 0.7361	0.0101	0.0525	0.1379	0.2602	0.4007
0.60	8.5765	– 0.3091	0.0277	0.1068	0.2284	0.3782	0.5400
0.70	11.6056	– 0.0625	0.0437	0.1589	0.3222	0.5119	0.7111
0.80	14.9066	– 0.2005	0.0540	0.1992	0.4094	0.6575	0.9200
0.90	18.3845	– 0.6088	0.0604	0.2312	0.4915	0.8107	1.1546
1.00	21.9395	– 0.9494	0.0682	0.2669	0.5784	0.9674	1.3901
1.10	25.4694	– 0.9404	0.0820	0.3164	0.6774	1.1231	1.6049
1.20	28.8727	– 0.5892	0.1012	0.3778	0.7855	1.2733	1.7931
1.30	32.0516	– 0.1865	0.1204	0.4382	0.8893	1.4137	1.9648
1.40	34.9147	– 0.0658	0.1335	0.4829	0.9738	1.5401	2.1327
1.50	37.3798	– 0.3269	0.1385	0.5067	1.0331	1.6487	2.2974
1.60	39.3760	– 0.7537	0.1388	0.5173	1.0732	1.7366	2.4426
1.70	40.8459	– 0.9929	0.1403	0.5281	1.1054	1.8013	2.5454
1.80	41.7473	– 0.8464	0.1464	0.5465	1.1360	1.8411	2.5923
1.90	42.0543	– 0.4355	0.1546	0.5673	1.1598	1.8548	2.5884
2.00	41.7581	– 0.1004	0.1593	0.5767	1.1639	1.8419	2.5519

2.5 $$\begin{bmatrix} \Delta\ddot{q}_1 \\ \Delta\ddot{q}_2 \end{bmatrix}_i = 400\begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}_i - 40\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}_i - 2\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}_i$$

2.6 With $\Delta\mathbf{q}_i$, $\Delta\dot{\mathbf{q}}_i$ and $\Delta\ddot{\mathbf{q}}_i$ known from steps 2.3, 2.4 and 2.5, $\mathbf{q}_i$, $\dot{\mathbf{q}}_i$ and $\ddot{\mathbf{q}}_i$ are updated to determine the responses $\mathbf{q}_{i+1}$, $\dot{\mathbf{q}}_{i+1}$ and $\ddot{\mathbf{q}}_{i+1}$ at the end of the time step. The modal displacements $\mathbf{q}_i$ for the first twenty steps are shown in Table P15.3a.

2.7 $$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}_{i+1} = 10^{-1}\begin{bmatrix} 0.0386 & 0.1727 \\ 0.1391 & 0.4020 \\ 0.2796 & 0.3697 \\ 0.4411 & 0.0041 \\ 0.6098 & -0.5502 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_{i+1}$$

These displacements are also presented in Table P15.3.

3. *Comments.*

Because of different period elongations in the two numerical methods, their comparison at each time instant is not especially meaningful. Table P15.3b shows that the average acceleration method gives a more accurate value for the peak response.

Table P15.3b: Comparison of average acceleration and central difference methods

Time	u_5 ($\Delta t = 0.1$) central diff.	u_5 ($\Delta t = 0.1$) avg. accel.	u_5 (Theoretical)
0.10	0.0675	0.0152	0.03747
0.20	0.1697	0.0709	0.11710
0.30	0.2744	0.1655	0.22582
0.40	0.3811	0.2784	0.33496
0.50	0.5187	0.4007	0.45697
0.60	0.7110	0.5400	0.61781
0.70	0.9743	0.7111	0.81922
0.80	1.1877	0.9200	1.06098
0.90	1.3991	1.1546	1.29169
1.00	1.5833	1.3901	1.50588
1.10	1.7687	1.6049	1.68850
1.20	1.9742	1.7931	1.87169
1.30	2.1834	1.9648	2.06365
1.40	2.3561	2.1327	2.25139
1.50	2.4647	2.2974	2.41334
1.60	2.5191	2.4426	2.51023
1.70	2.5525	2.5454	2.56092
1.80	2.5841	2.5923	2.56900
1.90	2.5977	2.5884	2.57275
2.00	2.5579	2.5519	2.55760

Problem 15.4

Equation (15.2.3) gives the modal equations with $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{P}(t)$ defined in the solution to Problem 15.3. We now implement the procedure of Table 15.2.2 as follows.

1.0 *Initial calculations.*

1.1 $\mathbf{u}_0 = \dot{\mathbf{u}}_0 = \mathbf{0}$; therefore, $\mathbf{q}_0 = \dot{\mathbf{q}}_0 = \mathbf{0}$

1.2 $\mathbf{p}_0 = \mathbf{0}$; therefore, $\mathbf{P}_0 = \mathbf{0}$

1.3 $\ddot{\mathbf{q}}_0 = \mathbf{0}$

1.4 $\Delta t = 0.05$ sec

1.5 Substituting $\mathbf{M}$, $\mathbf{K}$, Δt, $\mathbf{C} = \mathbf{0}$, and $\beta = 1/4$ in step 1.5 gives

$$\hat{\mathbf{K}} = \begin{bmatrix} 1602.9 & 0 \\ 0 & 1704.4 \end{bmatrix}$$

1.6 Substituting $\mathbf{M}$, Δt, $\mathbf{C} = \mathbf{0}$, $\beta = 1/4$ and $\gamma = 1/2$ in step 1.6 gives

$$\mathbf{a} = \begin{bmatrix} 80 & 0 \\ 0 & 80 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

2.0 *Calculations for each time step i.*

For the parameters of this problem, computational steps 2.1 through 2.5 are specialized and implemented for each time step i as follows.

2.1 $\mathbf{P}_i = \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix}$

2.2 $\Delta\hat{\mathbf{P}}_i = \Delta\mathbf{P}_i + \mathbf{a}\,\dot{\mathbf{q}}_i + \mathbf{b}\,\ddot{\mathbf{q}}_i$; $\Delta\mathbf{P}_1 = \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix}$

and $\Delta\mathbf{P}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $i > 1$; and

$$\begin{bmatrix} \Delta\hat{P}_1 \\ \Delta\hat{P}_2 \end{bmatrix}_i = \begin{bmatrix} \Delta P_1 + 80\dot{q}_1 + 2q_1 \\ \Delta P_2 + 80\dot{q}_2 + 2q_2 \end{bmatrix}_i$$

2.3 Solve: $\begin{bmatrix} 1602.9 & 0 \\ 0 & 1704.4 \end{bmatrix} \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}_i = \begin{bmatrix} \Delta\hat{P}_1 \\ \Delta\hat{P}_2 \end{bmatrix}_i$

$\Rightarrow \Delta\mathbf{q}_i$

2.4 $\begin{bmatrix} \Delta\dot{q}_1 \\ \Delta\dot{q}_2 \end{bmatrix}_i = 40 \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}_i - 2 \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}_i$

2.5 $\begin{bmatrix} \Delta\ddot{q}_1 \\ \Delta\ddot{q}_2 \end{bmatrix}_i = 1600 \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}_i - 80 \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}_i - 2 \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}_i$

2.6 With $\Delta\mathbf{q}_i$, $\Delta\dot{\mathbf{q}}_i$ and $\Delta\ddot{\mathbf{q}}_i$ known from steps 2.3, 2.4 and 2.5, $\mathbf{q}_i$, $\dot{\mathbf{q}}_i$ and $\ddot{\mathbf{q}}_i$ are updated to determine the responses $\mathbf{q}_{i+1}$, $\dot{\mathbf{q}}_{i+1}$ and $\ddot{\mathbf{q}}_{i+1}$ at the end of the time step. The modal displacements $\mathbf{q}_i$ for the first twenty steps are shown in Table P15.4a.

2.7 $\mathbf{u}_{i+1} = \mathbf{\Phi}\,\mathbf{q}_{i+1}$
where $\mathbf{\Phi}$ is given in step 2.7 of Problem 15.3.

These displacements are also presented in Table P15.4a.

3. *Comments.*

The solutions to Problems 15.3 and 15.4 are compared in Table P15.4b, which shows that the smaller time step leads to more accurate results.

Table P15.4b

Time	u_5 ($\Delta t = 0.1$)	u_5 ($\Delta t = 0.05$)	u_5 (Theoretical)
0.10	0.0152	0.0200	0.03747
0.20	0.0709	0.0916	0.11710
0.30	0.1655	0.1939	0.22582
0.40	0.2784	0.3054	0.33496
0.50	0.4007	0.4254	0.45697
0.60	0.5400	0.5720	0.61781
0.70	0.7111	0.7616	0.81922
0.80	0.9200	0.9895	1.06098
0.90	1.1546	1.2292	1.29169
1.00	1.3901	1.4518	1.50588
1.10	1.6049	1.6467	1.68850
1.20	1.7931	1.8259	1.87169
1.30	1.9648	2.0076	2.06365
1.40	2.1327	2.1939	2.25139
1.50	2.2974	2.3642	2.41334
1.60	2.4426	2.4896	2.51023
1.70	2.5454	2.5561	2.56092
1.80	2.5923	2.5743	2.56900
1.90	2.5884	2.5676	2.57275
2.00	2.5519	2.5496	2.55760

Table P15.4a: Numerical solution of modal equations by the average acceleration method

Time	q_1	q_2	u_1	u_2	u_3	u_4	u_5
0.05	0.0380	– 0.0323	– 0.0004	– 0.0008	– 0.0001	0.0017	0.0041
0.10	0.1899	– 0.1535	– 0.0019	– 0.0035	– 0.0004	0.0083	0.0200
0.15	0.4927	– 0.3662	– 0.0044	– 0.0079	0.0002	0.0216	0.0502
0.20	0.9440	– 0.6184	– 0.0070	– 0.0117	0.0035	0.0414	0.0916
0.25	1.5406	– 0.8481	– 0.0087	– 0.0127	0.0117	0.0676	0.1406
0.30	2.2784	– 0.9992	– 0.0085	– 0.0085	0.0268	0.1001	0.1939
0.35	3.1518	– 1.0346	– 0.0057	0.0022	0.0499	0.1386	0.2491
0.40	4.1546	– 0.9457	– 0.0003	0.0198	0.0812	0.1829	0.3054
0.45	5.2796	– 0.7542	0.0073	0.0431	0.1198	0.2326	0.3634
0.50	6.5187	– 0.5070	0.0164	0.0703	0.1635	0.2873	0.4254
0.55	7.8628	– 0.2648	0.0257	0.0987	0.2101	0.3467	0.4940
0.60	9.3023	– 0.0868	0.0344	0.1259	0.2569	0.4103	0.5720
0.65	10.8268	– 0.0166	0.0415	0.1499	0.3021	0.4776	0.6611
0.70	12.4252	– 0.0715	0.0467	0.1699	0.3448	0.5480	0.7616
0.75	14.0861	– 0.2380	0.0502	0.1863	0.3851	0.6212	0.8721
0.80	15.7973	– 0.4753	0.0527	0.2006	0.4241	0.6966	0.9895
0.85	17.5466	– 0.7253	0.0551	0.2148	0.4638	0.7737	1.1099
0.90	19.3214	– 0.9267	0.0585	0.2314	0.5060	0.8519	1.2292
0.95	21.1087	– 1.0302	0.0636	0.2521	0.5522	0.9307	1.3439
1.00	22.8957	– 1.0104	0.0708	0.2778	0.6029	1.0095	1.4518
1.05	24.6695	– 0.8722	0.0801	0.3080	0.6575	1.0878	1.5523
1.10	26.4173	– 0.6494	0.0906	0.3413	0.7147	1.1650	1.6467
1.15	28.1264	– 0.3967	0.1016	0.3752	0.7718	1.2405	1.7370
1.20	29.7844	– 0.1759	0.1118	0.4071	0.8263	1.3137	1.8259
1.25	31.3795	– 0.0411	0.1203	0.4347	0.8759	1.3841	1.9158
1.30	32.9001	– 0.0254	0.1264	0.4565	0.9190	1.4512	2.0076
1.35	34.3352	– 0.1326	0.1301	0.4721	0.9552	1.5145	2.1011
1.40	35.6744	– 0.3364	0.1318	0.4826	0.9851	1.5735	2.1939
1.45	36.9081	– 0.5869	0.1322	0.4897	1.0103	1.6278	2.2829
1.50	38.0273	– 0.8228	0.1324	0.4957	1.0329	1.6770	2.3642
1.55	39.0240	– 0.9862	0.1334	0.5030	1.0547	1.7209	2.4339
1.60	39.8910	– 1.0371	0.1359	0.5130	1.0771	1.7592	2.4896
1.65	40.6219	– 0.9631	0.1400	0.5262	1.1003	1.7914	2.5301
1.70	41.2116	– 0.7822	0.1454	0.5417	1.1234	1.8175	2.5561
1.75	41.6557	– 0.5388	0.1513	0.5576	1.1448	1.8372	2.5698
1.80	41.9511	– 0.2925	0.1567	0.5716	1.1622	1.8503	2.5743
1.85	42.0956	– 0.1037	0.1605	0.5812	1.1732	1.8568	2.5727
1.90	42.0882	– 0.0186	0.1620	0.5845	1.1762	1.8565	2.5676
1.95	41.9289	– 0.0581	0.1607	0.5807	1.1703	1.8495	2.5600
2.00	41.6189	– 0.2124	0.1568	0.5702	1.1559	1.8357	2.5496

Problem 15.5

Solution to this problem is available under Example 15.1 in the book. Table P15.5 shows these results using the linear acceleration method, those from the average acceleration method (Problem 15.3), and theoretical results.

Table P15.5

Time	u_5 ($\Delta t = 0.1$) avg. accel..	u_5 ($\Delta t = 0.1$) linear accel.	u_5 (Theoretical)
0.10	0.0152	0.0105	0.03747
0.20	0.0709	0.0693	0.11710
0.30	0.1655	0.1663	0.22582
0.40	0.2784	0.2777	0.33496
0.50	0.4007	0.3959	0.45697
0.60	0.5400	0.5335	0.61781
0.70	0.7111	0.7090	0.81922
0.80	0.9200	0.9258	1.06098
0.90	1.1546	1.1651	1.29169
1.00	1.3901	1.3974	1.50588
1.10	1.6049	1.6032	1.68850
1.20	1.7931	1.7854	1.87169
1.30	1.9648	1.9611	2.06365
1.40	2.1327	2.1412	2.25139
1.50	2.2974	2.3160	2.41334
1.60	2.4426	2.4596	2.51023
1.70	2.5454	2.5488	2.56092
1.80	2.5923	2.5812	2.56900
1.90	2.5884	2.5748	2.57275
2.00	2.5519	2.5508	2.55760

Comments.

Because of different period elongations in the two numerical methods, their comparison at each time instant is not especially meaningful. Table P15.5 shows that the linear acceleration method gives a more accurate value for the peak response.

Problem 15.6

Equation (15.2.3) gives the modal equations with $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{P}(t)$ defined in the solution to Problem 15.3. We now implement the procedure of Table 15.2.2 as follows.

1.0 *Initial calculations.*

1.1 $\mathbf{u}_0 = \dot{\mathbf{u}}_0 = \mathbf{0}$; therefore, $\mathbf{q}_0 = \dot{\mathbf{q}}_0 = \mathbf{0}$

1.2 $\mathbf{p}_0 = \mathbf{0}$; therefore, $\mathbf{P}_0 = \mathbf{0}$

1.3 $\ddot{\mathbf{q}}_0 = \mathbf{0}$

1.4 $\Delta t = 0.05 \sec$

1.5 Substituting $\mathbf{M}$, $\mathbf{K}$, Δt, $\mathbf{C} = \mathbf{0}$, and $\beta = 1/6$ in step 1.5 gives

$$\hat{\mathbf{K}} = \begin{bmatrix} 2402.9 & 0 \\ 0 & 2504.4 \end{bmatrix}$$

1.6 Substituting $\mathbf{M}$, Δt, $\mathbf{C} = \mathbf{0}$, $\beta = 1/6$ and $\gamma = 1/2$ in step 1.6 gives

$$\mathbf{a} = \begin{bmatrix} 120 & 0 \\ 0 & 120 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

2.0 *Calculations for each time step i.*

For the parameters of this problem, computational steps 2.1 through 2.5 are specialized and implemented for each time step i as follows.

2.1 $\mathbf{P}_i = \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix}$

2.2 $\Delta\hat{\mathbf{P}}_i = \Delta\mathbf{P}_i + \mathbf{a}\,\dot{\mathbf{q}}_i + \mathbf{b}\,\ddot{\mathbf{q}}_i$; $\Delta\mathbf{P}_1 = \begin{bmatrix} 60.98 \\ -55.02 \end{bmatrix}$

and $\Delta\mathbf{P}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $i > 1$; and

$$\begin{bmatrix} \Delta\hat{P}_1 \\ \Delta\hat{P}_2 \end{bmatrix}_i = \begin{bmatrix} \Delta P_1 + 120\dot{q}_1 + 3q_1 \\ \Delta P_2 + 120\dot{q}_2 + 3q_2 \end{bmatrix}_i$$

2.3 Solve: $\begin{bmatrix} 2402.9 & 0 \\ 0 & 2504.4 \end{bmatrix} \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}_i = \begin{bmatrix} \Delta\hat{P}_1 \\ \Delta\hat{P}_2 \end{bmatrix}_i$

$$\Rightarrow \Delta\mathbf{q}_i$$

2.4

$$\begin{bmatrix} \Delta\dot{q}_1 \\ \Delta\dot{q}_2 \end{bmatrix}_i = 60 \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}_i - 3 \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}_i - 0.025 \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}_i$$

2.5

$$\begin{bmatrix} \Delta\ddot{q}_1 \\ \Delta\ddot{q}_2 \end{bmatrix}_i = 2400 \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \end{bmatrix}_i - 120 \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}_i - 3 \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}_i$$

2.6 With $\Delta\mathbf{q}_i$, $\Delta\dot{\mathbf{q}}_i$ and $\Delta\ddot{\mathbf{q}}_i$ known from steps 2.3, 2.4 and 2.5, $\mathbf{q}_i$, $\dot{\mathbf{q}}_i$ and $\ddot{\mathbf{q}}_i$ are updated to determine the responses $\mathbf{q}_{i+1}$, $\dot{\mathbf{q}}_{i+1}$ and $\ddot{\mathbf{q}}_{i+1}$ at the end of the time step. The modal displacements $\mathbf{q}_i$ for the first twenty steps are shown in Table P15.6a.

2.7 $\mathbf{u}_{i+1} = \mathbf{\Phi}\,\mathbf{q}_{i+1}$

where $\mathbf{\Phi}$ is given in step 2.7 of Problem 15.3.

These displacements are also presented in Table P15.6a.

3. *Comments*

The solutions to Problems 15.5 and 15.6 are compared in Table P15.6b, which shows that the smaller time step leads to more accurate results.

Table P15.6a: Numerical solution of modal equations by the linear acceleration method

Time	q_1	q_2	u_1	u_2	u_3	u_4	u_5
0.05	0.0254	− 0.0220	− 0.0003	− 0.0005	− 0.0001	0.0011	0.0028
0.10	0.1775	− 0.1483	− 0.0019	− 0.0035	− 0.0005	0.0078	0.0190
0.15	0.4805	− 0.3693	− 0.0045	− 0.0082	− 0.0002	0.0210	0.0496
0.20	0.9324	− 0.6298	− 0.0073	− 0.0124	0.0028	0.0409	0.0915
0.25	1.5298	− 0.8646	− 0.0090	− 0.0135	0.0108	0.0671	0.1409
0.30	2.2684	− 1.0150	− 0.0088	− 0.0093	0.0259	0.0996	0.1942
0.35	3.1428	− 1.0433	− 0.0059	0.0018	0.0493	0.1382	0.2491
0.40	4.1469	− 0.9425	− 0.0003	0.0198	0.0811	0.1825	0.3047
0.45	5.2732	− 0.7378	0.0076	0.0437	0.1202	0.2323	0.3622
0.50	6.5136	− 0.4804	0.0168	0.0713	0.1644	0.2871	0.4236
0.55	7.8593	− 0.2347	0.0262	0.0999	0.2111	0.3466	0.4922
0.60	9.3004	− 0.0620	0.0348	0.1268	0.2578	0.4102	0.5706
0.65	10.8265	− 0.0057	0.0417	0.1503	0.3025	0.4775	0.6605
0.70	12.4266	− 0.0798	0.0465	0.1696	0.3445	0.5481	0.7622
0.75	14.0892	− 0.2657	0.0497	0.1852	0.3841	0.6214	0.8738
0.80	15.8022	− 0.5170	0.0520	0.1990	0.4227	0.6968	0.9921
0.85	17.5533	− 0.7708	0.0544	0.2131	0.4623	0.7740	1.1128
0.90	19.3297	− 0.9636	0.0579	0.2301	0.5049	0.8522	1.2318
0.95	21.1187	− 1.0473	0.0633	0.2516	0.5518	0.9311	1.3454
1.00	22.9073	− 1.0008	0.0710	0.2783	0.6035	1.0100	1.4519
1.05	24.6826	− 0.8359	0.0807	0.3096	0.6593	1.0884	1.5511
1.10	26.4317	− 0.5937	0.0917	0.3437	0.7171	1.1657	1.6445
1.15	28.1420	− 0.3348	0.1027	0.3779	0.7745	1.2412	1.7345
1.20	29.8012	− 0.1241	0.1128	0.4094	0.8287	1.3145	1.8241
1.25	31.3973	− 0.0141	0.1208	0.4361	0.8774	1.3849	1.9154
1.30	32.9186	− 0.0324	0.1264	0.4565	0.9193	1.4520	2.0092
1.35	34.3543	− 0.1744	0.1295	0.4707	0.9542	1.5153	2.1045
1.40	35.6939	− 0.4046	0.1306	0.4801	0.9831	1.5743	2.1989
1.45	36.9277	− 0.6654	0.1309	0.4868	1.0079	1.6286	2.2885
1.50	38.0470	− 0.8917	0.1313	0.4933	1.0309	1.6779	2.3692
1.55	39.0434	− 1.0267	0.1328	0.5017	1.0538	1.7218	2.4374
1.60	39.9099	− 1.0368	0.1360	0.5133	1.0776	1.7600	2.4908
1.65	40.6402	− 0.9194	0.1408	0.5282	1.1024	1.7923	2.5288
1.70	41.2290	− 0.7038	0.1468	0.5451	1.1268	1.8183	2.5529
1.75	41.6720	− 0.4441	0.1530	0.5616	1.1488	1.8380	2.5656
1.80	41.9661	− 0.2051	0.1583	0.5753	1.1659	1.8510	2.5704
1.85	42.1091	− 0.0466	0.1616	0.5837	1.1757	1.8574	2.5704
1.90	42.0999	− 0.0083	0.1622	0.5851	1.1769	1.8570	2.5677
1.95	41.9388	− 0.0997	0.1600	0.5792	1.1690	1.8499	2.5629
2.00	41.6267	− 0.2980	0.1554	0.5669	1.1529	1.8360	2.5548

Table P15.6b

Time	$u_5 (\Delta t = 0.1)$	$u_5 (\Delta t = 0.05)$	u_5 (Theoretical)
0.10	0.0105	0.0190	0.03747
0.20	0.0693	0.0915	0.11710
0.30	0.1663	0.1942	0.22582
0.40	0.2777	0.3047	0.33496
0.50	0.3959	0.4236	0.45697
0.60	0.5335	0.5706	0.61781
0.70	0.7090	0.7622	0.81922
0.80	0.9258	0.9921	1.06098
0.90	1.1651	1.2318	1.29169
1.00	1.3974	1.4519	1.50588
1.10	1.6032	1.6445	1.68850
1.20	1.7854	1.8241	1.87169
1.30	1.9611	2.0092	2.06365
1.40	2.1412	2.1989	2.25139
1.50	2.3160	2.3692	2.41334
1.60	2.4596	2.4908	2.51023
1.70	2.5488	2.5529	2.56092
1.80	2.5812	2.5704	2.56900
1.90	2.5748	2.5677	2.57275
2.00	2.5508	2.5548	2.55760

Problem 15.7

Solution to this problem is available under Example 15.2 in the textbook.

Problem 15.8

The **m** and **k** of the system were defined in Example 15.1; **c** is excluded because the system is undamped. For the system starting from rest, $\mathbf{u}_0 = \mathbf{0}$ and $\dot{\mathbf{u}}_0 = \mathbf{0}$. Because the system is linear, the computational steps of Table 15.3.1 are simplified in three ways: (1) Step 2.2 is eliminated because $\mathbf{k}_i = \mathbf{k}$. (2) Step 2.3 needs to be computed only once:

$$\hat{\mathbf{k}} = \mathbf{k} + \frac{2}{\Delta t}\mathbf{c} + \frac{4}{(\Delta t)^2}\mathbf{m} = \mathbf{k} + \frac{4}{(0.05)^2}\mathbf{m}$$

$$= \mathbf{k} + 1600\mathbf{m}$$

(3) In Step 2.4 no iteration is necessary and $\Delta\mathbf{u}_i$ is determined by solving the algebraic equations $\hat{\mathbf{k}}\,\Delta\mathbf{u}_i = \Delta\hat{\mathbf{p}}_i$.

We now implement the procedure of Table 15.3.1 as follows.

1.0 *Initial calculations.*

1.1 Solve $\mathbf{m}\ddot{\mathbf{u}}_0 = \mathbf{p}_0$ where $\mathbf{p}_0 = \mathbf{0}$ to obtain $\ddot{\mathbf{u}}_0 = \mathbf{0}$

1.2 $\Delta t = 0.05$ sec

1.3 $\mathbf{a} = \frac{4}{\Delta t}\mathbf{m} + 2\mathbf{c} = \frac{4}{0.05}\mathbf{m} + 2\mathbf{c} = 80\mathbf{m}$ and $\mathbf{b} = 2\mathbf{m}$

2.0 *Calculations for each time step i.*

Computational steps 2.1, 2.4 (modified as above), 2.5, and 2.6 are implemented for $i = 1, 2, 3, \ldots$ to obtain the displacements u_1, u_2, u_3, u_4, and u_5 presented in Table P15.8a.

3. *Comments.*

The solutions to Problems 15.7 and 15.8 are compared in Table P15.8b, which shows that the smaller time step leads to more accurate results.

Table P15.8a: Numerical solution by the average acceleration method

Time	u_1	u_2	u_3	u_4	u_5
0.05	0.0000	– 0.0001	– 0.0003	0.0008	0.0051
0.10	– 0.0002	– 0.0010	– 0.0012	0.0055	0.0235
0.15	– 0.0013	– 0.0041	– 0.0016	0.0179	0.0551
0.20	– 0.0044	– 0.0089	0.0020	0.0387	0.0952
0.25	– 0.0082	– 0.0117	0.0116	0.0662	0.1422
0.30	– 0.0087	– 0.0079	0.0270	0.0991	0.1949
0.35	– 0.0039	0.0042	0.0489	0.1366	0.2518
0.40	0.0030	0.0233	0.0794	0.1794	0.3101
0.45	0.0097	0.0464	0.1184	0.2291	0.3679
0.50	0.0172	0.0716	0.1631	0.2858	0.4273
0.55	0.0260	0.0990	0.2101	0.3462	0.4947
0.60	0.0354	0.1276	0.2564	0.4083	0.5744
0.65	0.0442	0.1533	0.3006	0.4740	0.6658
0.70	0.0499	0.1733	0.3429	0.5448	0.7661
0.75	0.0515	0.1880	0.3844	0.6194	0.8744
0.80	0.0524	0.2010	0.4245	0.6956	0.9905
0.85	0.0560	0.2161	0.4634	0.7722	1.1117
0.90	0.0614	0.2345	0.5043	0.8490	1.2332
0.95	0.0666	0.2558	0.5505	0.9269	1.3488
1.00	0.0723	0.2799	0.6021	1.0071	1.4548
1.05	0.0804	0.3085	0.6574	1.0872	1.5531
1.10	0.0912	0.3421	0.7145	1.1638	1.6481
1.15	0.1036	0.3780	0.7707	1.2374	1.7409
1.20	0.1151	0.4108	0.8244	1.3101	1.8308
1.25	0.1226	0.4371	0.8747	1.3817	1.9191
1.30	0.1265	0.4573	0.9191	1.4500	2.0089
1.35	0.1302	0.4729	0.9552	1.5134	2.1022
1.40	0.1340	0.4849	0.9839	1.5711	2.1971
1.45	0.1354	0.4934	1.0085	1.6241	2.2878
1.50	0.1345	0.4987	1.0317	1.6739	2.3682
1.55	0.1342	0.5040	1.0545	1.7198	2.4354
1.60	0.1361	0.5135	1.0770	1.7585	2.4904
1.65	0.1413	0.5282	1.0996	1.7891	2.5330
1.70	0.1485	0.5452	1.1217	1.8139	2.5609
1.75	0.1543	0.5608	1.1431	1.8342	2.5739
1.80	0.1576	0.5730	1.1618	1.8487	2.5763
1.85	0.1603	0.5817	1.1736	1.8558	2.5736
1.90	0.1633	0.5861	1.1754	1.8549	2.5697
1.95	0.1637	0.5841	1.1685	1.8462	2.5644
2.00	0.1596	0.5738	1.1544	1.8320	2.5544

Table P15.8b

Time	u_5 ($\Delta t = 0.1$)	u_5 ($\Delta t = 0.05$)	u_5 (Theoretical)
0.10	0.0172	0.0235	0.03747
0.20	0.0753	0.0952	0.11710
0.30	0.1682	0.1949	0.22582
0.40	0.2799	0.3101	0.33496
0.50	0.4044	0.4273	0.45697
0.60	0.5436	0.5744	0.61781
0.70	0.7127	0.7661	0.81922
0.80	0.9228	0.9905	1.06098
0.90	1.1588	1.2332	1.29169
1.00	1.3921	1.4548	1.50588
1.10	1.6068	1.6481	1.68850
1.20	1.7974	1.8308	1.87169
1.30	1.9676	2.0089	2.06365
1.40	2.1341	2.1971	2.25139
1.50	2.3012	2.3682	2.41334
1.60	2.4462	2.4904	2.51023
1.70	2.5469	2.5609	2.56092
1.80	2.5953	2.5763	2.56900
1.90	2.5926	2.5697	2.57275
2.00	2.5540	2.5544	2.55760

Problem 15.9

Solution to this problem is available under Example 15.3 in the textbook.

Problem 15.10

The **m** and **k** of the system were defined in Example 15.1; **c** is excluded because the system is undamped. For the system starting from rest, $\mathbf{u}_0 = \mathbf{0}$ and $\dot{\mathbf{u}}_0 = \mathbf{0}$. Because the system is linear, the computational steps of Table 15.3.3 are simplified in three ways: (1) Step 2.2 is eliminated because $\mathbf{k}_i = \mathbf{k}$. (2) Step 2.3 needs to be computed only once:

$$\hat{\mathbf{k}} = \mathbf{k} + \frac{3}{\theta\Delta t}\mathbf{c} + \frac{6}{(\theta\Delta t)^2}\mathbf{m}$$

$$= \mathbf{k} + \frac{6}{(1.42 \times 0.05)^2}\mathbf{m}$$

$$= \mathbf{k} + 1190.24\mathbf{m}$$

(3) In Step 2.4 no iteration is necessary and $\delta\mathbf{u}_i$ is determined by solving the algebraic equations (15.3.9).

We now implement the procedure of Table 15.3.1 as follows.

1.0 *Initial calculations.*

1.1 Solve $\mathbf{m}\ddot{\mathbf{u}}_0 = \mathbf{p}_0$ where $\mathbf{p}_0 = \mathbf{0}$ to obtain $\ddot{\mathbf{u}}_0 = \mathbf{0}$

1.2 $\Delta t = 0.05$ sec

1.3 $\mathbf{a} = \frac{6}{\theta\Delta t}\mathbf{m} + 3\mathbf{c} = \frac{6}{1.42\,(0.05)}\mathbf{m} = 84.51\mathbf{m}$

and $\mathbf{b} = 3\mathbf{m} + \frac{\theta\Delta t}{2}\mathbf{c} = 3\mathbf{m}$

2. *Calculations for each time step i.*

Computational steps 2.1, 2.4 (modified as above), 2.5, 2.6, and 2.7 are implemented for $i = 1, 2, 3, \ldots, 20$ to obtain the displacements u_1, u_2, u_3, u_4, and u_5 presented in Table P15.10a.

3. *Comments.*

The solutions to Problems 15.9 and 15.10 are compared in Table P15.10b, which shows that the smaller time step leads to more accurate results.

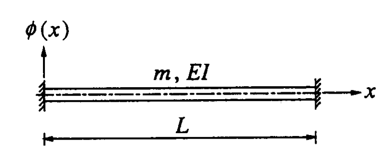
Table P15.10a: Numerical solution by Wilson's method

Time	u_1	u_2	u_3	u_4	u_5
0.05	0.0000	−0.0001	−0.0001	0.0004	0.0023
0.10	−0.0001	−0.0007	−0.0008	0.0037	0.0161
0.15	−0.0009	−0.0029	−0.0012	0.0132	0.0426
0.20	−0.0030	−0.0065	0.0011	0.0305	0.0789
0.25	−0.0060	−0.0095	0.0083	0.0550	0.1227
0.30	−0.0076	−0.0082	0.0213	0.0855	0.1728
0.35	−0.0058	−0.0001	0.0408	0.1213	0.2274
0.40	−0.0006	0.0151	0.0676	0.1623	0.2849
0.45	0.0064	0.0358	0.1020	0.2090	0.3439
0.50	0.0141	0.0602	0.1430	0.2618	0.4049
0.55	0.0223	0.0866	0.1882	0.3199	0.4704
0.60	0.0308	0.1137	0.2348	0.3819	0.5441
0.65	0.0391	0.1398	0.2808	0.4472	0.6283
0.70	0.0461	0.1631	0.3249	0.5156	0.7234
0.75	0.0511	0.1825	0.3673	0.5874	0.8278
0.80	0.0542	0.1986	0.4083	0.6621	0.9397
0.85	0.0566	0.2134	0.4489	0.7388	1.0566
0.90	0.0597	0.2294	0.4904	0.8164	1.1756
0.95	0.0642	0.2482	0.5341	0.8945	1.2932
1.00	0.0703	0.2707	0.5813	0.9729	1.4060
1.05	0.0778	0.2970	0.6326	1.0513	1.5121
1.10	0.0866	0.3266	0.6869	1.1291	1.6113
1.15	0.0965	0.3583	0.7426	1.2054	1.7052
1.20	0.1068	0.3904	0.7974	1.2794	1.7958
1.25	0.1164	0.4204	0.8492	1.3507	1.8850
1.30	0.1243	0.4463	0.8963	1.4190	1.9740
1.35	0.1299	0.4668	0.9376	1.4840	2.0633
1.40	0.1334	0.4819	0.9728	1.5450	2.1523
1.45	0.1353	0.4927	1.0022	1.6013	2.2397
1.50	0.1363	0.5007	1.0271	1.6526	2.3225
1.55	0.1371	0.5076	1.0491	1.6987	2.3974
1.60	0.1383	0.5149	1.0698	1.7393	2.4612
1.65	0.1405	0.5239	1.0903	1.7743	2.5118
1.70	0.1438	0.5350	1.1105	1.8031	2.5484
1.75	0.1482	0.5477	1.1299	1.8256	2.5717
1.80	0.1530	0.5606	1.1471	1.8414	2.5832
1.85	0.1574	0.5717	1.1603	1.8508	2.5851
1.90	0.1604	0.5790	1.1680	1.8535	2.5799
1.95	0.1614	0.5810	1.1687	1.8496	2.5695
2.00	0.1602	0.5772	1.1615	1.8390	2.5553

Table P15.10b

Time	$u_5(\Delta t = 0.1)$	$u_5(\Delta t = 0.05)$	u_5 (Theoretical)
0.10	0.0077	0.0161	0.03747
0.20	0.0520	0.0789	0.11710
0.30	0.1311	0.1728	0.22582
0.40	0.2339	0.2849	0.33496
0.50	0.3537	0.4049	0.45697
0.60	0.4895	0.5441	0.61781
0.70	0.6479	0.7234	0.81922
0.80	0.8360	0.9397	1.06098
0.90	1.0516	1.1756	1.29169
1.00	1.2805	1.4060	1.50588
1.10	1.5049	1.6113	1.68850
1.20	1.7123	1.7958	1.87169
1.30	1.8994	1.9740	2.06365
1.40	2.0691	2.1523	2.25139
1.50	2.2253	2.3225	2.41334
1.60	2.3659	2.4612	2.51023
1.70	2.4813	2.5484	2.56092
1.80	2.5598	2.5832	2.56900
1.90	2.5938	2.5799	2.57275
2.00	2.5840	2.5553	2.55760

Problem 16.1



$$\phi(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \quad \text{(a)}$$

$$u(0) = 0 \Rightarrow \phi(0) = 0 \Rightarrow C_2 + C_4 = 0 \Rightarrow C_4 = -C_2 \quad \text{(b)}$$

$$u'(0) = 0 \Rightarrow \phi'(0) = 0 \Rightarrow \beta(C_1 + C_3) = 0 \Rightarrow C_3 = -C_1 \quad \text{(c)}$$

$$u(L) = 0 \Rightarrow C_1(\sin \beta L - \sinh \beta L) + C_2(\cos \beta L - \cosh \beta L) = 0 \quad \text{(d)}$$

$$u'(L) = 0 \Rightarrow C_1(\cos \beta L - \cosh \beta L) + C_2(-\sin \beta L - \sinh \beta L) = 0 \quad \text{(e)}$$

Rewrite Eqs. (d) and (e) in matrix form:

$$\begin{bmatrix} (\sin \beta L - \sinh \beta L) & (\cos \beta L - \cosh \beta L) \\ (\cos \beta L - \cosh \beta L) & (-\sin \beta L - \sinh \beta L) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{(f)}$$

Setting the determinant of the coefficient matrix to zero gives

$$1 - \cos \beta L \cosh \beta L = 0 \quad \text{(g)}$$

Equation (g) is solved numerically to obtain

$$\beta_n L = 4.730, \ 7.853, \text{ and } 10.996$$

for $n = 1, 2,$ and 3. Eq. (16.3.8) then gives the natural frequencies:

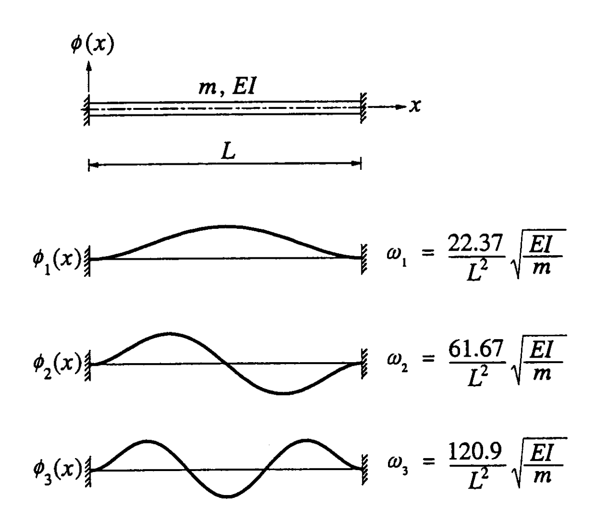
$$\omega_1 = \frac{22.37}{L^2}\sqrt{\frac{EI}{m}}; \quad \omega_2 = \frac{61.67}{L^2}\sqrt{\frac{EI}{m}}; \quad \omega_3 = \frac{120.9}{L^2}\sqrt{\frac{EI}{m}}$$

To determine the natural vibration modes corresponding to each value of $\beta_n L$, we express C_2 in terms of C_1 from Eq. (e) and substitute in Eq. (a) together with Eqs. (b) and (c). The result is

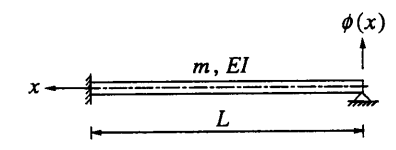
$$\phi_n(x) = C_1 \left[\sin \beta_n x - \sinh \beta_n x + \left(\frac{\cos \beta_n L - \cosh \beta_n L}{\sin \beta_n L + \sinh \beta_n L} \right) (\cos \beta_n x - \cosh \beta_n x) \right]$$

where C_1 is an arbitrary constant. The first three natural vibration modes are shown in the accompanying figure.

The natural vibration frequencies of the clamped beam are higher than for the simply supported beam.



Problem 16.2



$$\phi(x) = C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x \qquad \text{(a)}$$

$$u(0) = 0 \Rightarrow \phi(0) = 0 \Rightarrow C_2 + C_4 = 0 \qquad \text{(b)}$$

$$\mathcal{M}(0) = 0 \Rightarrow EI\,\phi''(0) = 0 \Rightarrow \beta^2(-C_2 + C_4) = 0 \qquad \text{(c)}$$

These two equations gives $C_2 = C_4 = 0$ and the general solution reduces to

$$\phi(x) = C_1 \sin\beta x + C_3 \sinh\beta x \qquad \text{(d)}$$

$$u(L) = 0 \Rightarrow \phi(L) = 0 \Rightarrow C_1 \sin\beta L + C_3 \sinh\beta L = 0 \qquad \text{(e)}$$

$$u'(L) = 0 \Rightarrow \phi'(L) = 0 \Rightarrow \beta(C_1 \cos\beta L + C_3 \cosh\beta L) = 0 \qquad \text{(f)}$$

Rewrite Eqs. (e) and (f) in matrix form:

$$\begin{bmatrix} \sin\beta L & \sinh\beta L \\ \cos\beta L & \cosh\beta L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \text{(g)}$$

For a nontrivial solution, the determinant of the coefficient matrix must be zero. Thus

$$\sin\beta L \cosh\beta L - \cos\beta L \sinh\beta L = 0$$

or

$$\tan\beta L = \tanh\beta L \qquad \text{(h)}$$

Equation (g) is solved numerically to obtain

$$\beta_n L = 3.927,\ 7.069,\ \text{and}\ 10.210$$

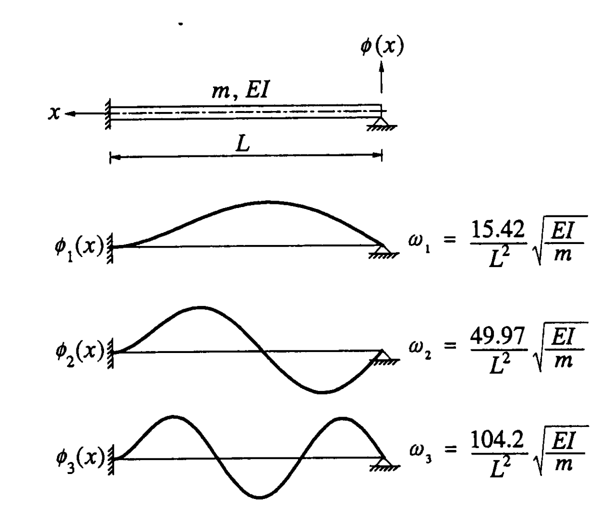
for n = 1, 2, and 3. Eq. (16.3.8) then gives the natural frequencies:

$$\omega_1 = \frac{15.42}{L^2}\sqrt{\frac{EI}{m}};\quad \omega_2 = \frac{49.97}{L^2}\sqrt{\frac{EI}{m}};\quad \omega_3 = \frac{104.2}{L^2}\sqrt{\frac{EI}{m}}$$

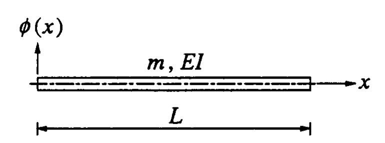
Corresponding to each value of $\beta_n L$, the natural vibration mode is obtained by expressing C_3 in terms of C_1 from Eq. (g) and substituting in Eq. (d):

$$\phi_n(x) = C_1\left(\sin\beta_n x - \frac{\sin\beta_n L}{\sinh\beta_n L}\sinh\beta_n x\right)$$

where C_1 is an arbitrary constant. The first three natural vibration modes are shown in the accompanying figure.



Problem 16.3



$$\phi(x) = C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x \quad \text{(a)}$$

$$\mathcal{M}(0) = 0 \Rightarrow EI\,\phi''(0) = 0 \Rightarrow \beta^2(-C_2 + C_4) = 0 \Rightarrow C_4 = C_2 \quad \text{(b)}$$

$$\mathcal{V}(0) = 0 \Rightarrow EI\,\phi'''(0) = 0 \Rightarrow \beta^3(-C_1 + C_3) = 0 \Rightarrow C_3 = C_1 \quad \text{(c)}$$

$$\mathcal{M}(L) = 0 \Rightarrow C_1(\sinh\beta L - \sin\beta L) + C_2(\cosh\beta L - \cos\beta L) = 0 \quad \text{(d)}$$

$$\mathcal{V}(L) = 0 \Rightarrow C_1(\cosh\beta L - \cos\beta L) + C_2(\sinh\beta L + \sin\beta L) = 0 \quad \text{(e)}$$

For a nontrivial solution of C_1 or C_2, the determinant of the coefficients must be zero. Thus

$$1 - \cos\beta L \cosh\beta L = 0 \quad \text{(f)}$$

Equation (f) is identical to the frequency equation for the clamped beam, but now it also possesses a zero root. Substituting $\beta = 0$ in Eq. (16.3.7) gives

$$\phi^{\text{iv}}(x) = 0 \quad \text{(g)}$$

The general solution of Eq. (g) is

$$\phi(x) = D_1 + D_2 x + D_3 x^2 + D_4 x^3 \quad \text{(h)}$$

Applying the boundary conditions at $x = 0$ and $x = L$, we get $D_3 = D_4 = 0$, and

$$\phi(x) = D_1 + D_2 x \quad \text{(i)}$$

Equation (i) in fact represents two eigenfunctions corresponding to two zero eigenvalues. We can select the two eigenfunctions as, say, D_1 and $D_2 x$, but any linear combination of these is also an eigenfunction. Let D_1 be the first eigenfunction and $D_1 + D_2 x$ be the second. In order to satisfy the condition of modal othogonality, $D_2 = -2D_1/L$. Therefore, the two functions corresponding to $\beta_1 = 0$ and $\beta_2 = 0$ are

$$\phi_1(x) = D_1 \quad \text{(j)}$$

$$\phi_2(x) = D_1\left(1 - \frac{x}{2L}\right) \quad \text{(k)}$$

where D_1 is an arbitrary constant.

The non-zero roots of Eq. (f) can be obtained numerically:

$$\beta_n L = 4.730,\ 7.853, \text{ and } 10.996$$

for n = 3, 4, and 5. Thus

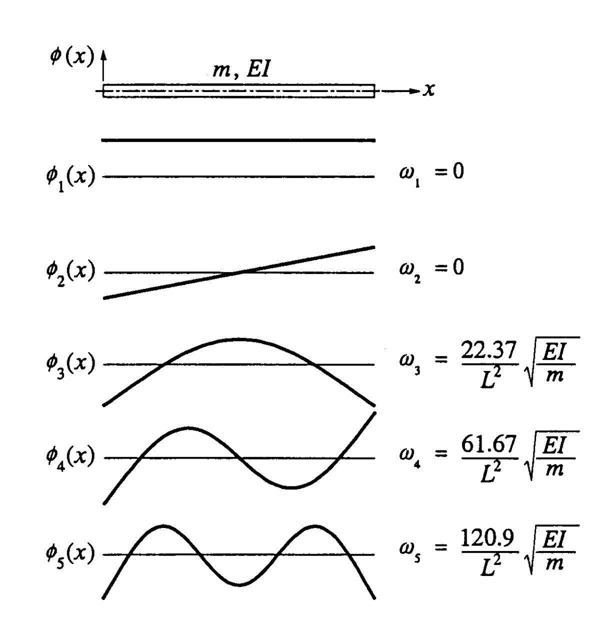
$$\omega_3 = \frac{22.37}{L^2}\sqrt{\frac{EI}{m}};\quad \omega_4 = \frac{61.67}{L^2}\sqrt{\frac{EI}{m}};\quad \omega_5 = \frac{120.9}{L^2}\sqrt{\frac{EI}{m}}$$

To determine the natural vibration mode corresponding to each non-zero value of $\beta_n L$, we express C_2 in terms of C_1 from Eq. (e) and substituted in Eq. (a) together with Eqs. (b) and (c) to obtain:

$$\phi_n(x) = C_1\left[\sin\beta_n x + \sinh\beta_n x + \frac{\cos\beta_n L - \cosh\beta_n L}{\sin\beta_n L + \sinh\beta_n L}(\cos\beta_n x + \cosh\beta_n x)\right]$$

where C_1 is an arbitrary constant.

The natural vibration modes are shown in the accompanying figure. The first two modes, with zero natural frequencies, are rigid body translation and rigid body rotation, respectively. Modes 3 to 5 have the same vibration frequencies as the first three modes of the beam clamped at both ends, although the mode shapes are not identical.



Problem 16.4

1. *Natural vibration frequencies and modes.*

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}} \qquad \phi_n(x) = \sin\frac{n\pi x}{L} \qquad \text{(a)}$$

2. *Set up modal equations.*

$$\ddot{q}_n(t) + \omega_n^2\, q_n(t) = 0 \qquad \text{(b)}$$

where $q_n(0)$ and $\dot{q}_n(0)$ are determined from $u(x,0)$ and $\dot{u}(x,0)$, respectively.

3. *Initial conditions*

The initial displacements due to the weight W are

$$u(x,0) = \frac{W}{48EI}\left(4x^3 - 3L^2x\right); \quad 0 \le x \le \frac{L}{2} \qquad \text{(c)}$$

The initial velocities are

$$\dot{u}(x,0) = 0$$

Determine $q_n(0)$ from $u(x,0)$:

$$u(x,0) = \sum_{r=1}^{N} \phi_r(x)\, q_r(0) \qquad \text{(d)}$$

Multiplying both sides of Eq. (d) by $m(x)\varphi_n(x)$ and integrating over the length of the beam:

$$\int_0^L m(x)\,\varphi_n(x)\,u(x,0)\,dx = \sum_{r=1}^{N} q_r(0)\int_0^L m(x)\,\varphi_n(x)\,\varphi_r(x)\,dx$$

Because of the modal othogonality relation of Eq. (16.4.6a), all terms in the summation on the right side vanish except the one for which $r = n$. Thus

$$\int_0^L m(x)\,\varphi_n(x)\,u(x,0)\,dx = q_n(0)\int_0^L m(x)\,[\varphi_n(x)]^2\,dx$$

or

$$q_n(0) = \frac{\int_0^L m(x)\,\varphi_n(x)\,u(x,0)\,dx}{M_n} \qquad \text{(e)}$$

Substituting $\varphi_n(x)$ in Eqs. (16.5.3a) and (e) gives

$$M_n = \int_0^L m\sin\left(\frac{n\pi x}{L}\right)^2 dx = \frac{mL}{2} \qquad \text{(f)}$$

$$q_n(0) = \frac{1}{M_n}\int_0^L m\sin\left(\frac{n\pi x}{L}\right)u(x,0)\,dx \qquad \text{(g)}$$

The integral in Eq. (g) vanishes for n even because $\varphi_n(x)$ is antisymmetric about $x = L/2$ but $u(x,0)$ is symmetric. For n odd, Eq. (g) is

$$q_n(0) = \frac{2m}{M_n}\frac{W}{48EI}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\left(4x^3 - 3L^2x\right)dx$$

$$= \begin{cases} -\dfrac{2WL^3}{\pi^4 EI}\dfrac{1}{n^4} & n = 1, 5, 9, \cdots \\[2ex] \dfrac{2WL^3}{\pi^4 EI}\dfrac{1}{n^4} & n = 3, 7, 11, \cdots \end{cases} \qquad \text{(h)}$$

Because the initial velocity is zero,

$$\dot{q}_n(0) = 0 \qquad \text{(i)}$$

4. *Modal responses.*

$$q_n(t) = q_n(0)\cos\omega_n t + \frac{\dot{q}_n(0)}{\omega_n}\sin\omega_n t$$

where $q_n(0)$ and $\dot{q}_n(0)$ are given by Eqs. (h) and (i), respectively. Therefore,

$$q_n(t) = q_n(0)\cos\omega_n t \qquad \text{(j)}$$

5. *Total response.*

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)\, q_n(t)$$

Substituting Eqs. (a) and (j) gives

$$u(x,t) = \frac{2WL^3}{\pi^4 EI}\left(-\sin\frac{\pi x}{L}\cos\omega_1 t + \frac{1}{81}\sin\frac{3\pi x}{L}\cos\omega_3 t - \frac{1}{625}\sin\frac{5\pi x}{L}\cos\omega_5 t + \frac{1}{2401}\sin\frac{7\pi x}{L}\cos\omega_7 t - \cdots\right) \qquad \text{(k)}$$

6. *Specialize for mid-span deflection.*

$$u\left(\frac{L}{2}, t\right) = -\frac{2WL^3}{\pi^4 EI}\left(\cos\omega_1 t + \frac{1}{81}\cos\omega_3 t + \frac{1}{625}\cos\omega_5 t + \frac{1}{2401}\cos\omega_7 t + \cdots\right) \qquad \text{(l)}$$

Problem 16.5

1. *Natural vibration frequencies and modes.*

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}} \qquad \phi_n(x) = \sin\frac{n\pi x}{L} \qquad \text{(a)}$$

2. *Set up modal equations.*

$$M_n = \frac{mL}{2} \qquad K_n = \frac{n^4\pi^4 EI}{2L^3}$$

$$P_n(t) = \int_0^L p(t)\,\phi_n(x)\,dx = \begin{cases} 0 & n = 2,4,6,\cdots \\ \dfrac{2pL}{n\pi} & n = 1,3,5,\cdots \end{cases}$$

(b)

The nth modal equation (where n is an odd number) is

$$M_n\ddot{q}_n + K_n q_n = \frac{2pL}{n\pi} \qquad \text{(c)}$$

3. *Determine dynamic response.*

The solution to Eq. (c) is:

$$q_n(t) = \frac{2pL}{n\pi K_n}(1 - \cos\omega_n t) = \frac{4pL^4}{\pi^5 EI}\frac{1}{n^5}(1 - \cos\omega_n t)$$

(d)

$$u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \phi_n(x)\,q_n(t) \qquad \text{(e)}$$

where $\phi_n(x)$ and $q_n(t)$ are given in Eqs. (a) and (d). Note that the modes with antisymmetric mode shape $(n = 2,4,6,\cdots)$ do not contribute to the response.

4. *Specialize for* $x = L/2$.

$$\phi_n\left(\frac{L}{2}\right) = \begin{cases} 1 & n = 1,5,9,\cdots \\ -1 & n = 3,7,11,\cdots \end{cases}$$

Substituting these in Eq. (e) and using Eqs. (a) and (d) gives

$$u\left(\frac{L}{2},t\right) = \frac{4pL^4}{\pi^5 EI}\left(\frac{1-\cos\omega_1 t}{1} - \frac{1-\cos\omega_3 t}{243} + \frac{1-\cos\omega_5 t}{3125} - \frac{1-\cos\omega_7 t}{16807} + \cdots\right)$$

Problem 16.6

1. *Natural vibration frequencies and modes.*

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}} \qquad \phi_n(x) = \sin\frac{n\pi x}{L} \qquad \text{(a)}$$

2. *Set up modal equations.*

$$M_n = \frac{mL}{2} \qquad K_n = \frac{n^4\pi^4 EI}{2L^3}$$

$$P_n(t) = \int_0^L p(x,t)\,\phi_n(x)\,dx$$

$$= \int_0^{L/2} p\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}(-p)\sin\frac{n\pi x}{L}\,dx$$

$$= \begin{cases} 0 & n = 1,3,4,5,7,8,\cdots \\ \dfrac{4pL}{n\pi} & n = 2,6,10,\cdots \end{cases} \qquad \text{(b)}$$

For $n = 2,6,10,\cdots$, the modal equation is

$$M_n\ddot{q}_n + K_n q_n = \frac{4pL}{n\pi} \qquad \text{(c)}$$

3. *Determine dynamic response.*

The solution to Eq. (c) is

$$q_n(t) = \frac{4pL}{n\pi K_n}(1 - \cos\omega_n t) = \frac{8pL^4}{\pi^5 EI}\frac{1}{n^5}(1 - \cos\omega_n t) \qquad \text{(d)}$$

$$u(x,t) = \sum_{n=2,6,10,\cdots}^{\infty}\phi_n(x)\,q_n(t) \qquad \text{(e)}$$

where $\phi_n(x)$ and $q_n(t)$ are given in Eqs. (a) and (d), respectively.

Note that only modes $n = 2,6,10,\cdots$ contribute to the response. Modes $n = 1,3,5,\cdots$, which are symmetric about $x = L/2$, do not respond because the applied force is antisymmetric about $x = L/2$. Antisymmetric modes $n = 4,8,12,\cdots$ do not respond because each half of the beam with constant force contains one or more complete sine waves, and thus provides zero contribution to $P_n(t)$.

4. *Specialize for* $x = L/4$.

$$\phi_n\left(\frac{L}{4}\right) = \begin{cases} 1 & n = 2,10,18,\cdots \\ -1 & n = 6,14,22,\cdots \end{cases}$$

Substituting these in Eq. (e) and using Eqs. (a) and (d) gives

$$u\left(\frac{L}{4},t\right) = \frac{8pL^4}{\pi^5 EI}\left(\frac{1-\cos\omega_2 t}{32} - \frac{1-\cos\omega_6 t}{7776} + \frac{1-\cos\omega_{10} t}{100{,}000} - \ldots\right)$$

Problem 16.7

1. *Determine the natural vibration frequencies and modes.*

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}} \qquad \phi_n(x) = \sin\frac{n\pi x}{L} \tag{a}$$

2. *Setup the modal equations.*

Substituting $\phi_n(x)$ in Eq. (16.5.3a) gives M_n, which is substituted in Eq. (16.5.5) together with ω_n^2 to get K_n:

$$M_n = \frac{mL}{2} \qquad K_n = \frac{n^4\pi^4 EI}{2L^3} \tag{b}$$

From Problem 8.25, the applied force is

$$p(x,t) = \begin{cases} p_o & 0 \le x \le vt \quad 0 \le t \le t_d \\ 0 & vt < x \le L \quad 0 \le t \le t_d \\ p_o & 0 \le x \le L \quad t \ge t_d \end{cases} \tag{c}$$

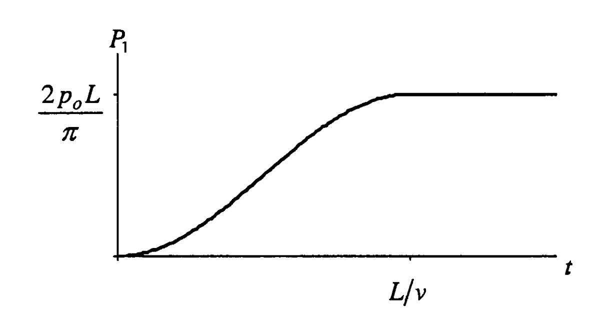
Substituting for $p(x,t)$ in Eq. (16.5.3c) gives

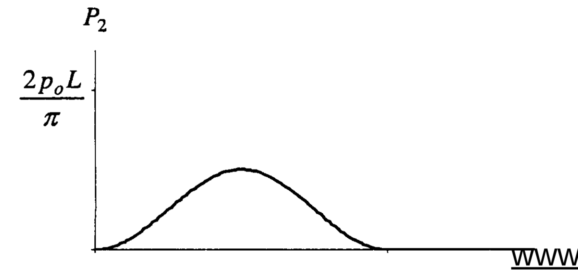
$$P_n(t) = \int_0^L p(x,t)\,\phi_n(x)\,dx$$

$$= \begin{cases} \int_0^{vt} p_o \sin(n\pi x/L)dx & 0 \le t \le t_d \\ \int_0^{L} p_o \sin(n\pi x/L)dx & t \ge t_d \end{cases} \tag{d}$$

$$= \begin{cases} \dfrac{p_o L}{n\pi}\left[1-\cos\dfrac{n\pi v}{L}t\right] & 0 \le t \le t_d \\ \dfrac{p_o L}{n\pi}\left[1-(-1)^n\right] & t \ge t_d \end{cases}$$

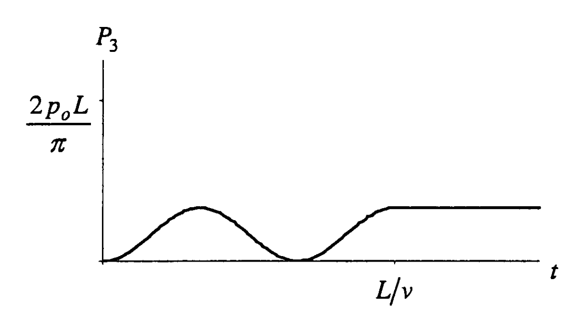
Equation (d) is plotted next for n = 1, 2, and 3:





t

L/v



The nth modal equation of motion is

$$M_n\ddot{q}_n(t) + K_n q_n(t) = P_n(t) \tag{e}$$

with M_n, K_n, and $P_n(t)$ as given above.

3. *Solve the modal equations.*

Response for $0 \le t \le t_d$. The particular solution to Eq. (e) can be obtained by superposing the steady-state responses to the constant and the cosine term on the right-hand side of Eq. (d). The steady-state response to the constant term is $(P_n)_o/K_n$, where $(P_n)_o = p_o L/n\pi$, and that for the cosine term is adapted from Eqs. (3.2.3) and (3.2.26), noting that $\varsigma = 0$ and replacing p_o and k by $(P_n)_o$ and K_n respectively. The complete solution is obtained by adding the complementary solution, given by Eq. (3.1.4), to the particular solution, and determining the constants A and B by imposing zero initial conditions.

The result is

$$q_n(t) = \frac{2p_o}{n\pi m}\frac{1}{\omega_n^2}\left[1 - \frac{\omega_n^2}{\omega_n^2 - (n\pi v/L)^2}\cos\frac{n\pi v}{L}t + \frac{(n\pi v/L)^2}{\omega_n^2 - (n\pi v/L)^2}\cos\omega_n t\right] \qquad t \le L/v \tag{f}$$

Equation (g) is valid if $\omega_n \ne n\pi v/L$; otherwise the particular solution to the cosine term should formulated similar to that in Eq. (3.1.12), noting that the forcing function is a cosine function instead of a sine function.

Response for $t \ge t_d$. The motion is described by Eq. (4.5.3) with q_n instead of u, t_d instead of t_r, and $q_n(t_d)$ and $\dot{q}_n(t_d)$ determined from Eq. (f) :

$$q_n(t_d) = \frac{2p_o}{n\pi m}\frac{1}{\omega_n^2}\left[1 - \frac{\omega_n^2}{\omega_n^2 - (n\pi v/L)^2}(-1)^n + \frac{(n\pi v/L)^2}{\omega_n^2 - (n\pi v/L)^2}\cos\omega_n t_d\right] \quad \text{(g)}$$

$$\dot{q}_n(t_d) = -\frac{2p_o}{n\pi m}\frac{1}{\omega_n}\frac{(n\pi v/L)^2}{\omega_n^2 - (n\pi v/L)^2}\sin\omega_n t_d \quad \text{(h)}$$

Noting that $P_n(t) = 2p_o L/n\pi$ for $n = 1,3,5,\ldots$ and $P_n(t) = 0$ for $n = 2,4,6,\ldots$, substituting these in Eq. (4.5.3), using trigonometric identities and manipulating the mathematical quantities, we obtain

$$q_n(t) = \frac{2p_o}{n\pi m}\frac{1}{\omega_n^2}\left[2 - \left\{1 + \frac{\omega_n^2}{\omega_n^2 - (n\pi v/L)^2}(-1)^n\right\}\cos\omega_n(t - t_d) + \frac{(n\pi v/L)^2}{\omega_n^2 - (n\pi v/L)^2}\cos\omega_n t\right]$$

$$n = 1,3,5,\ldots \qquad t \ge L/v \quad \text{(i)}$$

and

$$q_n(t) = \frac{2p_o}{n\pi m}\frac{1}{\omega_n^2}\left[\left\{1 - \frac{\omega_n^2}{\omega_n^2 - (n\pi v/L)^2}(-1)^n\right\}\cos\omega_n(t - t_d) + \frac{(n\pi v/L)^2}{\omega_n^2 - (n\pi v/L)^2}\cos\omega_n t\right]$$

$$n = 2,4,6,\ldots \qquad t \ge L/v \quad \text{(j)}$$

Thus, the modal response $q_n(t)$ is given by Eq. (f) while the moving load is on the bridge span and by Eqs. (i) and (j) after the load has crossed the span.

4. *Determine the displacement response.*

From Eq. (16.5.7) the displacement response is

$$u(x,t) = \sum_{n=1}^{\infty}\phi_n(x)q_n(t) = \sum_{n=1}^{\infty}q_n(t)\sin\frac{n\pi x}{L} \quad \text{(k)}$$

where $q_n(t)$ is given by Eqs. (f), (i), and (j).

5. *Specialize for* $x = L/2$.

At midspan, $x = L/2$ and

$$\sin\frac{n\pi(L/2)}{L} = \sin\frac{n\pi}{2} = \begin{cases} 0 & n = 2,4,6,\ldots \\ 1 & n = 1,5,9,\ldots \\ -1 & n = 3,7,11,\ldots \end{cases} \quad \text{(l)}$$

Substituting Eq. (l) into Eq. (k) gives the deflection at midspan:

$$u\left(\frac{L}{2},t\right) = q_1(t) - q_3(t) + q_5(t) - q_7(t) + q_9(t) - \cdots \quad \text{(m)}$$

where $q_n(t)$ is given by Eq. (f) for $t \le L/v$ and by Eq. (i) for $t \ge L/v$.

Problem 16.8

1. *Determine the natural vibration frequencies and modes.*

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}} \qquad \phi_n(x) = \sin\frac{n\pi x}{L} \qquad \text{(a)}$$

2. *Setup the modal equations.*

Substituting $\phi_n(x)$ in Eq. (16.5.3a) gives M_n, which is substituted in Eq. (16.5.5) together with ω_n^2 to get K_n:

$$M_n = \frac{mL}{2} \qquad K_n = \frac{n^4\pi^4 EI}{2L^3} \qquad \text{(b)}$$

From Problem 8.26, the applied force is

$$p(x,t) = \begin{cases} p_o \cos\omega t\, \delta(x - vt) & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(c)}$$

where $\delta(x - vt)$ is the Dirac delta function centered at $x = vt$. Substituting for $p(x,t)$ in Eq. (16.5.3c) gives

$$P_n(t) = \int_0^L p(x,t)\phi_n(x)\,dx$$

$$= \begin{cases} \int_0^L p_o \cos\omega t\, \delta(x - vt)\sin(n\pi x/L)dx & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases}$$

$$= \begin{cases} p_o \cos\omega t \sin(n\pi x/L) & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases}$$

$$= \begin{cases} p_o \cos\omega t \sin(n\pi t/t_d) & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases}$$

This can be re-written as

$$P_n(t) = \begin{cases} \frac{p_o}{2}[\sin(\omega + n\pi/t_d) - \sin(\omega - n\pi/t_d)] & 0 \le t \le t_d \\ 0 & t \ge t_d \end{cases} \qquad \text{(d)}$$

The nth modal equation of motion is

$$M_n\ddot{q}_n(t) + K_n q_n(t) = P_n(t) \qquad \text{(e)}$$

with M_n, K_n, and $P_n(t)$ as given above.

3. *Solve the modal equations.*

Forced vibration phase. The response $q_n(t)$ can be obtained by superposing the responses due to the two sine terms in the right-hand side of Eq. (d). The individual responses are adapted from Eq. (3.1.6b) by changing the notation from $u(t)$ to $q_n(t)$, substituting for ω with $\omega + n\pi/t_d$ in one case and with $\omega - n\pi/t_d$ in the other, and noting that

$$t_d = \frac{L}{v} \qquad (u_{st})_o = \frac{p_o}{2\tilde{k}} = \frac{p_o}{mL\omega_n^2}$$

The result is

$$q_n(t) = \frac{p_o}{mL}\left[\frac{1}{\omega_n^2 - (\omega + n\pi v/L)^2}\left\{\sin\left(\omega + \frac{n\pi v}{L}\right)t - \left(\frac{\omega + n\pi v/L}{\omega_n}\right)\sin\omega_n t\right\} - \frac{1}{\omega_n^2 - (\omega - n\pi v/L)^2}\left\{\sin\left(\omega - \frac{n\pi v}{L}\right)t - \left(\frac{\omega - n\pi v/L}{\omega_n}\right)\sin\omega_n t\right\}\right]$$

$$t \le L/v \qquad \text{(f)}$$

Based on Eq. (3.1.6b), Eq. (g) is valid if $\omega_n \ne \omega + n\pi v/L$ and $\omega_n \ne \omega - n\pi v/L$; otherwise Eq. (3.1.13a) should be used instead of Eq. (3.1.6b).

Free vibration phase. The motion is described by Eq. (4.7.3) with $q_n(t)$ instead of $u(t)$, and $q_n(t_d)$ and $\dot{q}_n(t_d)$ determined from Eq. (f) :

$$q_n(t_d) = \frac{p_o}{mL}\left[(-1)^n\left\{\frac{1}{\omega_n^2 - (\omega + n\pi v/L)^2} - \frac{1}{\omega_n^2 - (\omega - n\pi v/L)^2}\right\} \times \sin\omega L/v - \frac{1}{\omega_n}\left\{\frac{(\omega + n\pi v/L)}{\omega_n^2 - (\omega + n\pi v/L)^2} - \frac{(\omega - n\pi v/L)}{\omega_n^2 - (\omega - n\pi v/L)^2}\right\}\sin\omega_n L/v\right] \qquad \text{(g)}$$

$$\dot{q}_n(t_d) = \left[\left\{\frac{(\omega + n\pi v/L)}{\omega_n^2 - (\omega + n\pi v/L)^2} - \frac{(\omega - n\pi v/L)}{\omega_n^2 - (\omega - n\pi v/L)^2}\right\} \times \left\{(-1)^n\cos\frac{\omega L}{v} - \cos\frac{\omega_n L}{v}\right\}\right] \qquad \text{(h)}$$

Substituting these in Eq. (4.7.3), using trigonometric identities, and manipulating the mathematical quantities, we obtain

$$q_n(t) = \frac{p_o}{mL}\left[(-1)^n \left\{\frac{1}{\omega_n^2 - (\omega + n\pi v/L)^2} - \frac{1}{\omega_n^2 - (\omega - n\pi v/L)^2}\right\} \times \sin\frac{\omega L}{v}\cos\omega_n\left(t - \frac{L}{v}\right)\right.$$

$$\left. + \frac{1}{\omega_n}\left\{\frac{(\omega + n\pi v/L)}{\omega_n^2 - (\omega + n\pi v/L)^2} - \frac{(\omega - n\pi v/L)}{\omega_n^2 - (\omega - n\pi v/L)^2}\right\} \times \left\{(-1)^n \cos\frac{\omega L}{v}\sin\omega_n\left(t - \frac{L}{v}\right) - \sin\omega_n t\right\}\right]$$

$$t \ge L/v \qquad \text{(i)}$$

Thus, the modal response $q_n(t)$ is given by Eq. (f) while the moving load is on the bridge span and by Eq. (i) after the load has crossed the span.

4. *Determine the displacement response.*

From Eq. (16.5.7), the displacement response is

$$u(x,t) = \sum_{n=1}^{\infty}\phi_n(x)q_n(t) = \sum_{n=1}^{\infty} q_n(t)\sin\frac{n\pi x}{L} \qquad \text{(j)}$$

where $q_n(t)$ is given by Eqs. (f) and (i).

5. *Specialize for* $x = L/2$

At midspan, $x = L/2$ and

$$\sin\frac{n\pi(L/2)}{L} = \sin\frac{n\pi}{2} = \begin{cases} 0 & n = 2,4,6,\ldots \\ 1 & n = 1,5,9,\ldots \\ -1 & n = 3,7,11,\ldots \end{cases} \qquad \text{(k)}$$

Substituting Eq. (k) into Eq. (j) gives the deflection at midspan:

$$u\left(\frac{L}{2}, t\right) = q_1(t) - q_3(t) + q_5(t) - q_7(t) + q_9(t) - \cdots \qquad \text{(l)}$$

where $q_n(t)$ is given by Eq. (f) for $t \le L/v$ and by Eq. (i) for $t \ge L/v$.

Problem 16.9

Equation (16.6.3) ⇒

$$m(x) = \sum_{n=1}^{\infty} \Gamma_n \, m(x) \, \phi_n(x) \qquad \text{(a)}$$

Integrating over the length of the cantilever beam gives

$$\int_0^L m(x)\, dx = \sum_{n=1}^{\infty} \Gamma_n \int_0^L m(x)\, \phi_n(x)\, dx = \sum_{n=1}^{\infty} \Gamma_n \, L_n^h \qquad \text{(b)}$$

By definition, $\Gamma_n L_n^h = M_n^*$ and Eq. (b) becomes

$$\sum_{n=1}^{\infty} M_n^* = \int_0^L m(x)\, dx \qquad \text{(c)}$$

which is the same as Eq. (16.6.19).

Problem 16.10

Equation (16.6.3) ⇒

$$m(x) = \sum_{n=1}^{\infty} \Gamma_n \, m(x) \, \phi_n(x) \qquad \text{(a)}$$

Multiplying both side by x and integrating over the length of the cantilever gives

$$\int_0^L x \, m(x) \, dx = \sum_{n=1}^{\infty} \Gamma_n \int_0^L x \, m(x) \, \phi_n(x) \, dx = \sum_{n=1}^{\infty} \Gamma_n \, L_n^{\theta} \qquad \text{(b)}$$

By definition, $M_n^* = \Gamma_n L_n^h$ and $h_n^* = L_n^{\theta} / L_n^h$ and Eq. (b) becomes

$$\sum_{n=1}^{\infty} h_n^* \, M_n^* = \int_0^L x \, m(x) \, dx \qquad \text{(c)}$$

which is the same as Eq. (16.6.20).

Problem 16.11

1. *Tower properties.*

$L = 200\,\text{ft}$

$$m = \pi\left[(12.5)^2 - (11.25)^2\right]\frac{0.15}{32.2}$$

$$= 0.4345\ \text{kip-sec}^2/\text{ft}$$

$$EI = (3600 \times 144)(\pi/4)\left[(12.5)^4 - (11.25)^4\right]$$

$$= 3.4184 \times 10^9\ \text{kip-ft}^2$$

2. *Natural vibration periods and modes.*

Equation (16.3.22) gives the natural vibration frequencies and the corresponding periods in seconds are

$$T_1 = 0.806 \qquad T_2 = 0.129$$

$$T_3 = 0.046, T_4 = 0.023, \text{etc.}$$

3. *Determine the modal static responses.*

The modal properties given in Table E16.2 are valid for a uniform cantilever. Substituting these in the equations of Table 16.6.1 gives Table P16.9a for the modal static responses.

Table P16.11a

Mode	$u_n^{st}(L)$	$\mathcal{V}_{bn}^{st}$	$\mathcal{M}_{bn}^{st}$
1	0.0258	53.274	7,740
2	-3.66×10^{-4}	16.365	684.77
3	2.73×10^{-5}	5.625	143.27
4	-4.87×10^{-6}	2.873	51.83

4. *Determine spectrum ordinates.*

From Fig. 6.9.5, scaled to $\ddot{u}_{go} = 1/3\,g$, the design spectrum ordinates are

$$\frac{A_1}{g} = \frac{1.80\,T_1^{-1}}{3} = 0.744$$

$$\frac{A_2}{g} = \frac{2.71}{3} = 0.903$$

$$\frac{A_3}{g} = \frac{11.70\,T_3^{0.704}}{3} = 0.446$$

$$\frac{A_4}{g} = \frac{1}{3} = 0.333$$

5. *Determine peak responses.*

For each response quantity r, substituting r_n^{st} and A_n in

$$r_{no} = r_n^{st} A_n$$

gives the peak modal responses for the first four modes.

Table P16.11b

Mode	$u_n(L)$ (in.)	$\mathcal{V}_{bn}$ (kips)	$\mathcal{M}_{bn}$ (kip-ft)
1	7.409	1,276.2	185,425
2	– 0.128	475.8	19,911
3	0.0047	80.8	2,057.5
4	0.0006	30.8	555.8

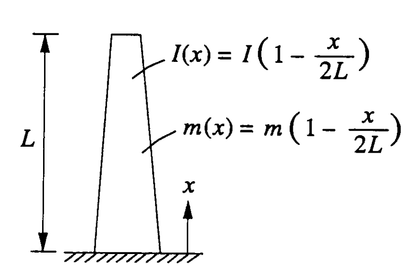
These peak modal responses are combined according to the SRSS rule to obtain:

Top displacement $\quad u_o(L) = 7.410$ in.

Base shear $\quad \mathcal{V}_{bo} = 1{,}365$ kips

Base overturning moment $\quad \mathcal{M}_{bo} = 186{,}503$ kip-ft

Problem 17.1



The shape functions selected are

$$\psi_1(x) = 1 - \cos\frac{\pi x}{2L}$$

$$\psi_2(x) = 1 - \cos\frac{3\pi x}{2L}$$

1. *Set up* $\tilde{\mathbf{k}}$ *and* $\tilde{\mathbf{m}}$.

For the selected $\psi_j(x)$, the stiffness coefficients are computed from Eq. (17.1.4a):

$$\tilde{k}_{11} = \int_0^L EI\left(1 - \frac{x}{2L}\right)\left[\left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi x}{2L}\right]^2 dx = 2.59145\frac{EI}{L^3}$$

$$\tilde{k}_{22} = \int_0^L EI\left(1 - \frac{x}{2L}\right)\left[\left(\frac{3\pi}{2L}\right)^2 \cos\frac{3\pi x}{2L}\right]^2 dx = 187.7\frac{EI}{L^3}$$

$$\tilde{k}_{12} = \int_0^L EI\left(1 - \frac{x}{2L}\right)\left[\left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi x}{2L}\right]\left[\left(\frac{3\pi}{2L}\right)^2 \cos\frac{3\pi x}{2L}\right] dx$$
$$= 2.7758\frac{EI}{L^3}$$

Similarly the mass coefficients are determined from Eq. (17.1.4b):

$$\tilde{m}_{11} = \int_0^L m\left(1 - \frac{x}{2L}\right)\left(1 - \cos\frac{\pi x}{2L}\right)^2 dx = 0.13376\, mL$$

$$\tilde{m}_{22} = \int_0^L m\left(1 - \frac{x}{2L}\right)\left(1 - \cos\frac{3\pi x}{2L}\right)^2 dx = 1.2978\, mL$$

$$\tilde{m}_{12} = \int_0^L m\left(1 - \frac{x}{2L}\right)\left(1 - \cos\frac{\pi x}{2L}\right)\left(1 - \cos\frac{3\pi x}{2L}\right) dx$$
$$= 0.363296\, mL$$

Thus

$$\tilde{\mathbf{k}} = \frac{EI}{L^3}\begin{bmatrix} 2.59145 & 2.7758 \\ 2.7758 & 187.7 \end{bmatrix}$$

$$\tilde{\mathbf{m}} = mL\begin{bmatrix} 0.13376 & 0.363296 \\ 0.363296 & 1.2978 \end{bmatrix}$$

2. *Solve eigenvalue problem.*

$$\tilde{\mathbf{k}}\,\mathbf{z} = \tilde{\omega}^2\,\tilde{\mathbf{m}}\,\mathbf{z}$$

$$\tilde{\omega}_1 = 4.3178\sqrt{\frac{EI}{mL^4}} \qquad \tilde{\omega}_2 = 24.8415\sqrt{\frac{EI}{mL^4}}$$

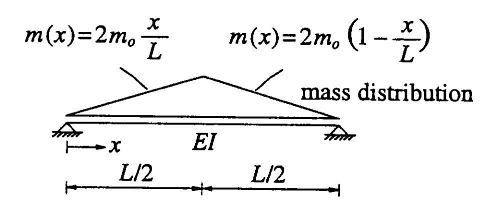
$$\mathbf{z}_1 = \begin{bmatrix} 0.9997 \\ 0.0244 \end{bmatrix} \qquad \mathbf{z}_2 = \begin{bmatrix} -0.9406 \\ 0.3396 \end{bmatrix}$$

3. *Determine natural modes from Eq.* (17.1.8).

$$\tilde{\phi}_1(x) = 0.9997\left(1 - \cos\frac{\pi x}{2L}\right) + 0.0244\left(1 - \cos\frac{3\pi x}{2L}\right)$$
$$= 1.0241 - 0.9997\cos\frac{\pi x}{2L} - 0.0244\cos\frac{3\pi x}{2L}$$

$$\tilde{\phi}_2(x) = -0.9406\left(1 - \cos\frac{\pi x}{2L}\right) + 0.3396\left(1 - \cos\frac{3\pi x}{2L}\right)$$
$$= -0.6010 + 0.9406\cos\frac{\pi x}{2L} - 0.3396\cos\frac{3\pi x}{2L}$$

Problem 17.2



1. *Select shape functions.*

$$\psi_1(x) = \sin\frac{\pi x}{L} \qquad \psi_2(x) = \sin\frac{3\pi x}{L}$$

$$\psi_1'(x) = \frac{\pi}{L}\cos\frac{\pi x}{L} \qquad \psi_2'(x) = \frac{3\pi}{L}\cos\frac{3\pi x}{l}$$

$$\psi_1''(x) = -\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi x}{L} \qquad \psi_2''(x) = -\left(\frac{3\pi}{L}\right)^2 \sin\frac{3\pi x}{L}$$

2. *Set up* $\tilde{\mathbf{k}}$.

$$\tilde{k}_{11} = \left(\frac{\pi}{L}\right)^4 \int_0^L EI\sin^2\frac{\pi x}{L}dx$$

$$= EI\left(\frac{\pi}{L}\right)^3\left[\frac{\pi x/L}{2} - \frac{\sin(\pi x/L)\cos(\pi x/L)}{2}\right]_0^L$$

$$= EI\left(\frac{\pi}{L}\right)^3\frac{\pi}{2} = \frac{\pi^4}{2}\frac{EI}{L^3} = 48.705\frac{EI}{L^3}$$

$$\tilde{k}_{12} = \left(\frac{\pi}{L}\right)^2\left(\frac{3\pi}{L}\right)^2\int_0^L EI\sin\frac{\pi x}{L}\sin\frac{3\pi x}{L}dx$$

$$= EI\left(\frac{\pi}{L}\right)^2\left(\frac{3\pi}{L}\right)^2\left[\frac{\sin(-2\pi x/L)}{2(-2\pi/L)} - \frac{\sin(4\pi x/L)}{2(4\pi/L)}\right]_0^L$$

$$= 0$$

$$\tilde{k}_{21} = \tilde{k}_{12} = 0$$

$$\tilde{k}_{22} = \left(\frac{3\pi}{L}\right)^4\int_0^L EI\sin^2\frac{3\pi x}{L}dx$$

$$= EI\left(\frac{3\pi}{L}\right)^3\left[\frac{3\pi x/L}{2} - \frac{\sin(6\pi x/L)}{4}\right]_0^L$$

$$= EI\left(\frac{3\pi}{L}\right)^3\frac{3\pi}{2} = \frac{(3\pi)^4}{2}\frac{EI}{L^3}$$

$$= 3945.07\frac{EI}{L^3}$$

Thus

$$\tilde{\mathbf{k}} = \frac{EI}{L^3}\begin{bmatrix} 48.705 & 0 \\ 0 & 3945.07 \end{bmatrix}$$

3. *Set up* $\tilde{\mathbf{m}}$.

$$\tilde{m}_{11} = \int_0^{L/2} 2m_o\frac{x}{L}\sin^2\left(\frac{\pi x}{L}\right)dx + \int_{L/2}^{L} 2m_o\left(1-\frac{x}{L}\right)\sin^2\left(\frac{\pi x}{L}\right)dx$$

$$= \frac{2m_o}{\pi}\frac{L}{\pi}\int_0^{L/2}\frac{\pi x}{L}\sin^2\left(\frac{\pi x}{L}\right)d\left(\frac{\pi x}{L}\right) + \frac{2m_o}{\pi/L}\int_{L/2}^{L}\sin^2\left(\frac{\pi x}{L}\right)d\left(\frac{\pi x}{L}\right) - 2m_o\frac{1}{\pi}\frac{L}{\pi}\int_{L/2}^{L}\frac{\pi x}{L}\sin^2\left(\frac{\pi x}{L}\right)d\left(\frac{\pi x}{L}\right)$$

$$= \frac{2m_oL}{\pi^2}\left[\frac{(\pi x/L)^2}{4} - \frac{(\pi x/L)\sin(2\pi x/L)}{4} - \frac{\cos(2\pi x/L)}{8}\right]_0^{L/2} + \frac{2m_o}{\pi/L}\left[\frac{\pi x/L}{2} - \frac{\sin(2\pi x/L)}{4}\right]_{L/2}^{L} - \frac{2m_oL}{\pi^2}\left[\frac{(\pi x/L)^2}{4} - \frac{(\pi x/L)\sin(2\pi x/L)}{4} - \frac{\cos(2\pi x/L)}{8}\right]_{L/2}^{L}$$

$$= \frac{2m_oL}{\pi^2}\left[\frac{(\pi/2)^2}{4} + \frac{1}{8} + \frac{1}{8}\right] + \frac{2m_oL}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] - \frac{2m_oL}{\pi^2}\left[\frac{\pi^2}{4} - \frac{1}{8} - \frac{(\pi/2)^2}{4} - \frac{1}{8}\right]$$

$$= \frac{2m_o L}{\pi^2}\left[\frac{(\pi/2)^2}{4} + \frac{1}{4} + \frac{\pi^2}{4} - \frac{\pi^2}{4} + \frac{1}{4} + \frac{(\pi/2)^2}{4}\right]$$

$$= \frac{2m_o L}{\pi^2}\left[\frac{\pi^2}{8} + \frac{1}{2}\right]$$

$$= 0.3513 m_o L$$

$$\tilde{m}_{12} = \int_0^{L/2} 2m_o \frac{x}{L}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right)dx + \int_{L/2}^{L} 2m_o\left(1 - \frac{x}{L}\right)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{3\pi x}{L}\right)dx$$

$$= \frac{2m_o}{L}\left\{x\left[\frac{\sin(2\pi x/L)}{2(2\pi/L)} - \frac{\sin(4\pi x/L)}{2(4\pi/L)}\right]_0^{L/2} - \int_0^{L/2}\frac{\sin(2\pi x/L)}{2(2\pi/L)}dx + \int_0^{L/2}\frac{\sin(4\pi x/L)}{2(4\pi/L)}dx\right\} + 2m_o\left[\frac{\sin(2\pi x/L)}{2(2\pi/L)} - \frac{\sin(4\pi x/L)}{2(4\pi/L)}\right]_{L/2}^{L} -$$

$$\frac{2m_o}{L}\left\{-\int_{L/2}^{L}\frac{\sin(2\pi x/L)}{2(2\pi/L)}dx + \int_{L/2}^{L}\frac{\sin(4\pi x/L)}{2(4\pi/L)}dx\right\}$$

$$= \frac{2m_o}{L}\left\{\left[\frac{\cos(2\pi x/L)}{2(2\pi/L)^2}\right]_0^{L/2} - \left[\frac{\cos(4\pi x/L)}{2(4\pi/L)^2}\right]_0^{L/2}\right\} + 2m_o\{0\} - \frac{2m_o}{L}\left\{\left[\frac{\cos(2\pi x/L)}{2(2\pi/L)^2}\right]_{L/2}^{L} - \left[\frac{\cos(4\pi x/L)}{2(4\pi/L)^2}\right]_{L/2}^{L}\right\}$$

$$= \frac{2m_o}{L}\left[\frac{-2}{2(2\pi/L)^2} - \frac{1}{2(2\pi/L)^2}(2)\right]$$

$$= -0.1013 m_o L$$

$$\tilde{m}_{21} = \tilde{m}_{12} = -0.1013 m_o L$$

$$\tilde{m}_{22} = \int_0^{L/2} 2m_o \frac{x}{L}\sin^2\left(\frac{3\pi x}{L}\right)dx + \int_{L/2}^{L} 2m_o\left(1 - \frac{x}{L}\right)\sin^2\left(\frac{3\pi x}{L}\right)dx$$

$$= \frac{2m_o}{3\pi}\frac{L}{3\pi}\int_0^{L/2}\frac{3\pi x}{L}\sin^2\left(\frac{3\pi x}{L}\right)d\left(\frac{3\pi x}{L}\right) + \frac{2m_o}{3\pi/L}\int_{L/2}^{L}\sin^2\left(\frac{3\pi x}{L}\right)d\left(\frac{3\pi x}{L}\right) - \frac{2m_o}{3\pi}\frac{L}{3\pi}\int_{L/2}^{L}\frac{3\pi x}{L}\sin^2\left(\frac{3\pi x}{L}\right)d\left(\frac{3\pi x}{L}\right)$$

$$= \frac{2m_o L}{(3\pi)^2}\left[\frac{(3\pi x/L)^2}{4} - \frac{(3\pi x/L)\sin(6\pi x/L)}{4} - \frac{\cos(6\pi x/L)}{8}\right]_0^{L/2} + \frac{2m_o}{3\pi/L}\left[\frac{3\pi x/L}{2} - \frac{\sin(6\pi x/L)}{4}\right]_{L/2}^{L} - \frac{2m_o L}{(3\pi)^2}\left[\frac{(3\pi x/L)^2}{4} - \frac{(3\pi x/L)\sin(6\pi x/L)}{4} - \frac{\cos(6\pi x/L)}{8}\right]_{L/2}^{L}$$

$$= \frac{2m_o L}{(3\pi)^2}\left[\frac{(3\pi/2)^2}{4} + \frac{1}{8} + \frac{1}{8}\right] + \frac{2m_o L}{3\pi}\left[\frac{3\pi}{2} - \frac{3\pi}{4}\right] - \frac{2m_o L}{(3\pi)^2}\left[\frac{(3\pi)^2}{4} - \frac{1}{8} - \frac{(3\pi/2)^2}{4} - \frac{1}{8}\right]$$

$$= \frac{2m_o L}{(3\pi)^2}\left[\frac{(3\pi/2)^2}{4} + \frac{1}{4} + \frac{(3\pi)^2}{4} - \frac{(3\pi)^2}{4} + \frac{1}{4} + \frac{(3\pi/2)^2}{4}\right]$$

$$= \frac{2m_o L}{(3\pi)^2}\left[\frac{9\pi^2}{8} + \frac{1}{2}\right]$$

$$= 0.2613 m_o L$$

Thus

$$\tilde{\mathbf{m}} = m_o L \begin{bmatrix} 0.3513 & -0.1013 \\ -0.1013 & 0.2613 \end{bmatrix}$$

4. *Solve reduced eigenvalue problem.*

$$\tilde{\mathbf{k}}\,\mathbf{z} = \tilde{\omega}^2\,\tilde{\mathbf{m}}\,\mathbf{z}$$

$$\tilde{\omega}_1 = \frac{11.765}{L^2}\sqrt{\frac{EI}{m_o}} \qquad \tilde{\omega}_2 = \frac{130.467}{L^2}\sqrt{\frac{EI}{m_o}}$$

$$\mathbf{z}_1 = \begin{Bmatrix} 1.0 \\ -0.0036 \end{Bmatrix} \qquad \mathbf{z}_2 = \begin{Bmatrix} 0.2790 \\ 0.9603 \end{Bmatrix}$$

5. *Determine natural vibration modes from Eq.* (17.1.8).

$$\tilde{\phi}_1(x) = 1.0\sin\left(\frac{\pi x}{L}\right) - 0.0036\sin\left(\frac{3\pi x}{L}\right)$$

$$\tilde{\phi}_2(x) = 0.2790\sin\left(\frac{\pi x}{L}\right) + 0.9603\sin\left(\frac{3\pi x}{L}\right)$$

Note that the first mode of this beam with nonuniform mass is only slightly different from that for a uniform beam; the second mode differs more between the two beams.

Problem 17.3

1. *Stiffness and mass matrices.*

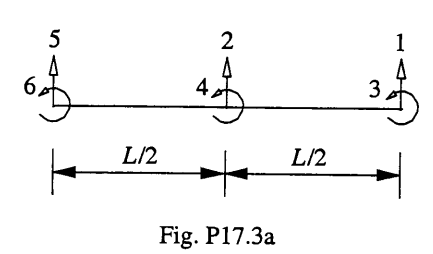


Fig. P17.3a

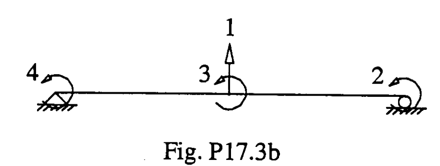


Fig. P17.3b

The 6×6 stiffness and mass matrices with reference to the DOFs in Fig. P17.3a are given in Example 17.2. Impose boundary conditions $u_1 = u_5 = 0$ by eliminating the corresponding rows and columns to obtain $\mathbf{k}$ and $\mathbf{m}$ with reference to the DOFs in Fig. P17.3b:

$$\mathbf{k} = \frac{8EI}{L^3}\left[\begin{array}{c|ccc} 24 & 3L & 0 & -3L \\ \hline 3L & L^2 & L^2/2 & 0 \\ 0 & L^2/2 & 2L^2 & L^2/2 \\ -3L & 0 & L^2/2 & L^2 \end{array}\right] = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{0t} & \mathbf{k}_{00} \end{bmatrix}$$

$$\mathbf{m} = \frac{mL}{840}\begin{bmatrix} 312 & -6.5L & 0 & 6.5L \\ -6.5L & L^2 & -0.75L^2 & 0 \\ 0 & -0.75L^2 & 2L^2 & -0.75L^2 \\ 6.5L & 0 & -0.75L^2 & L^2 \end{bmatrix}$$

2. *Solve eigenvalue problem.*

$$(\mathbf{k} - \omega^2 \mathbf{m})\phi = 0$$

$$\omega_1 = 9.9086\sqrt{\frac{EI}{mL^4}} \qquad \omega_2 = 43.818\sqrt{\frac{EI}{mL^4}}$$

$$\omega_3 = 110.14\sqrt{\frac{EI}{mL^4}} \qquad \omega_4 = 200.80\sqrt{\frac{EI}{mL^4}}$$

$$\Phi = \begin{bmatrix} 0.2196 & 0 & -0.0386 & 0 \\ -0.6898/L & -0.5774/L & -0.7066/L & -0.5774/L \\ 0/L & 0.5774/L & 0/L & -0.5774/L \\ 0.6898/L & -0.5774/L & 0.7066/L & -0.5774/L \end{bmatrix}$$

3. *Solution using lumped mass matrix*

$$\mathbf{m} = mL\begin{bmatrix} 0.5 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

Eliminate the three rotational DOFs by static condensation to obtain

$$\hat{k}_{lat} = \mathbf{k}_{tt} - \mathbf{k}_{t0}\,\mathbf{k}_{00}^{-1}\,\mathbf{k}_{0t} = \frac{48EI}{L^3}$$

Natural frequency:

$$\omega_1 = \sqrt{\frac{\hat{k}_{lat}}{0.5mL}} = 9.798\sqrt{\frac{EI}{mL^4}}$$

Natural mode after using static condensation equations:

$$\phi_1^T = \langle 0.2196 \quad -0.6588/L \quad 0/L \quad 0.6588/L \rangle$$

4. *Compare with exact frequencies.*

For a simply-supported beam, the exact values of the first four natural frequencies are

$$\omega_1 = 9.8696\sqrt{\frac{EI}{mL^4}} \qquad \omega_2 = 39.478\sqrt{\frac{EI}{mL^4}}$$

$$\omega_3 = 88.826\sqrt{\frac{EI}{mL^4}} \qquad \omega_4 = 157.914\sqrt{\frac{EI}{mL^4}}$$

The finite element method using consistent mass provides an excellent result for the fundamental frequency, but the accuracy deteriorates for higher modes. The lumped mass approximation provides only the fundamental frequency and the result is less accurate. Note that the finite element method overestimates the fundamental frequency whereas the lumped mass approximation provides an under estimate.

Problem 17.4

1. *Stiffness and mass matrices.*

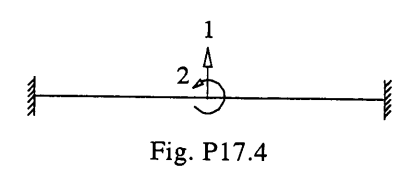


Fig. P17.4

The 6×6 stiffness and mass matrices are given in Example 17.2. Impose boundary conditions $u_1 = u_2 = u_5 = u_6 = 0$ by eliminating the corresponding rows and columns to obtain **k** and **m** with reference to the two DOFs in Fig. P17.4:

$$\mathbf{k} = \frac{8EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 2L^2 \end{bmatrix} \qquad \mathbf{m} = \frac{mL}{840}\begin{bmatrix} 312 & 0 \\ 0 & 2L^2 \end{bmatrix}$$

2. *Solve eigenvalue problem.*

$$(\mathbf{k} - \omega^2\mathbf{m})\phi = 0$$

$$\omega_1 = 22.74\sqrt{\frac{EI}{mL^4}} \qquad \omega_2 = 81.98\sqrt{\frac{EI}{mL^4}}$$

$$\phi_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} 0 \\ 1/L \end{Bmatrix}$$

3. *Solution using lumped mass matrix.*

$$\mathbf{m} = mL\begin{bmatrix} 0.5 & \\ & 0 \end{bmatrix}$$

Eliminate the DOF 2 by static condensation to obtain

$$\hat{k}_{lat} = \frac{8EI}{L^3}(24) = \frac{192EI}{L^3}$$

Natural frequency:

$$\omega_1 = \sqrt{\frac{\hat{k}_{lat}}{0.5mL}} = 19.60\sqrt{\frac{EI}{mL^4}}$$

Natural mode after calculating rotations using static condensation equations:

$$\phi_1^T = \langle 1 \quad 0 \rangle$$

4. *Compare with exact frequencies.*

For a beam clamped at both ends, the exact values of the first two natural frequencies are

$$\omega_1 = 22.37\sqrt{\frac{EI}{mL^4}} \qquad \omega_2 = 61.67\sqrt{\frac{EI}{mL^4}}$$

The finite element method using consistent mass provides an excellent result for the fundamental frequency, but the accuracy deteriorates for the second mode. The lumped mass approximation provides only the fundamental frequency, but the result is less accurate and lower than the exact value.

Problem 17.5

1. *Stiffness and mass matrices.*

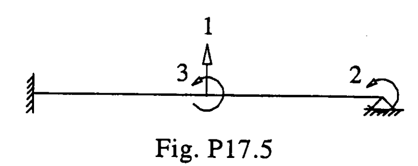


Fig. P17.5

The 6×6 stiffness and mass matrices are given in Example 17.2. Impose boundary conditions $u_1 = u_5 = u_6 = 0$ by eliminating the corresponding rows and columns to obtain **k** and **m** with reference to the three DOFs in Fig. 17.5:

$$\mathbf{k} = \frac{8EI}{L^3}\begin{bmatrix} 24 & 3L & 0 \\ 3L & L^2 & L^2/2 \\ 0 & L^2/2 & 2L^2 \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{0t} & \mathbf{k}_{00} \end{bmatrix}$$

$$\mathbf{m} = \frac{mL}{840}\begin{bmatrix} 312 & -6.5L & 0 \\ -6.5L & L^2 & -0.75L^2 \\ 0 & -0.75L^2 & 2L^2 \end{bmatrix}$$

2. *Solve eigenvalue problem.*

$$(\mathbf{k} - \omega^2\mathbf{m})\phi = 0$$

$$\omega_1 = 15.56\sqrt{\frac{EI}{mL^4}} \qquad \omega_2 = 58.41\sqrt{\frac{EI}{mL^4}}$$

$$\omega_3 = 155.64\sqrt{\frac{EI}{mL^4}}$$

$$\Phi = \begin{bmatrix} 0.2375 & -0.0349 & -0.0205 \\ -0.9370/L & -0.7449/L & -0.8516/L \\ 0.2561/L & 0.6662/L & -0.5237/L \end{bmatrix}$$

3. *Solution using lumped mass matrix.*

$$\mathbf{m} = mL\begin{bmatrix} 0.5 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

Eliminate the two rotational DOFs by static condensation to obtain

$$\hat{k}_{lat} = \mathbf{k}_{tt} - \mathbf{k}_{t0}\,\mathbf{k}_{00}^{-1}\,\mathbf{k}_{0t} = \frac{109.71EI}{L^3}$$

Natural frequency:

$$\omega_1 = \sqrt{\frac{\hat{k}_{lat}}{0.5mL}} = 14.81\sqrt{\frac{EI}{mL^4}}$$

Natural mode after calculating rotation using static condensation equations:

$$\phi_1^T = \langle 0.2375 \quad -0.8143/L \quad 0.2036/L \rangle$$

4. *Compare with exact frequencies.*

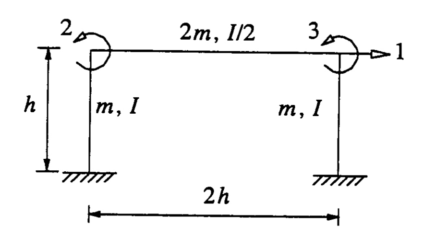
For a beam clamped at one end and simply supported at the other, the exact values of the first three natural frequencies are

$$\omega_1 = 15.42\sqrt{\frac{EI}{mL^4}} \qquad \omega_2 = 49.97\sqrt{\frac{EI}{mL^4}}$$

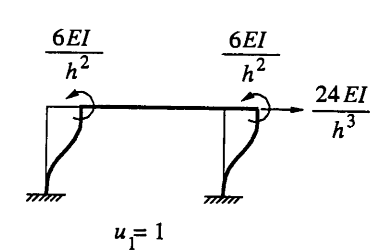
$$\omega_3 = 104.2\sqrt{\frac{EI}{mL^4}}$$

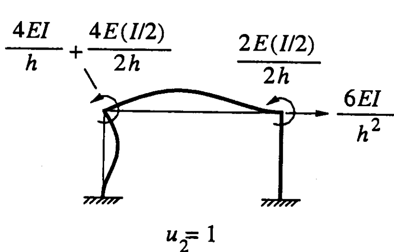
The finite element method using consistent mass provides an excellent result for the fundamental frequency, but the accuracy deteriorates increasingly for the second and third modes. The lumped mass approximation provides only the fundamental frequency; the result is less accurate and lower than the exact value.

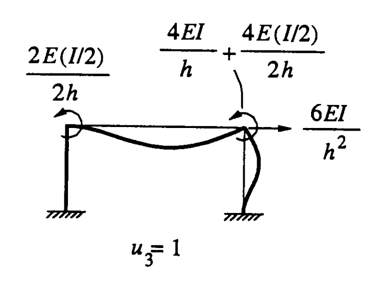
Problem 17.6



1. *Stiffness matrix.*



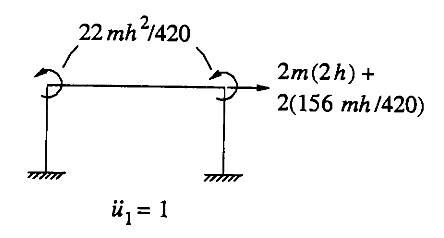


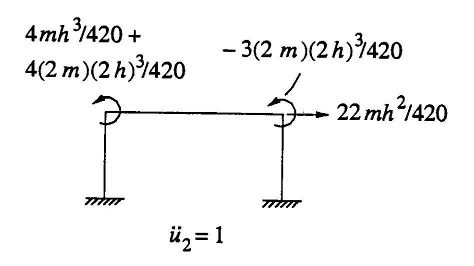


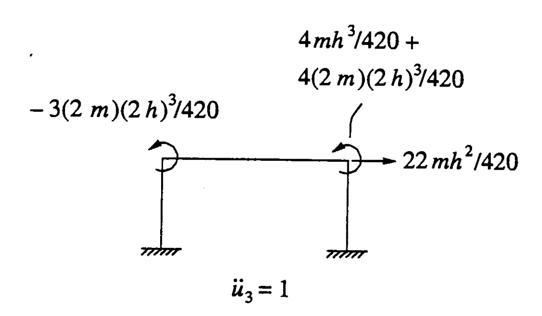
Thus

$$\mathbf{k} = \frac{EI}{h^3}\begin{bmatrix} 24 & 6h & 6h \\ & 5h^2 & h^2/2 \\ (sym) & & 5h^2 \end{bmatrix}$$

2. *Mass matrix.*







Thus

$$\mathbf{m} = \frac{mh}{420}\begin{bmatrix} 1992 & 22h & 22h \\ & 68h^2 & -48h^2 \\ (sym) & & 68h^2 \end{bmatrix}$$

3. *Solve eigenvalue problem.*

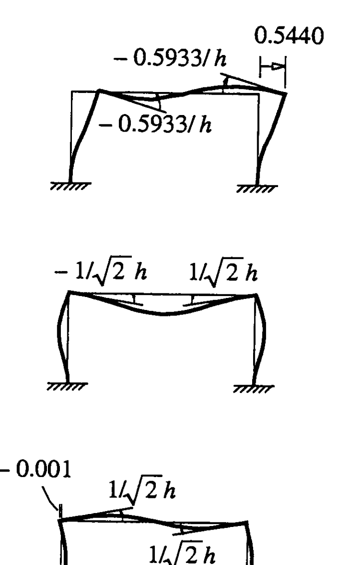
$$(\mathbf{k} - \omega^2\mathbf{m})\phi = 0$$

$$\omega_1 = 1.5354\sqrt{\frac{EI}{mh^4}} \qquad \omega_2 = 4.0365\sqrt{\frac{EI}{mh^4}}$$

$$\omega_3 = 10.7471\sqrt{\frac{EI}{mh^4}}$$

$$\Phi = \begin{bmatrix} 0.5440 & 0 & -0.0001 \\ -0.5933/h & -1/\sqrt{2}h & 1/\sqrt{2}h \\ -0.5933/h & 1/\sqrt{2}h & 1/\sqrt{2}h \end{bmatrix}$$

4. *Plot modes.*



Problem 17.7

1. *Determine lateral stiffness.*

Statically condense joint rotations in the stiffness matrix of Problem 17.6:

$$\hat{k}_{lat} = \mathbf{k}_{tt} - \mathbf{k}_{t0}\,\mathbf{k}_{00}^{-1}\,\mathbf{k}_{0t}$$

$$= \frac{EI}{h^3}\left(24 - \langle 6 \quad 6\rangle \begin{bmatrix} 5 & 1/2 \\ 1/2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 6 \end{bmatrix}\right)$$

$$= 10.9091\frac{EI}{h^3}$$

2. *Calculate lumped mass.*

$$m_{lat} = \underbrace{2m(2h)}_{\text{beam}} + \underbrace{(1/2)(mh + mh)}_{\text{columns}} = 5mh$$

3. *Calculate natural frequency.*

$$\omega_1 = \sqrt{\frac{\hat{k}_{lat}}{m_{lat}}} = 1.477\sqrt{\frac{EI}{mh^4}}$$

The exact value is

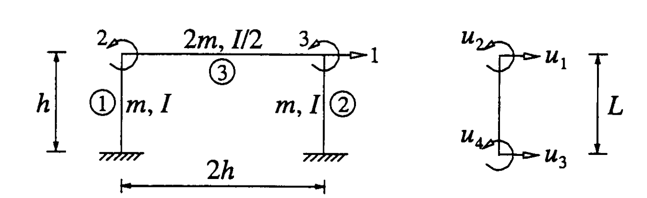
$$\omega_1 = 1.5354\sqrt{\frac{EI}{mh^4}}$$

$$\phi = \begin{bmatrix} 0.544 \\ \hline -\mathbf{k}_{00}^{-1}\,\mathbf{k}_{0t}\,(0.544) \end{bmatrix} = \begin{bmatrix} 0.544 \\ -0.594/h \\ -0.594/h \end{bmatrix}$$

4. *Comment on accuracy.*

The accuracy of the lumped mass procedure is satisfactory; the fundamental frequency is estimated with approximately 4% error and the first mode is very accurate (less than 0.03% error). Using the lumped mass procedure we have reduced the order of the system to obtain only the fundamental frequency and mode.

Problem 17.8



1. *Identify DOFs.*

The frame (assemblage) DOFs that correspond to the element DOFs are identified for each element in Table P17.8.

Table P17.8

Element		
①	②	③
1	0	1
2	2	3
0	0	0
0	3	0

2. *Element stiffness matrix*

$$\bar{\mathbf{k}}_e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{matrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 3 & 0 & 0 \end{matrix}$$

For elements ① and ②, $L = h$; for element ③, $L = 2h$.

3. *Assemble element stiffness matrices.*

$$\mathbf{k} = \begin{bmatrix} \frac{12EI}{h^3} + \frac{12EI}{h^3} & \frac{6EI}{h^2} & \frac{6EI}{h^2} \\ \frac{6EI}{h^2} & \frac{4EI}{h} + \frac{4E\,(I/2)}{h} & \frac{2E\,(I/2)}{2h} \\ \frac{6EI}{h^2} & \frac{2E\,(I/2)}{2h} & \frac{4E\,(I/2)}{h} + \frac{4EI}{h} \end{bmatrix}$$

$$= \frac{EI}{h^3}\begin{bmatrix} 24 & 6h & 6h \\ 6h & 5h^2 & h^2/2 \\ 6h & h^2/2 & 5h^2 \end{bmatrix}$$

This is the same as determined in Problem 17.6.

4. *Element mass matrix*

$$\bar{\mathbf{m}}_e = \frac{mL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & -6L & -3L^2 \\ & & 156 & -22L \\ (sym) & & & 4L^2 \end{bmatrix} \begin{matrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 3 & 0 & 0 \end{matrix}$$

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